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Practice for Qualifying Exams

Problem Source: **CMU August 2003 Qualifying Exam**

Consider a hollow body at temperature T containing electromagnetic radiation in equilibrium with its walls. The energy spectrum of this so-called blackbody radiation can be measured by observing it through a negligibly small hole in the cavity wall. As argued by Planck, the energy levels of the radiation are quantized and are the same as those of a simple harmonic oscillator of angular frequency ω.

(a) What are the quantized energy levels of a radiation mode of angular frequency ω? Use the Boltzmann distribution to find the thermal average number of photons for such a mode.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$(ignore \quad zero \quad point)$$

$$E_n = \hbar\omega n$$

$$Z = \sum_n e^{-\beta E_n} = \sum_n e^{-\beta\hbar\omega n} = \sum_n \left(e^{-\beta\hbar\omega}\right)^n = \frac{1}{1-e^{-\beta\hbar\omega}}$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\langle E\rangle = \hbar\omega\langle N\rangle$$

$$\langle N\rangle = \frac{1}{e^{\beta\hbar\omega}-1}$$

(b) The density of modes in $\vec{k}$ space is constant. Thus for a cavity of volume V, the number of modes in a volume d^3k near a wave number $\vec{k}$ is just

$$2\frac{V}{(2\pi)^3}d^3k$$

If c is the speed of light, how many modes are there having angular frequency between ω and $\omega + d\omega$?

In terms of wavelength and frequency, $\omega = ck$. Then, converting the volume-wise density into an integral only over the surface, I have:

$$\to 2\frac{V}{(2\pi)^3}\left(4\pi k^2\right)dk$$

$$k=\frac{\omega}{c}\quad dk=\frac{1}{c}d\omega$$

for

$$\to 2\frac{V}{(2\pi)^3}\left(4\pi\frac{\omega^2}{c^2}\right)\frac{1}{c}d\omega=\frac{V\omega^2}{\pi^2c^3}d\omega$$

(c) Express the total internal energy U for this radiation inside the cavity as an integral over ω. Discuss analytically for very high and very low ω how the integral depends on ω. Sketch the integrand as a function of ω for two different values of T.

Using the equipartition of energy, I have classically

$$E=\int_0^\infty kT\left(\frac{V\omega^2}{\pi^2c^3}\right)d\omega$$

This integral clearly diverges as ω becomes very large, and scales linearly in T.

(d) Show that $U=aVT^4$ and give an integral expression for the constant a.

$$\langle E\rangle=\frac{\sum_i E_i e^{-\beta E_i}}{Z}=\frac{-\sum_i\frac{\partial}{\partial\beta}e^{-\beta E_i}}{Z}=-\frac{1}{Z}\frac{\partial}{\partial\beta}Z$$

$$Z=\frac{1}{1-e^{-\hbar\omega\beta}}\quad \frac{\partial Z}{\partial\beta}=-\frac{\hbar\omega e^{-\hbar\omega\beta}}{\left(1-e^{-\hbar\omega\beta}\right)^2}$$

$$-\frac{1}{Z}\frac{\partial Z}{\partial\beta}=\frac{\hbar\omega e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}=\frac{\hbar\omega}{e^{\hbar\omega\beta}-1}$$

$$U=\int_0^\infty\frac{\hbar\omega}{e^{\hbar\omega\beta}-1}\left(\frac{V\omega^2}{\pi^2c^3}\right)d\omega$$

$$x=\hbar\omega\beta\quad dx=\hbar\beta d\omega$$

$$U=\int_0^\infty\frac{\hbar}{e^x-1}\left(\frac{V}{\pi^2c^3}\right)\left(\frac{x}{\hbar\beta}\right)^3\frac{1}{\hbar\beta}dx=\frac{V}{\pi^2c^3\hbar^3\beta^4}\int_0^\infty\frac{x^3}{e^x-1}dx=\left[\frac{k_B^4}{\pi^2c^3\hbar^3}\int_0^\infty\frac{x^3}{e^x-1}dx\right]VT^4$$

(e) Use part (d) to compute the entropy S and pressure P as functions of V and T. Determine any constants that appear in terms of the symbols introduced above. Show that your results are thermodynamically consistent with the formula in part (d).

With $U=TS-PV$, $dU=TdS$ at constant V.

$$Let \quad a = \frac{k_B^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$\frac{dU}{T} = dS$$

$$\frac{4aVT^3 dT}{T} = dS$$

$$\frac{4}{3} aVT^3 = S$$

With $U = TS - PV$, $P = -\left(\frac{\partial U}{\partial V}\right)_S$

$$U = aVT^4 = a\left(\frac{3}{4}\frac{S}{a}\right)^{\frac{4}{3}} V^{-\frac{1}{3}}$$

$$P = -\left(\frac{\partial U}{\partial V}\right)_S = \frac{1}{3} a\left(\frac{3}{4}\frac{S}{V}\right)^{\frac{4}{3}} = \frac{1}{3} aT^4$$

And from these results, clearly $U = TS - PV$ holds.