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Practice for Qualifying Exams

Problem Source: **CMU August 2003 Qualifying Exam**

A three-level quantum system, which might describe a spin-1 particle, for example, has a Hamiltonian $\hat{H}$ whose matrix representation in the orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$ is given by

$$\hat{H} = \begin{pmatrix} \langle 1|\hat{H}|1\rangle & \langle 1|\hat{H}|2\rangle & \langle 1|\hat{H}|3\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle & \langle 2|\hat{H}|3\rangle \\ \langle 3|\hat{H}|1\rangle & \langle 3|\hat{H}|2\rangle & \langle 3|\hat{H}|3\rangle \end{pmatrix} = \hbar \begin{pmatrix} 0 & -i\omega_1 & 0 \\ i\omega_1 & \omega_2 & 0 \\ 0 & 0 & \omega_2 \end{pmatrix}$$

Here, $\omega_1 > 0, \omega_2 > 0$ are two angular frequencies.

(a) A general solution to the Schrodinger equation $i\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$ can be written as $|\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle$. Write down the set of coupled differential equations satisfied by the (possibly complex) coefficients $c_{1,2,3}(t)$ in terms of ω_1, ω_2.

$$i\frac{d}{dt}\left(c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle\right) = \hat{H}\left(c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle\right)$$
$$i\left(\dot{c}_1(t)|1\rangle + \dot{c}_2(t)|2\rangle + \dot{c}_3(t)|3\rangle\right) = \hat{H}\left(c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle\right)$$
$$i\dot{c}_1(t)|1\rangle = -i\hbar\omega_1 c_2(t)|1\rangle$$

e.g.,

$$i\dot{c}_1(t) = -i\hbar\omega_1 c_2(t)$$
$$i\dot{c}_2(t) = i\hbar\omega_1 c_1(t) + \hbar\omega_2 c_2(t)$$
$$i\dot{c}_3(t) = \hbar\omega_2 c_3(t)$$

(b) Assuming that $|\Psi(t)\rangle$ is normalized to unity, find the probability in terms of $c_{1,2,3}(t)$ that an appropriate measurement at time t will find the system in:

(i) $|\phi\rangle = |3\rangle \rightarrow |\langle\phi|\Psi(t)\rangle|^2 = c_3(t)c_3^+(t)$

(ii) $|\phi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \rightarrow |\langle\phi|\Psi(t)\rangle|^2 = \frac{1}{2}(c_1(t) + c_2(t))(c_1^+(t) + c_2^+(t))$

(iii) $|\phi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle) \rightarrow |\langle\phi|\Psi(t)\rangle|^2 = \frac{1}{2}(c_1(t) + ic_2(t))(c_1^+(t) - ic_2^+(t))$

Answer the next four questions assuming that $\omega_1 = 2\omega_0, \omega_2 = 3\omega_0$, in terms of some fixed frequency ω_0. In this case, the Hamiltonian $\hat{H}$ takes the form

$$\hat{H} = \hbar\omega_0 \hat{M} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \cdot \hat{M}^+ \quad with \quad \hat{M} = \begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{-i}{\sqrt{5}} \\ \frac{-i}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{pmatrix}$$

(c) Find the matrix $\hat{M}^+$ and show that (or indicate why) $\hat{M}$ is unitary.

To be unitary, $\hat{M}\hat{M}^+ = \hat{M}^+\hat{M} = I$. In fact, such a diagonalization can be produced from the Hamiltonian using a matrix of eigenvectors of the Hamiltonian. To verify that this is a matrix of Eigenvectors, I have:

$$\hat{H} = \hbar\omega_0 \begin{pmatrix} 0 & -2i & 0 \\ 2i & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2i & 0 \\ 2i & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-i}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{4i}{\sqrt{5}} - \frac{3i}{\sqrt{5}} \\ 0 \end{pmatrix} = -1 \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-i}{\sqrt{5}} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2i & 0 \\ 2i & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2i & 0 \\ 2i & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{i}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{4i}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}} \\ 0 \end{pmatrix} = 4 \begin{pmatrix} -\frac{i}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$$

Just as expected.

(d) What are the possible energies of the system?

These energies correspond to the eigenvalues of the Hamiltonian: $-\hbar\omega_0, 3\hbar\omega_0, 4\hbar\omega_0$.

(e) Suppose that $|\Psi_0\rangle = |\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$. What is the probability that a measurement at time t will show that the system is in its ground state?

