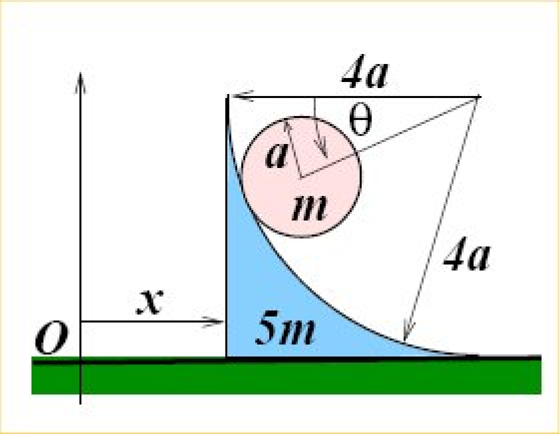
Ben Sauerwine
Practice for Qualifying Exams

Problem Source: **CMU Qualification Exam Day 1 (August 2005)**

(1) A spherical ball of radius a and mass m which is not uniformly distributed throughout its volume rolls without slipping down the circular surface of radius of curvature 4a of a wedge of mass 5m which is free to slide frictionlessly across a smooth horizontal surface, as shown. The density $\rho(r)$ of the ball varies only with the distance r from its center according to $\frac{\rho(r)}{\rho(0)} = 1 - \frac{4r}{5a}$.



(a) Show that the moment of inertia of the ball about an axis coming out of the page through the center of the ball is $I = \frac{1}{3}ma^2$.

First I need to know $\rho(0)$ in terms of ball mass M (capitalized for clarity) and radius a:

$$M = 4\pi\int_0^a \rho(r) r^2 dr = 4\pi\rho(0)\int_0^a \left[1 - \frac{4r}{5a}\right] r^2 dr = 4\pi\rho(0)\left[\frac{1}{3}a^3 - \frac{4}{5a}\frac{1}{4}a^4\right]$$

$$\rho(0) = \frac{M}{4\pi\left[\frac{1}{3}a^3 - \frac{1}{5}a^3\right]} = \frac{15M}{8\pi a^3}$$

Certainly,

$$I = \int_M r_{axis}{}^2 dm = \int_M r_{axis}{}^2 \frac{dm}{dx^3} dx^3 = \int_M [r\sin\phi]^2 \rho(0)\left[1 - \frac{4r}{5a}\right] r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \rho(0)\int_0^a \int_0^{2\pi} \int_0^{\pi} \left[r^4 - \frac{4r^5}{5a}\right] \sin^3\phi \, dr\, d\theta\, d\phi = 2\pi\rho(0)\left[\frac{1}{5}a^5 - \frac{1}{6}\cdot\frac{4a^6}{5a}\right]\int_0^{\pi} \sin^3\phi \, d\phi$$

In order to complete this integral, I take:

$$\int_0^{\pi} \sin^3 \phi d\phi = \int_0^{\pi} \left(1-\cos^2 \phi\right)\sin \phi d\phi = \int_0^{\pi} \sin \phi d\phi - \int_0^{\pi} \sin \phi \cos^2 \phi d\phi$$

$$= 2 - \int_0^{\pi} \sin \phi \cos^2 \phi d\phi$$

$$u = \cos \phi \quad du = -\sin \phi$$

$$= 2 + \int_1^{-1} u^2 du = 2 + \frac{1}{3}u^3 \Big|_1^{-1} = 2 - \frac{2}{3} = \frac{4}{3}$$

Collecting these results, then, I have

$$I = 2\pi \frac{15M}{8\pi a^3}\left[\frac{1}{5}a^5 - \frac{1}{6}\cdot\frac{4a^6}{5a}\right]\frac{4}{3} = \frac{M}{a^3}\left[a^5 - \frac{2}{3}a^5\right] = \frac{1}{3}Ma^2$$

(b) Let x denote the rightward horizontal displacement of the left vertical edge of the wedge from the origin O, and θ denote the angle which a line connecting the center of curvature of the surface to the center of the ball makes with the horizontal. Determine the Lagrangian L of the system in terms of $x, \theta, \dot{x}, \dot{\theta}$.

