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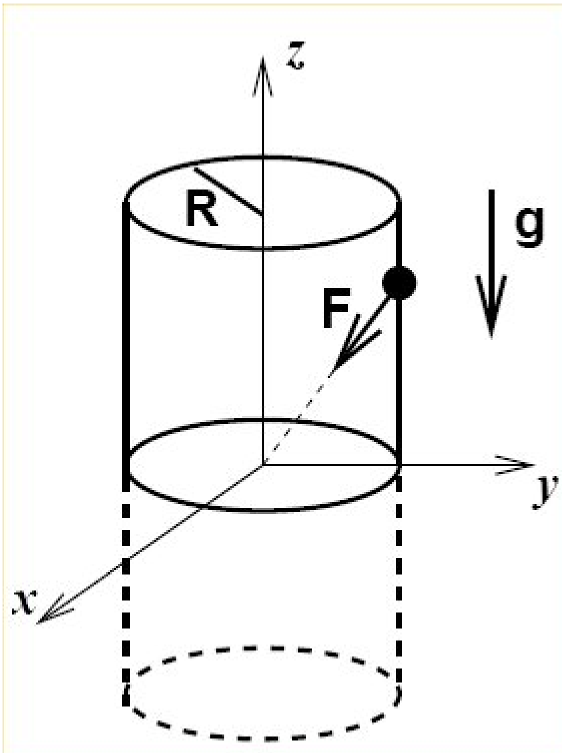
Practice for Qualifying Exams

Problem Source: **CMU Qualification Exam Day 2 (August 2003)**

**(7)**




**A particle of mass m is acted upon by a constant gravitational field with acceleration g. The particle is also acted upon by a force $\vec{F}$ that is directed towards the origin. The magnitude of this force is $F = kr$ where r is the distance from the particle to the origin. There is a cylinder of radius R into which the particle cannot enter. The particle is constrained to slide on the cylinder with no friction.**
**(a) Write the kinetic energy K and the potential energy V of the particle in the cylindrical coordinate system $(\rho, \phi, z)$.**

Note well that I assume here that k is negative.

$$V_g = mgz$$

$$V_c = \int_0^{\sqrt{\rho^2+z^2}} krdr = \frac{1}{2}k\left(\rho^2 + z^2\right) = \frac{1}{2}k\left(R^2 + z^2\right)$$

$$V = V_g + V_c$$

$$K = \frac{1}{2}m\left(\dot{z}^2 + \left(\rho\dot{\phi}\right)^2 + \dot{\rho}^2\right) = \frac{1}{2}m\left(\dot{z}^2 + \left(R\dot{\phi}\right)^2\right)$$

**(b) Write the equations of motion for this system independent of initial conditions. Use the symbol $\vec{F}_{cyl}$ to represent the force of the cylinder on this particle.**

$$L = K - V$$

$$= \frac{1}{2}m\left(\dot{z}^2 + \left(\rho\dot{\phi}\right)^2 + \dot{\rho}^2\right) - mgz - \frac{1}{2}k\left(\rho^2 + z^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z}$$

$$\frac{d}{dt}\left(m\dot{z}\right) = -mg - kz$$

$$m\ddot{z} = -mg - kz$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = 0$$

$$\vec{F}_{cyl} = \hat{\rho}\frac{\partial L}{\partial \rho} = \hat{\rho}\left(m\rho\dot{\phi}^2 - k\rho\right)$$

This force is the familiar central potential plus the centripetal acceleration.

**(c) Calculate the magnitude of $\vec{F}_{cyl}$ on the particle.**

In this case, it would be

$$\vec{F}_{cyl} \cdot \hat{\rho} = F_{cyl} = \left(mR\dot{\phi}^2 - kR\right)$$

**(d) Identify all relevant conservation laws that are valid for the motion of the particle when it remains on the surface of the cylinder. Justify why the laws you have identified apply to this specific motion.**

From the second equation of motion, it is clear that angular momentum is conserved: this is certainly always the case, and in this case the angular momentum of the particle will always remain such as no force ever acts in the rotational direction.

Further, energy of the particle is conserved as this is a frictionless scenario. The "normal" force acting perpendicular to the allowed motion of the particle can do no work.

**(e) Given the initial conditions $z_0 = 0, \dot{z}_0 = 0, \phi_0 = 0, \dot{\phi}_0 \neq 0$ at t = 0, determine the position of the particle on the surface of the cylinder at any subsequent time.**

Certainly, $\rho = R$ as the particle remains on the surface of the cylinder. Since angular momentum is conserved, $\frac{d}{dt}\left(mR^2\dot{\phi}\right) = 0, \dot{\phi} = const, \phi(t) = \dot{\phi}_0 t$. All that remains is to solve the differential equation in z. I have: $m\ddot{z} = -mg - kz$. The solution to the homogeneous equation is clearly of the form $m\ddot{z} = -kz, z \approx A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$. A particular solution is of the form $z = -\frac{mg}{k}$. Solutions to the full equation are then $z \approx A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t - \frac{mg}{k}$. Solving this for my boundary conditions, then, I have:

$$0 = A\cos\sqrt{\frac{k}{m}}0 + B\sin\sqrt{\frac{k}{m}}0 - \frac{mg}{k} = A - \frac{mg}{k}$$

$$0 = -A\sin\sqrt{\frac{k}{m}}0 + B\cos\sqrt{\frac{k}{m}}0 = B$$

$$z(t) = \frac{mg}{k}\cos\sqrt{\frac{k}{m}}t - \frac{mg}{k}$$

$$\rho(t) = R$$

$$\phi(t) = \dot{\phi}_0 t$$