This will be given by the measurement on the time-developed system: $e^{-\frac{i\hat{H}t}{\hbar}}\left|\Psi_0\right\rangle$

This is not a stationary state of the system, since it is not an eigenstates. The time development operator is found in part (f), but another means to measure the ground state I is to derive the time dependence of the amplitude from the differential equations in (a). Recalling and simplifying a bit, I have:

$$\dot{c}_1(t) = -\hbar\omega_1 c_2(t)$$
$$\dot{c}_2(t) = \hbar\omega_1 c_1(t) + -i\hbar\omega_2 c_2(t)$$
$$\dot{c}_3(t) = -i\hbar\omega_2 c_3(t)$$

Only the first two are relevant here.

$$\dot{c}_1(t) = -2\hbar\omega_0 c_2(t)$$
$$\dot{c}_2(t) = 2\hbar\omega_0 c_1(t) + -3i\hbar\omega_0 c_2(t)$$

Which I write:

$$\vec{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$\dot{\vec{c}}(t) = \hbar\omega_0 \begin{bmatrix} 0 & -2 \\ 2 & -3i \end{bmatrix} \vec{c}(t)$$

$$\vec{c}(t) = e^{\hbar\omega_0 t\begin{bmatrix} 0 & -2 \\ 2 & -3i \end{bmatrix}} \cdot \vec{c}(0)$$

$$\begin{bmatrix} 0 & -2 \\ 2 & -3i \end{bmatrix} \to eigenvalues \quad -\lambda(-3i-\lambda)+4 = \lambda^2 + 3i\lambda + 4 = 0, \lambda = \frac{-3i \pm \sqrt{-9-16}}{2} = \{-4i, i\}$$

$$Null\begin{bmatrix} -i & -2 \\ 2 & -4i \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -i \\ 2 \end{bmatrix} \quad and \quad Null\begin{bmatrix} 4i & -2 \\ 2 & i \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & -3i \end{bmatrix} = \frac{1}{5}\begin{bmatrix} i & 2 \\ -2 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -4i \end{bmatrix}\begin{bmatrix} -i & -2 \\ 2 & i \end{bmatrix}$$

$$e^{\hbar\omega_0\begin{bmatrix} 0 & -2 \\ 2 & -3i \end{bmatrix}} = \frac{1}{5}\begin{bmatrix} i & 2 \\ -2 & -i \end{bmatrix}\begin{bmatrix} e^{\hbar\omega_0 it} & 0 \\ 0 & e^{-4\hbar\omega_0 it} \end{bmatrix}\begin{bmatrix} -i & -2 \\ 2 & i \end{bmatrix}$$

So in the basis of Eigenstates, I then have:

$$P = \left|\left\langle E = -\hbar\omega_0 \left|T\right|\Psi_0\right\rangle\right|^2 = \left|\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} \end{bmatrix}\frac{1}{5}\begin{bmatrix} i & 2 \\ -2 & -i \end{bmatrix}\begin{bmatrix} e^{\hbar\omega_0 it} & 0 \\ 0 & e^{-4\hbar\omega_0 it} \end{bmatrix}\begin{bmatrix} -i & -2 \\ 2 & i \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}\right|^2$$

$$= \frac{1}{250}\left|\begin{bmatrix} 2 & i \end{bmatrix}\begin{bmatrix} i & 2 \\ -2 & -i \end{bmatrix}\begin{bmatrix} e^{\hbar\omega_0 it} & 0 \\ 0 & e^{-4\hbar\omega_0 it} \end{bmatrix}\begin{bmatrix} -i & -2 \\ 2 & i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}\right|^2$$

(f) Finally, construct the matrix representation of the time evolution operator

$$\hat{T}(t) = e^{\frac{-i\hat{H}t}{\hbar}}$$ **in the original basis** $|1\rangle, |2\rangle, |3\rangle$ **.**

From Unitarity, I can consider the Taylor expansion of the exponential as such:

$$\hat{H} = \hat{M}D\hat{M}^{+}$$

$$e^{a\hat{H}} \rightarrow 1 + a\hat{M}D\hat{M}^{+} + \frac{1}{2}a^{2}\hat{M}D\hat{M}^{+}\hat{M}D\hat{M}^{+} + \ldots \rightarrow 1 + a\hat{M}D\hat{M}^{+} + \frac{1}{2}a^{2}\hat{M}D^{2}\hat{M}^{+} +$$

For

$$\hat{T}(t) = e^{\frac{-i\hat{H}t}{\hbar}} = \hat{M}\begin{bmatrix} e^{\frac{it}{\hbar}} & 0 & 0 \\ 0 & e^{-\frac{3it}{\hbar}} & 0 \\ 0 & 0 & e^{-\frac{4it}{\hbar}} \end{bmatrix}\hat{M}^{+}$$