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Practice for Qualifying Exams

Problem Source: **CMU August 2005 Qualifying Exam**

(a) A 10^4 kg spacecraft is launched from Earth and is to be radially accelerated away from the Sun using a circular solar sail. The initial acceleration of the sail is to be 1g. Assuming a flat sail, determine the radius of the sail if it is:
(i) Black, so that it absorbs the Sun's light $\left(1.3 \, {}^{kW}\!/_{m^2}\right)$.

The radiation pressure is given by $\left(\frac{1.3 \, {}^{kW}\!/_{m^2}}{c}\right) = \frac{1300}{c}\frac{J}{m^3} = \frac{1300}{c}\frac{Nm}{m^3} = \frac{1300}{c}\frac{N}{m^2}$

Here, c is the speed of light.

Then this force is

$$m \cdot g = \frac{1300}{c} \cdot \pi r^2$$

$$\sqrt{\frac{m \cdot g \cdot c}{1300\pi}} = r$$

(ii) Shiny, so that it reflects the Sun's light.

Now the radiation pressure is doubled since the light is reflected. At double the force, then, I have

$$m \cdot g = 2 \cdot \frac{1300}{c} \cdot \pi r^2$$

$$\sqrt{\frac{m \cdot g \cdot c}{2 \cdot 1300\pi}} = r$$

(a) A small part of the solar sail collects radiation emanating from an extremely large number of atoms in the solar photosphere, and these sources are incoherent. At any given instant, there is a near-infinite number of phasors in the complex plane, all of different magnitudes and different phases, rotating at different velocities. Their vector sum is zero, so perhaps the net field on the sail should be zero. How do you explain this paradox?

It is highly improbable that the vector sum of a number of random vectors would be zero, though the expected mean probably would be: in fact, it would have an expected magnitude based on the coherence, density, and magnitude of incident waves. Thus, at any particular point an essentially random and probably nonzero vector gives the electric and magnetic field.

(b) Given the electromagnetic wave
$\vec{E} = \hat{i}E_0 \cos(kz - \omega t) + \hat{j}E_0 \sin(kz - \omega t)$,
where E_0 is a constant. Find the corresponding magnetic field $\vec{B}$ and the Poynting vector.

Using $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, I have

$$\vec{\nabla} \times \vec{E} = -k\hat{j}E_0 \sin(kz - \omega t) - k\hat{i}E_0 \cos(kz - \omega t)$$
$$\frac{\partial \vec{B}}{\partial t} = k\hat{j}E_0 \sin(kz - \omega t) + k\hat{i}E_0 \cos(kz - \omega t)$$
$$\vec{B} = -\frac{k}{\omega}\hat{j}E_0 \cos(kz - \omega t) + \frac{k}{\omega}\hat{i}E_0 \sin(kz - \omega t)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$
$$= \frac{1}{\mu_0}\hat{k}\frac{kE_0^2}{\omega}\left(-\cos^2(kz = \omega t) - \sin^2(kz = \omega t)\right) = -\frac{1}{\mu_0}\frac{kE_0^2}{\omega}\hat{k}$$

(c) Show that in free space with $\rho = 0, \vec{J} = 0$, Maxwell's equations are correctly obtained from a single vector function $\vec{A}$ satisfying
$\vec{\nabla} \cdot \vec{A} = 0, \nabla^2 \vec{A} - \frac{1}{c^2}\frac{d^2 \vec{A}}{dt^2} = 0$.

In the Coulomb gauge,
$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$
And the Maxwell's equations are
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$
In charge and current free space, I have:

$$\vec{\nabla}\cdot\vec{E}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Consider $\vec{E}=\vec{\nabla}\cdot V$, taking $\vec{A}_0=\vec{A}+\vec{\nabla}V$, chosen so that $\vec{A}_0$ is divergenceless and so $-\vec{\nabla}\cdot\vec{A}=0=\nabla^2 V$ for this charge-free region.
I see that whatever $\vec{A}$ represents, $\vec{B}=\vec{\nabla}\times\vec{A}$ selects this curl part and since the divergence of a curl is always zero, $\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$.

Now in order for $\vec{\nabla}\cdot\vec{A}=0$, I need $\vec{\nabla}\cdot\vec{A}_0+\nabla^2 V=0$. I've already shown that the definition $\vec{B}=\vec{\nabla}\times\vec{A}$ works for the divergenceless part $\vec{A}_0$, and so I have $\nabla^2 V=\vec{\nabla}\cdot\vec{E}=0$, as expected.

Next, write:

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$$

$$\vec{\nabla}\times\vec{E}=\vec{\nabla}\times\left(-\vec{\nabla}V\right)-\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)$$

$$\vec{\nabla}\times\left(-\vec{\nabla}V\right)\to 0$$

$$\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)\to\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right)$$

$$\vec{\nabla}\times\vec{E}=-\left(\frac{\partial\vec{B}}{\partial t}\right)$$

Finally, write

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{\nabla}^2\vec{A}=-\vec{\nabla}^2\vec{A}\quad\left(in\quad this\quad gauge\right)$$

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{d^2\vec{A}}{dt^2}=0$$

$$\vec{\nabla}\times\vec{B}=-\frac{1}{c^2}\frac{d^2\vec{A}}{dt^2}=-\mu_0\varepsilon_0\frac{d^2\vec{A}}{dt^2}$$

$$\vec{E}=-\vec{\nabla}V-\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}\vec{E}=-\vec{\nabla}\frac{dV}{dt}-\frac{d^2\vec{A}}{dt^2}$$

$$\left(ch\arg efree\quad space\right):V\to 0$$

$$\frac{d}{dt}\vec{E}=-\frac{d^2\vec{A}}{dt^2}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{d\vec{E}}{dt}$$

(d) As shown by Arthur Eddington, there is a simple limit to the luminosity of a star or the X-ray emitting plasma around a neutron star or black hole. The limit derives from the case where the radiation pressure on an electron in the plasma is greater than the gravitational pull of the central astrophysical object. Assuming that the plasma is uniformly spherically distributed around the central object of mass M, derive an expression for this limiting luminosity, the Eddington limit, in terms of subatomic parameters. Note: The luminosity is defined as the energy per unit time radiated off by an astrophysical object. Also, you may find it useful to know the classical electron radius,

$$r_e=\frac{e^2}{m_ec^2}=2.8\cdot 10^{-13}cm$$

and/or the Thomson scattering cross section,

$$\sigma_T=\frac{8\pi}{3}r_e^2=6.65\times 10^{-25}cm^2.$$

The radiation pressure in terms of luminosity L is then given by $\frac{L}{4\pi R_{object}{}^2c}$. Now I take the force $\frac{L}{4\pi R_{object}{}^2c}\sigma_T$. This proposes that in a stable star, all matter must stay in this region and so only the cross-section of the surface electron facing inward matters, since this face absorbs from the electrons slightly more inside than this one. Then, the maximum luminosity occurs where

$$\frac{L}{4\pi R_{object}{}^{2} c}\sigma_T = G\frac{m_e M}{R^2_{object}}, \text{ or } L = \frac{4\pi}{\sigma_T} M m_e G c .$$