For the purposes of tracking the different values through this problem, let

$$R = 4a$$

$$M = 5m$$

$$I = \frac{1}{3}ma^2$$

And of course, $L = K - V$

There are four terms to consider here:

-The kinetic energy of the center-of-mass motion of the wedge.

$$K_{CM \quad wedge} = \frac{1}{2}M\dot{x}^2$$

-The kinetic energy of the center-of-mass motion of the ball.

$$K_{CM \quad ball} = \frac{1}{2}m\left[\left(\dot{x} + (R-a)\dot{\theta}\sin\theta\right)^2 + \left((R-a)\dot{\theta}\cos\theta\right)^2\right]$$

$$K_{CM \quad ball} = \frac{1}{2}m\left[\dot{x}^2 + 2(R-a)\dot{\theta}\dot{x}\sin\theta + (R-a)^2\dot{\theta}^2\right]$$

-The rotational kinetic energy of the ball.

Again, consider the rotational kinetic energy of the ball: recall the no-slipping condition.

$$K_{Rot\quad ball} = \frac{1}{2} I\omega^2$$

$$\omega = \left(\frac{R}{a}\dot{\theta} - \dot{\theta} \right)$$

$$K_{Rot\quad ball} = \frac{1}{2} I\left(\frac{R}{a}\dot{\theta} - \dot{\theta} \right)^2$$

Why does this discrepancy from what one might expect exist? Here is the fallacy that one might fall into:

There is always a one-to-one correspondence between points on the ball and points on the surface of the wedge such that an equal distance is traversed by the contact point about the outside of the ball as the length of the surface. However, it is a fallacy to believe that this length is then equal to the actual rotation that has occurred with respect to a Cartesian coordinate system. I must correct for this precession about the outside of the ball by subtracting off $\dot{\theta}$ in the central coordinate system, as this point will undoubtedly precess exactly once per rotation about the large circle. (Imagine, if you will, an enormous inside circle rolling inside a slightly larger outer circle. Will it rotate very nearly once as it rolls about the inside of the outer circle? No, in fact it rotates very slowly.)

-*The gravitational potential energy of the ball.*

Assume that the scale of this picture is small enough such that the acceleration due to gravity is constant.

$$V_{ball} = (R-a)mg(1-\sin\theta)$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\left(m(R-a)^2 + I\left(\frac{R}{a} - 1 \right)^2 \right)\dot{\theta}^2 + m(R-a)\dot{\theta}\dot{x}\sin\theta - (R-a)mg(1-\sin\theta)$$

$$= 3m\dot{x}^2 + 6ma^2\dot{\theta}^2 + 3am\dot{\theta}\dot{x}\sin\theta - 3amg(1-\sin\theta)$$

c) From the equations of motion, find expressions for the accelerations $\ddot{x}$ and $\ddot{\theta}$ in terms of $x, \theta, \dot{x}, \dot{\theta}$.

Using:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = \left(\frac{\partial L}{\partial q_i} \right)$$

I have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = \left(\frac{\partial L}{\partial x} \right)$$

$$\frac{d}{dt}\Big((M+m)\dot{x} + m(R-a)\dot{\theta}\sin\theta \Big) = 0$$

$$(M+m)\ddot{x} + m(R-a)\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \right) = 0$$

Further, I have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\left(\frac{\partial L}{\partial\theta}\right)$$

$$\frac{d}{dt}\left(\left(m(R-a)^2+I\left(\frac{R}{a}-1\right)^2\right)\dot{\theta}+m(R-a)\dot{x}\sin\theta\right)=m(R-a)\cos\theta\left(\dot{\theta}\dot{x}+g\right)$$

$$\left(m(R-a)^2+I\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}+m(R-a)\left(\ddot{x}\sin\theta+\dot{x}\dot{\theta}\cos\theta\right)=m(R-a)\cos\theta\left(\dot{\theta}\dot{x}+g\right)$$

Simplifying these a bit, then,

$$\ddot{x}=-\frac{m(R-a)}{(M+m)}\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)$$

$$\ddot{x}=\frac{\cos\theta}{\sin\theta}\left(\dot{\theta}\dot{x}+g\right)-\frac{1}{\sin\theta}\left((R-a)+\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}-\dot{x}\dot{\theta}\frac{\cos\theta}{\sin\theta}$$

for

$$\frac{\cos\theta}{\sin\theta}\left(\dot{\theta}\dot{x}+g\right)-\frac{1}{\sin\theta}\left((R-a)+\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}-\dot{x}\dot{\theta}\frac{\cos\theta}{\sin\theta}=-\frac{m(R-a)}{(M+m)}\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)$$

