

Problem Source: **CMU Qualification Exam Day 1 (August 2004)**

(3) This problem concerns the precession of perihelion predicted by general relativity and how it might be generated by modifications of the distribution of mass in the sun.

(a) Write down the Lagrangian for the central force problem for a particle restricted to the equatorial plane ($\theta = \frac{\pi}{2}$) and show that the Euler-Lagrange equations are:

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V(r)}{\partial r} \quad \text{and} \quad \frac{d}{dt}(mr^2\dot{\phi}) = 0$$

where r is the magnitude of the radius vector connecting the sun to the planet (still in equatorial plane), ϕ is the angle measured from the perihelion or closest approach of the planet to the sun and $V(r)$ is an arbitrary central potential. What is the physical interpretation of the second equation?

$$L = K - V$$

$$L = \frac{1}{2}m\left((r\dot{\phi})^2 + \dot{r}^2\right) - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

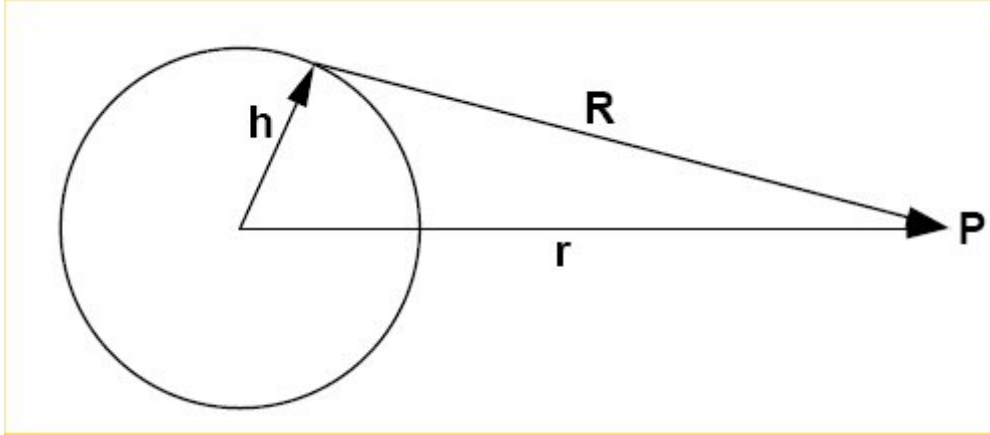
$$m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V(r)}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0$$

This latter equation represents conservation of angular momentum.

(b) Consider a thin uniform ring of matter of radius h and mass M_{ring} as shown below.



Write down an integral expression, including limits of integration, for the gravitational potential energy per unit mass at the point P in the figure. Note that P lies in the plane of the ring. Do not compute the integral at this time.

$$\begin{aligned}
 - \int_0^{2\pi} G \frac{M_{ring}}{2\pi R} d\theta &= - \int_0^{2\pi} G \frac{M_{ring}}{2\pi \sqrt{(r-h\cos\theta)^2 + (h\sin\theta)^2}} d\theta \\
 &= - \int_0^{2\pi} G \frac{M_{ring}}{2\pi \sqrt{r^2 + h^2 - 2rh\cos\theta}} d\theta \\
 &= -2 \int_0^{\pi} \frac{GM_{ring}}{2\pi \sqrt{r^2 + h^2 - 2rh\cos\theta}} d\theta
 \end{aligned}$$

(c) Expand the integrand that you found above in the limit that $r \gg h$, keeping terms up to order $\frac{h^2}{r^2}$. Integrate this expression to find the potential at P and check that you have arrived at the expected answer in the limit $r \rightarrow \infty$.

Let $x = \frac{h}{r}$. Now:

$$\begin{aligned}
 &-2 \int_0^{\pi} \frac{GM_{ring}}{2\pi \sqrt{r^2 + h^2 - 2rh\cos\theta}} d\theta \\
 &= \frac{-GM_{ring}}{\pi r} \int_0^{\pi} \frac{1}{\sqrt{1+x^2-2x\cos\theta}} d\theta
 \end{aligned}$$

Through Taylor expansion, I get

$$\frac{\partial}{\partial x} \left[\frac{1}{\sqrt{1+x^2-2x\cos\theta}} \right] = \left(\frac{-1}{2} \right) \left(\frac{2x-2\cos\theta}{(1+x^2-2x\cos\theta)^{\frac{3}{2}}} \right)$$

