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Practice for Qualifying Exams

Problem Source: **CMU August 2004 Qualifying Exam**

The Fourier transform $\hat{f}(k)$ of a function $f(x)$ is defined by

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f(x)dx$$

(a) Show that the Fourier transform of $f'(x)=\frac{df}{dx}$ can be expressed in a simple way in terms of $\hat{f}(k)$. Provide a derivation or indicate your reasoning.

$$\frac{\partial}{\partial k}\hat{f}(k)=\frac{-ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f(x)dx$$

$$dv=-ike^{-ikx}dx \qquad v=e^{-ikx}$$

$$u=f(x) \qquad du=f'(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f'(x)dx$$

$$\frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}\to 0 \quad (\text{int}\,egrable)$$

$$=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f'(x)dx=\frac{\partial}{\partial k}\hat{f}(k)=-ik\hat{f}(k)$$

is one simple way of expressing the Fourier transform of $f'(x)=\frac{df}{dx}$.

(b) State the conditions under which a contour integral of a function analytic in a suitable region in the complex plane can be expressed in terms of the residues of that function, and define what is meant by a residue. Be explicit in discussing what kinds of singularities a function may possess in order for this method to work.

First, the function must be analytic (e.g., in complex space, its derivative at a point is not a function of how that point is approached and further its derivatives are continuous) in an open region of complex space. This is equivalent to saying that if $f(x+iy)=u(x,y)+iv(x,y)\to\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ over a region of space that does not include its borders. The contour must be analytic in this entire region, except perhaps at some poles or singularities.

Second, the function must be single-valued. One way of ensuring that the function remains single-valued is to place “branch cuts” that no contour may cross in such a way as to prevent the function from ever encountering a situation where the value across the contour would be discontinuous.

There are several types of poles and singularities. The simplest to deal with are poles, which are zeroes of the denominator of a rational function. The "residue formula" is sufficient to deal with these. Also simple to deal with are removable singularities, where the function may be discontinuous. These removable singularities are dealt with by simply defining the value at a point to be its limit as approached from its surroundings. Difficult to deal with are essential singularities, where a function's Laurent series contains arbitrarily high powers of $\frac{1}{x}$.

The residue formula for poles is:

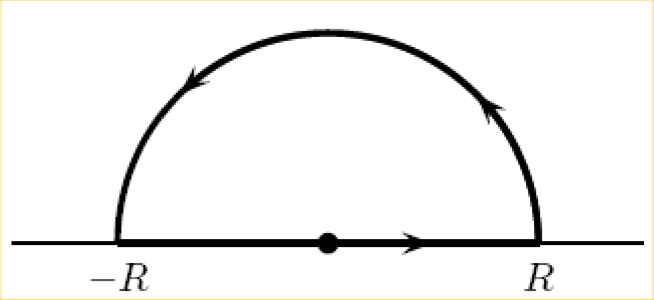
$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]_{z=z_0}$$

where m is the multiplicity of the pole. Clearly, then it won't work for essential singularities where this coefficient must be found by other means.

(c) Indicate how the Fourier transform $\hat{f}(k)$ of the function

$$f(x) = \frac{1}{1+x^2}$$

can be obtained for k in a suitable range (you should state what it is) by using a contour integral in the complex plane as indicated in the following figure, where the contour consists of a segment $[-R, R]$ of the real axis and a semicircle of radius R, centered at the origin, in the upper half-plane.



What is the minimum value of R such that the integral will have the desired value? Give an argument why the integral is equal to a constant times $\hat{f}(k)$ in the limit as R tends to infinity; pay particular attention to the contribution to the integral coming from the semicircle.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^2} dx$$

In the complex plane, this has poles of order 1 only at $x = \pm i$, where the denominator becomes zero. Thus, I see that R must be at least 1 in order to capture the upper singularity. Then, the value of the integral:

$$\int_{-R}^{R} \frac{e^{-ikx}}{1+x^2} dx + \int semicircle = 2\pi i \operatorname{Re} s\left[\frac{e^{-ikx}}{1+x^2}, x = i\right] = 2\pi i \frac{e^k}{2i} = \pi e^k$$

Now consider the contribution from the semicircle. I notice that the contribution looks something like this:

$$\int_{\pi}^{-\pi} \frac{e^{-ik(R\cos\theta + Ri\sin\theta)}}{1+R^2(R\cos\theta + Ri\sin\theta)} Rd\theta$$

I notice that the dominant term is $e^{kR\sin\theta}$, which will vanish at large radii as long as k is negative. Thus, $\hat{f}(k<0) = \sqrt{\frac{2}{\pi}} e^k$.

- **(d) Find an explicit formula for $\hat{f}(k)$ in the appropriate range (if you have already evaluated the integral in part (c), simply state the answer here.)**

$$\hat{f}(k<0) = \sqrt{\frac{2}{\pi}} e^k$$

- **(e) For values of k in a different range (state what it is), the contour in part (c) is not appropriate. Explain why this is so, indicate a different contour in the complex plane which can be used in place of the one given above, and how this integral is related to a residue or residues of an analytic function. Then find an explicit formula for $\hat{f}(k)$ for k values in this range.**

As I explained above, the contour from part c is appropriate only for negative values of k. For positive values of k, the contour must be closed in the lower half-plane where the exponential vanishes at large radii. Thus, I select the counterclockwise contour running across the real line from R to –R, then looping around in a counterclockwise circle in the negative quadrants of the imaginary plane. Here, I have:

$$\int_{R}^{-R} \frac{e^{-ikx}}{1+x^2} dx + \int semicircle = 2\pi i \operatorname{Re} s\left[\frac{e^{-ikx}}{1+x^2}, x = -i\right] = 2\pi i \frac{e^{-k}}{-2i} = -\pi e^{-k}$$

$$\int_{-R}^{R} \frac{e^{-ikx}}{1+x^2} dx = -\left[-\pi e^{-k}\right] = \pi e^{-k}$$

thus, I see that once the semicircle portion vanishes via an argument similar to the one from part (c), I’ll be left with

$$\hat{f}(k>0)=\sqrt{\frac{2}{\pi}}e^{-k}$$