

Ben Sauerwine
Practice for Qualifying Exams
Mathematical Methods

Problem Source: CMU August 2005 Qualifying Exam

Suppose we are interested in studying the motion of an an-harmonic oscillator whose Lagrangian is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + g(t)x^3$$

we will assume that $g(t)$ is small and acts over time as δt which is much shorter than the time scale $\frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$. That is, we will treat the tri-linear term as a perturbation.

We wish to find the motion of the oscillator to linear order in g at a time well after the perturbation is turned off. To do this we will use the Green's function method.

(a) Write down the equations of motion.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$m\ddot{x} = -kx + 3g(t)x^2$$

(b) The Green's function obeys

$$\ddot{G}(t-t') + \omega_0^2 G(t-t') = \delta(t-t') \quad (*)$$

Show that the motion of x is given by

$$x(t) = x_0(t) + \frac{3}{m} \int dt' G(t-t') g(t') x^2(t')$$

where $x_0(t)$ is a solution to the simple harmonic oscillator motion, (i.e., ignoring the anharmonic term.)

Specifically, $\int G(t-t') \left[\frac{\partial^2}{\partial t'^2} + \omega_0^2 \right] x(t') dt'$

$$\ddot{x} + \frac{k}{m}x = \frac{3}{m}g(t)x^2$$

$$\left(\frac{\partial^2}{\partial t'^2} + \omega_0^2\right)x(t') = \frac{3}{m}g(t')x(t')^2$$

By the property of Green's functions as an inverse of this linear operator, I have

$$x(t) = x_0(t) + \int G(t-t') \frac{3}{m}g(t')x(t')^2 dt'$$

where the solution $x_0(t)$ to the harmonic part represents a portion of the kernel of this linear operator.

(c) Write $G(t-t') = \frac{1}{2\pi} \int d\omega e^{i\omega(t-t')} \tilde{G}(\omega)$ and solve for $\tilde{G}(\omega)$.

$$\ddot{G}(t-t') + \omega_0^2 G(t-t') = \delta(t-t')$$

$$\left(\frac{\partial^2}{\partial t'^2} + \omega_0^2\right) \int d\omega e^{i\omega(t-t')} \tilde{G}(\omega)$$

$$= \int d\omega e^{i\omega(t-t')} [-\omega^2 + \omega_0^2] \tilde{G}(\omega)$$

$$\tilde{G}(\omega) = \frac{1}{\omega_0^2 - \omega^2}$$

where this is chosen to ensure that the integral will result in the desired delta function.

(d) Now we would like to solve for $G(t-t')$ but we find that the integral is not well defined due to the existence of poles on the real axis. To deal with this contribution we add an infinitesimal contribution to the action such that

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k + i\varepsilon)x^2 + g(t)x^3$$

where $\varepsilon > 0$. How does this addition to the Lagrangian change equation (*) from part b?

The equation of motion now becomes

$$m\ddot{x} + (k + i\varepsilon)x = 3g(t)x^2$$

$$\ddot{G}(t-t') + (\omega_0^2 + i\varepsilon)G(t-t') = \delta(t-t')$$

$$\rightarrow \ddot{G}(t-t') + (\omega_0 + i\varepsilon)^2 G(t-t') = \delta(t-t')$$

(e) This prescription makes the integral for $G(t-t')$ well defined. Perform this integration for the cases (a) $t > t'$ and (b) $t < t'$.

For case (a), the exponential has positive sign and must be closed in the upper half-plane. In the complex plane, this clearly has poles at $\omega = \pm\omega_0$. Now, however, with this addition I see that the poles are pushed slightly into the upper half-plane, so that $\omega = \pm(\omega_0 + i\varepsilon)$

$$\begin{aligned} \frac{1}{2\pi} \int d\omega \frac{1}{\omega_0^2 + i\varepsilon - \omega^2} e^{i\omega(t-t')} &\rightarrow \frac{1}{2\pi} \left[2\pi i \operatorname{Re} s \left[\left\{ \omega = \omega_0 \right\}, \frac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2} \right] + 2\pi i \operatorname{Re} s \left[\left\{ \omega = -\omega_0 \right\}, \frac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2} \right] \right] \\ &= i \left[\frac{e^{i\omega_0(t-t')}}{(\omega_0 + \omega_0)} + \frac{e^{-i\omega_0(t-t')}}{(\omega_0 + \omega_0)} \right] = \frac{i \cos(\omega_0(t-t'))}{\omega_0} \theta(t-t') \end{aligned}$$

For case (b), the green's function is zero since the poles were pushed into the upper half-plane.

(f) To leading order in $g(t)$, write down an equation of motion of the oscillator assuming that the motion of the oscillator prior to the turning on of the perturbation is given by $x_0(t)$

Assume that the perturbation is turned on at time zero.

$$x(t) = x_0(t) + \int_{-\infty}^{\infty} \frac{i \cos(\omega_0(t-t'))}{\omega_0} \theta(t-t') \frac{3}{m} g(t') x(t')^2 dt'$$

$$x(t) = x_0(t) + \frac{3i}{\omega_0 m} \int_{-\infty}^t \cos(\omega_0(t-t')) g(t') x(t')^2 dt'$$

Leading Order: $g(t') = g \text{ const}$

$$x(t) = x_0(t) + \frac{3ig}{\omega_0 m} \int_{-\infty}^t \cos(\omega_0(t-t')) g(t') x(t')^2 dt'$$

$$x(t') \approx A e^{i\omega_0 t'}$$

$$x(t) = x_0(t) + \frac{3igA^2}{\omega_0 m} \int_{-\infty}^t \cos(\omega_0(t-t')) e^{2i\omega_0 t'} dt' = x_0(t) + \frac{3igA^2}{2\omega_0 m} \int_{-\infty}^t \left[e^{i\omega_0(t-t')} + e^{-i\omega_0(t-t')} \right] e^{2i\omega_0 t'} dt'$$

$$= x_0(t) + \frac{3igA^2}{2\omega_0 m} \left[e^{i\omega_0 t} \int_{-\infty}^t e^{i\omega_0 t'} e^{2i\omega_0 t'} dt' + e^{-i\omega_0 t} \int_{-\infty}^t e^{-i\omega_0 t'} e^{2i\omega_0 t'} dt' \right]$$

$$\rightarrow x_0(t) + \frac{3igA^2}{2\omega_0 m} \left[e^{i\omega_0 t} e^{3i\omega_0(t+\delta)} + e^{-i\omega_0 t} e^{i\omega_0(t+\delta)} \right]$$

δ : phase shift