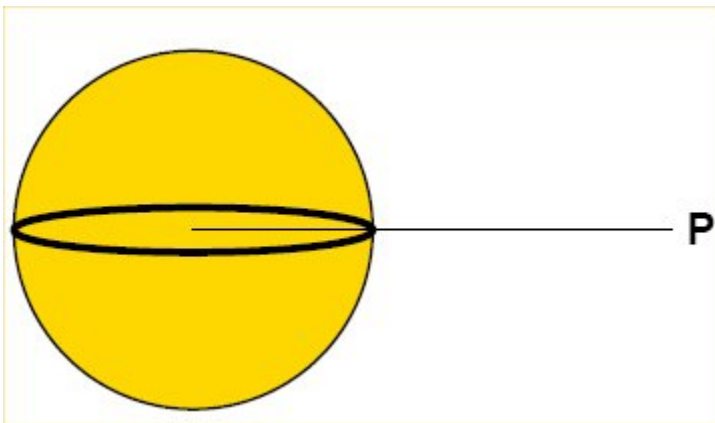
$$\frac{\partial^2}{\partial x^2} \left[\frac{1}{\sqrt{1+x^2-2x\cos\theta}} \right] = \left(\frac{-1}{2} \right) \left(\frac{2}{(1+x^2-2x\cos\theta)^{\frac{3}{2}}} \right) + \left(\frac{-1}{2} \right) \left(\frac{-3}{2} \right) \left(\frac{(2x-2\cos\theta)^2}{(1+x^2-2x\cos\theta)^{\frac{5}{2}}} \right)$$

and so the power series becomes:

$$\begin{aligned} &= \frac{-GM_{ring}}{\pi r} \int_0^\pi \frac{1}{\sqrt{1+x^2-2x\cos\theta}} d\theta \\ &= \frac{-GM_{ring}}{\pi r} x \int_0^\pi \left(1 + \left(\frac{-1}{2} \right) (-2\cos\theta)x + \frac{1}{2} \left(\frac{-1}{2} \right) \left(2 + \left(\frac{-3}{2} \right) (-2\cos\theta)^2 \right) x^2 + \dots \right) d\theta \\ &= \frac{-GM_{ring}}{\pi r} x \left(\pi + \frac{1}{2} \left(\frac{-1}{2} \right) \left(2\pi + \left(\frac{-3}{2} \right) (4) \left(\frac{\pi}{2} \right) \right) x^2 + \dots \right) \\ &= \frac{-GM_{ring}}{\pi r} x \left(\pi + \left(-\frac{\pi}{2} + \left(\frac{3}{2} \right) \left(\frac{\pi}{2} \right) \right) x^2 + \dots \right) = -\frac{GM_{ring}}{r} \left(1 + \frac{1}{4} \left(\frac{h}{r} \right)^2 + \dots \right) \end{aligned}$$

this clearly goes to the expected value of zero as r goes to infinity.

- (d) Suppose we model a deviation from a spherically symmetric mass distribution for the sun by adding a “bulge” to it given by a thin ring of mass as above at the Sun’s equator. Take the mass and radius of the sun to be M_{sun}, R_{sun} , while the mass of the bulge will be M_{bulge} and its radius to be that of the sun. Add the gravitational potentials from both the spherical mass distribution as well as that of the bulge to find the gravitational potential energy of a mass M at a point P. The distance r is described in part (b).**



$$U = -M_p G \left[\frac{M_{bulge}}{r} \left(1 + \frac{1}{4} \left(\frac{R_{sun}}{r} \right)^2 \right) + \frac{M_{sun}}{r} \right]$$

- (e) Use the above potential to derive the force acting on a planet of mass m moving in equatorial orbit around the sun.

$$\vec{F} = -\vec{\nabla}U = -\frac{MG}{r^2}(M_{bulge} + M_{sun})\hat{r} - \frac{3MG}{4r^4}M_{bulge}R_{sun}^2\hat{r}$$

- (f) Use the equations of motion above together with the change of variables

$u = \frac{1}{r}$ to find that for a general central potential $V(r)$ we can write

$$\frac{d^2u(\phi)}{d\phi^2} + u(\phi) = -\frac{m}{u^2L^2}F(u)$$

where we have changed the independent variable from t to the angle ϕ , $F(u)$

is the force $F(r)$ derived from the potential $V(r)$ after the substitution $u = \frac{1}{r}$,

and L is the constant angular momentum about the axis perpendicular to the plane of the orbit. Apply this formula to the potential found for the sun together with the bulge.

