

Problem Source: **CMU August 2004 Qualifying Exam Day 2**

An electromagnetic wave with fields

$$\vec{E}_0 = \hat{x}E_0 e^{i(kz - \omega t)}$$

$$\vec{B}_0 = \frac{1}{c} \hat{z} \times \vec{E}_0$$

is incident upon a particle of mass m and charge q at the origin. Assume that the motion is non-relativistic and that the particle is massive so that it stays near the origin.

- (a) What is the force of the \vec{E}_0 field on the particle, and what is the position of the particle as a function of time?

$$\vec{F} = \vec{E}_0 q = m\ddot{x}\hat{x} = qE_0 e^{-i\omega t} \hat{x}$$

$$\ddot{x} = \frac{qE_0}{m} e^{-i\omega t}$$

$$x(t) = -\frac{qE_0}{m\omega^2} e^{-i\omega t}$$

- (b) Show that an electric dipole

$$\vec{p}(t) = -\frac{q^2}{m\omega^2} E_0 e^{-i\omega t} \hat{x}$$

is induced.

The displacement in Coulomb-meters is then given by $\vec{p}(t) = qx(t) = -\frac{q^2}{m\omega^2} E_0 e^{-i\omega t} \hat{x}$.

- (c) For a current density of the form $\vec{J}(\vec{x}, t) = \vec{J}(x) e^{-i\omega t}$ the vector potential in the Lorentz gauge can be shown to be given by

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3 x' \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}, \text{ with } k = \frac{\omega}{c}.$$

Define the dipole limit for radiation and show that in this limit \vec{A} can be approximated by

$$\vec{A}(\vec{x}) \approx \frac{\mu_0}{4\pi r} e^{ikr} \int d^3 x' \vec{J}(\vec{x}'),$$

with $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x}) e^{-i\omega t}$

The dipole radiation is the portion of the electromagnetic fields that dies off as $\frac{1}{r}$, and as such is the portion that survives in the far-zone limit. Ignoring, then, lower order contributions I have:

$$|\bar{x} - \bar{x}'| = \sqrt{[\bar{x} - \bar{x}']^2} \approx r$$

$$\frac{1}{|\bar{x} - \bar{x}'|} \approx \frac{1}{r}$$

$$\bar{A}(\bar{x}, t) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \bar{J}(\bar{x}')$$

(d) For the problem of part (a), the vector potential can be approximated by

$$\bar{A}(\bar{x}) \approx i \frac{\mu_0 \omega}{4\pi} \bar{p} \frac{e^{ikr}}{r}$$

$$\bar{p}(t) = \bar{p} e^{-i\omega t}$$

Find the \bar{E} and \bar{B} fields radiated by the particle in part (a) at a distance $r \gg$ the wave length in the dipole approximation.

$$\bar{E} = -\bar{\nabla}V - \frac{\partial \bar{A}}{\partial t}$$

However, in this far zone I have determined that the fields die off like $\frac{1}{r}$, and certainly the dipole and charge fields are going to die faster than this. Thus, in the far zone I have

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t}$$

for

$$\bar{E} = \frac{\mu_0 \omega^2}{4\pi} \bar{p} \frac{e^{ikr}}{r}$$

Similarly,

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$$= i \frac{\mu_0 \omega}{4\pi} \left(\bar{\nabla} \frac{e^{ikr}}{r} \right) \times \bar{p} = i \frac{\mu_0 \omega}{4\pi} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{e^{ikr}}{r} \right] \hat{r} \right) \times \bar{p}$$

$$= i \frac{\mu_0 \omega}{4\pi} \left(\left[\frac{e^{ikr}}{r^2} + ik \frac{e^{ikr}}{r} \right] \right) \hat{r} \times \bar{p} \approx -\frac{k\mu_0 \omega}{4\pi} \frac{e^{ikr}}{r} [\hat{r} \times \bar{p}]$$

(e) Recalling that the Poynting vector $\bar{S} = \bar{E} \times \bar{H}$ is the power per unit area, find the time-averaged power per unit solid angle radiated by the particle in terms of given quantities.

θ : angle between \hat{p} and \hat{r}

$$\begin{aligned}\bar{S} &= \frac{1}{\mu_0} \frac{\mu_0 \omega^2}{4\pi} \frac{e^{ikr}}{r} \bar{p} \times \left[-\frac{k\mu_0 \omega}{4\pi} \frac{e^{ikr}}{r} [\hat{r} \times \bar{p}] \right] = \frac{1}{\mu_0} \frac{\mu_0^2 \omega^3 k}{(4\pi)^2} \frac{e^{2ikr}}{r^2} \bar{p} \times [\hat{r} \times \bar{p}] \\ &= \frac{1}{\mu_0} \frac{\mu_0^2 \omega^3 k}{(4\pi)^2} \frac{e^{2ikr}}{r^2} ([\bar{p} \cdot \bar{p}] \hat{r} - [\bar{p} \cdot \hat{r}] \bar{p}) = \frac{\mu_0 \omega^3 k}{(4\pi)^2} \frac{e^{2ikr}}{r^2} |\bar{p}|^2 (\hat{r} - \hat{p} \cos \theta)\end{aligned}$$

of course, the area of this large sphere is $4\pi r^2$ and so

$$\begin{aligned}dS &= \frac{\mu_0 \omega^3 k}{(4\pi)^2} e^{2ikr} |\bar{p}|^2 (\hat{r} - \hat{p} \cos \theta) \cdot \hat{r} \sin \theta d\theta d\phi \\ &= \frac{\mu_0 \omega^3 k}{(4\pi)^2} e^{2ikr} |\bar{p}|^2 (1 - \cos^2 \theta) \sin \theta d\theta d\phi \\ &= \frac{\mu_0 \omega^3 k}{(4\pi)^2} e^{2ikr} |\bar{p}|^2 \sin^3 \theta d\theta d\phi\end{aligned}$$

(f) Find the differential cross section for the scattering of the incident wave by the particle in the dipole approximation.

And of course this cross section must be given in terms of incident amplitude.

$$\frac{d\sigma}{d\Omega} = \frac{\mu_0 \omega^3 k}{(4\pi)^2} e^{2ikr} |\bar{p}|^2 \sin^2 \theta \cdot \frac{1}{\text{Incident Power}}$$

In this case, the incident power is $|S| = \frac{1}{\mu_0 c} E_0^2 e^{-2i\omega t}$

For overall,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\mu_0 \omega^3 k}{(4\pi)^2} e^{2ikr} |\bar{p}|^2 \sin^2 \theta \left[\frac{\mu_0 c}{E_0^2} \right] \\ &= \frac{\mu_0^2 \omega^4}{(4\pi)^2 E_0^2} e^{2ikr} |\bar{p}|^2 \sin^2 \theta \\ &= \frac{\mu_0^2 \omega^4}{(4\pi)^2 E_0^2} e^{2ikr} \left| -\frac{q^2}{m\omega^2} E_0 \right|^2 \sin^2 \theta = \frac{\mu_0^2 q^4}{(4\pi)^2 m^2} e^{2ikr} \sin^2 \theta\end{aligned}$$