$$\cos\theta\left(\dot{\theta}\dot{x}+g\right)-\left((R-a)+\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}-\dot{x}\dot{\theta}\cos\theta=-\sin\theta\frac{m(R-a)}{(M+m)}\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)$$

$$\left(\sin^2\theta\frac{m(R-a)}{(M+m)}-(R-a)-\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}=-\sin\theta\frac{m(R-a)}{(M+m)}\left(\dot{\theta}^2\cos\theta\right)-\cos\theta\left(\dot{\theta}\dot{x}+g\right)+\dot{x}\dot{\theta}\cos\theta$$

$$\left(\sin^2\theta\frac{m(R-a)}{(M+m)}-(R-a)-\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2\right)\ddot{\theta}=-\sin\theta\frac{m(R-a)}{(M+m)}\left(\dot{\theta}^2\cos\theta\right)-g\cos\theta$$

$$\ddot{\theta}=\frac{\sin\theta\frac{m(R-a)}{(M+m)}\left(\dot{\theta}^2\cos\theta\right)+g\cos\theta}{(R-a)+\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2-\sin^2\theta\frac{m(R-a)}{(M+m)}}\rightarrow\left[\frac{\dot{\theta}^2\sin\theta+\frac{2g}{a}}{8-\sin^2\theta}\right]\cos\theta$$

Further, then, I have:

$$\ddot{x}=-\frac{m(R-a)}{(M+m)}\left(\frac{\sin\theta\frac{m(R-a)}{(M+m)}\left(\dot{\theta}^2\cos\theta\right)+g\cos\theta}{(R-a)+\frac{I}{m(R-a)}\left(\frac{R}{a}-1\right)^2-\sin^2\theta\frac{m(R-a)}{(M+m)}}\sin\theta+\dot{\theta}^2\cos\theta\right)$$

$$\rightarrow-\frac{a}{2}\left(\left[\frac{\dot{\theta}^2\sin\theta+\frac{2g}{a}}{8-\sin^2\theta}\right]\cos\theta\sin\theta+\dot{\theta}^2\cos\theta\right)$$

(d) If both the ball and the wedge start initially at rest with the ball positioned such that $\theta = 0$, eventually the wedge ends up moving with uniform speed v_f to the left. Determine an expression for v_f, simplified as much as possible.

Solving these differential equations seems unnecessarily complex. Instead, I'll use a time-honored bit of treachery to determine the solution. Look at my total Kinetic and Potential energies: Certainly, the ball has traveled a certain height downward at this time:

$$K + V = const$$

$$\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\left(m(R-a)^2 + I\left(\frac{R}{a}-1\right)^2\right)\dot{\theta}^2 + m(R-a)\dot{\theta}\dot{x}\sin\theta = K$$

$$(R-a)mg(1-\sin\theta) = V$$

at the bottom of the slope when the potential energy has become kinetic, this becomes

$$\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\left(m(R-a)^2 + I\left(\frac{R}{a}-1\right)^2\right)\dot{\theta}^2 + m(R-a)\dot{\theta}\dot{x} = (R-a)mg$$

or

$$3m\dot{x}^2 + \frac{1}{2}\left(m(3a)^2 + \frac{1}{3}ma^2 9^2\right)\dot{\theta}^2 + m(3a)\dot{\theta}\dot{x} = (3a)mg$$

$$3\dot{x}^2 + 6a^2\dot{\theta}^2 + 3a\dot{\theta}\dot{x} = 3ag$$

Further, I know that net momentum in the x-direction is conserved:

$$M\dot{x} + m\left(\dot{x} + (R-a)\dot{\theta}\sin\theta\right) = 0$$

$$\dot{\theta} = -\frac{(M+m)}{m(R-a)\sin\theta}\dot{x} \to -\frac{2}{a}\dot{x}$$

Overall, then, I have

$$3\dot{x}^2 + 6a^2\left(\frac{2}{a}\right)^2\dot{x}^2 - 3a\frac{2}{a}\dot{x}^2 = 3ag$$

$$(3 + 24 - 6)\dot{x}^2 = 3ag$$

$$\dot{x} = -\sqrt{\frac{ag}{7}}$$

Here, I have chosen the negative root since the wedge will clearly end up going to the left. Note that the ball will start slipping after it leaves the wedge: the angular velocity required to roll without slipping is discontinuous here.