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V(r)}{\partial r}$$

$$mr^2\dot{\phi} = L$$

$$\frac{d}{dt} = \frac{\partial\phi}{\partial t} \frac{d}{d\phi} = \frac{L}{mr^2} \frac{d}{d\phi} = \frac{Lu^2}{m} \frac{d}{d\phi}$$

$$\frac{d^2}{dt^2} = \frac{Lu^2}{m} \frac{d}{d\phi} \left(\frac{Lu^2}{m} \frac{d}{d\phi} \right) = 2 \frac{L^2u^3}{m^2} \frac{du}{d\phi} \frac{d}{d\phi} + \frac{L^2u^4}{m^2} \frac{d^2}{d\phi^2}$$

$$m \frac{\partial^2}{\partial t^2} r = m \frac{\partial^2}{\partial t^2} \frac{1}{u} = m \left[2 \frac{L^2u^3}{m^2} \frac{du}{d\phi} \frac{d}{d\phi} + \frac{L^2u^4}{m^2} \frac{d^2}{d\phi^2} \right] \frac{1}{u}$$

$$= m \left[2 \frac{L^2u^3}{m^2} \frac{du}{d\phi} \left(\frac{-1}{u^2} \right) \frac{du}{d\phi} + \frac{L^2u^4}{m^2} \left(\frac{2}{u^3} \left(\frac{du}{d\phi} \right)^2 + \frac{-1}{u^2} \frac{d^2u}{d\phi^2} \right) \right]$$

$$= m \left[-2 \frac{L^2u}{m^2} \left(\frac{du}{d\phi} \right)^2 + \left(\frac{2L^2u}{m^2} \left(\frac{du}{d\phi} \right)^2 - \frac{L^2u^2}{m^2} \frac{d^2u}{d\phi^2} \right) \right] = -\frac{L^2u^2}{m} \frac{d^2u}{d\phi^2}$$

And for the other portion:

$$mr\dot{\phi}^2 - \frac{\partial V(r)}{\partial r} = \frac{m}{u} \left(\frac{Lu^2}{m} \right)^2 + F(r)$$

For:

$$-\frac{L^2 u^2}{m} \frac{d^2 u}{d\phi^2} = \frac{m}{u} \left(\frac{Lu^2}{m} \right)^2 + F(r)$$

$$\frac{d^2 u}{d\phi^2} = -u - \frac{m}{u^2 L^2} F(r)$$

$$\frac{d^2 u}{d\phi^2} + u = -\frac{m}{u^2 L^2} F(r) \rightarrow -\frac{m}{u^2 L^2} F(u)$$

$$F(u) = -MGu^2 (M_{bulge} + M_{sun}) - \frac{3}{4} MM_{bulge} Gu^4 R_{sun}^2$$

Just as expected.

- (g) **The force law found by adding the bulge to the mass distribution of the sun is of the same form as the modification given by Einstein's theory of general relativity. The full equation of motion can be written as:**

$$\frac{d^2 u}{d\phi^2} + u = \frac{1}{\alpha} + \delta u^2$$

where δ is constant. We consider the limit in which $\frac{\delta}{\alpha} \ll 1$. Solve this

equation in the limit that the u^2 term can be neglected. Describe a procedure that you might use to get an approximate solution to the full equation. How could you use this to get the precession of perihelion due to the bulge?

First, try solving:

$$\frac{d^2 u}{d\phi^2} + u = \frac{1}{\alpha}$$

Both homogeneous and particular solutions are necessary. Clearly, homogeneous solutions are of the form $u = c_1 \cos \theta + c_2 \sin \theta$. A particular solution is of the form

$$u = \frac{1}{\alpha}, \text{ so that the general solution is the sum } u = \frac{1}{\alpha} + c_1 \cos \theta + c_2 \sin \theta.$$

One way to approximate the solution to the general equation would be to utilize the power-series method, writing the solution as $u = \sum_{x=0}^{\infty} c_x \theta^x$. This is complicated a bit by

the presence of the u^2 term, so that in developing the recursion one uses

$$u^2 = \sum_{x=0}^{\infty} \sum_{i=0}^{\lfloor \frac{x}{2} \rfloor} \left[i = \frac{x}{2} \rightarrow 1 \quad \text{else} \quad 2 \right] c_x c_{x-i} \theta^x.$$

This allows one to get the short-term solution.

Through analytic continuation, taking steps within the radius of convergence of this solution one could then extrapolate through one cycle in order to determine the offset of

the next perihelion one cycle later. This would give the precession of perihelion due to the bulge.