Problem Source: **<u>CMU Qualification Exam Day 1 (August 2003)</u>**

(2) Consider a stretched string of variable density (mass per unit length) $\rho(x)$. If the transverse displacement is $u(x,t)$, then in the limit of small displacements $u(x,t)$ satisfies the wave equation

$$T\frac{\partial^2 u(x,t)}{\partial x^2} = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}.$$

Here, T represents the constant tension of the string.

(a) Write $u(x,t) = y(x)e^{-i\omega t}$ to find a differential equation for $y(t)$ in terms of the density per length ρ, the tension T and the frequency ω.

$$T\frac{\partial^2 u(x,t)}{\partial x^2} = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$Te^{-i\omega t}\frac{\partial^2 y(x)}{\partial x^2} = \rho(x)y(x)\left(-\omega^2\right)e^{-i\omega t}$$

$$Te^{-i\omega t}\frac{\partial^2 y(x)}{\partial x^2} + \omega^2\rho(x)e^{-i\omega t}y(x) = 0$$

$$T\frac{\partial^2 y(x)}{\partial x^2} + \omega^2\rho(x)y(x) = 0$$

Now let's specialize to the case of a string of length L with fixed endpoints at $x = \pm\frac{L}{2}$. The string has a uniform density ρ_0 and a point mass m attached in the middle.

(b) Why is it reasonable to take $\rho(x) = \rho_0 + m\delta(x)$?

This accurately gives the effective mass at each point along the length of the string: the delta function gives the additional mass imparted by the centrally located weight.

Consider each region: in the non-mass regions, I have the same density as the string. In the mass region, I have m mass. Integrate to verify:

$\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho_0 + m\delta(x) = L\rho_0 + m$, just as expected.

(c) Show that the equation for $y(x)$ with this density has solutions that can be taken to be either even or odd under $x \to -x$.

$$T\frac{\partial^2 y(x)}{\partial x^2} + \omega^2\left(\rho_0 + m\delta(x)\right)y(x) = 0$$

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(x) + \frac{\omega^2}{T}m\delta(x)y(x) = 0$$

Odd Variety: Assume $y(-x) = -y(x)$.

Under the Parity operator, then, I have:

$$\tilde{P} \rightarrow \frac{\partial^2 y(-x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(-x) + \frac{\omega^2}{T} m\delta(x)y(-x) = 0$$

$$-\frac{\partial^2 y(x)}{\partial x^2} - \frac{\omega^2}{T}\rho_0 y(x) - \frac{\omega^2}{T} m\delta(x)y(x) = 0$$

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(x) + \frac{\omega^2}{T} m\delta(x)y(x) = 0 \quad check!$$

Even Variety: Assume $y(x) = -y(x)$.
Under the Parity operator, then, I have:

$$\hat{P} \rightarrow \frac{\partial^2 y(-x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(-x) + \frac{\omega^2}{T} m\delta(x)y(-x) = 0$$

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(x) + \frac{\omega^2}{T} m\delta(x)y(x) = 0 \quad check!$$

(d) Determine the boundary condition on $y(x)$ and $\frac{dy}{dx}$ at x = 0, allowing for the possibility that $\frac{dy}{dx}$ could have a discontinuity there.

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{\omega^2}{T}\rho_0 y(x) + \frac{\omega^2}{T} m\delta(x)y(x) = 0$$

Consider a tiny interval near x = 0:

$$\frac{\partial^2 y(0)}{\partial x^2} = \frac{y'(\varepsilon) - y'(-\varepsilon)}{2\varepsilon}$$

If, as in the odd case, $y(0) = 0$, this does not provide any useful constraint on the first derivative other than the constraint found in part c, where

$$y(x) = c_{s1} \sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}\, x\right)$$

$$y'(x) = c_{s1}\sqrt{\frac{\omega^2}{T}\rho_0}\, \cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}\, x\right)$$

$$y(0) = 0$$

However, in the even case, a useful constraint does arise from the delta-function here: namely, that

$$y'(\varepsilon) - y'(-\varepsilon) = \frac{\omega^2}{T} m y(0)$$

or

$$y'(0) = \frac{1}{2}\frac{\omega^2}{T} m y(0)\delta(0)$$

(e) Find the equation that determines the allowed frequencies ω for both the even and odd solutions, so that they satisfy the relevant boundary conditions.

Odd solutions are relatively simple: with allowed frequencies at integer modes of n such that:

$$y(x) = c_{s1}\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}\, x\right)$$

$$\sqrt{\frac{\omega^2}{T}\rho_0}\,\frac{L}{2} = n\pi$$

$$\omega_{odd,n} = \sqrt{\frac{T}{\rho_0}}\,\frac{2}{L} n\pi$$

For the even solutions, however, one takes $y(-x) = y(x)$: substituting this into the original differential equation gives the same equation back, indicating the characteristic mirror symmetry of even solutions. However, a delta-function discontinuity must exist at x = 0 as listed above.

Solutions of this variety will then look something like

$$y(x) = c\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}\,(|x| + f)\right).$$

Simplifying a bit,

$$y(x)=c_{s1}\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}(|x|+f)\right)=c_{s1}\left[\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}|x|\right)\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)+\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}|x|\right)\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\right]$$

$$y(0)=c_{s1}\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)$$

boundary

$$y'(x)=\frac{\partial}{\partial x}c_{s1}\left[\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}|x|\right)\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)+\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}x\right)\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\right]$$

$$=c_{s1}\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}|x|\right)\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\sqrt{\frac{\omega^2}{T}\rho_0}2\delta(x)+c_{s1}\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}x\right)\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\sqrt{\frac{\omega^2}{T}\rho_0}$$

$$y'(0)=c_{s1}\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\sqrt{\frac{\omega^2}{T}\rho_0}2\delta(x)=\frac{1}{2}\frac{\omega^2}{T}my(0)=c_{s1}\frac{1}{2}\frac{\omega^2}{T}m\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\delta(x)$$

$$c_{s1}2\cos\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)\sqrt{\frac{\omega^2}{T}\rho_0}=c_{s1}\frac{1}{2}\frac{\omega^2}{T}m\sin\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)$$

$$4\frac{T}{m\omega^2}\sqrt{\frac{\omega^2}{T}\rho_0}=\tan\left(\sqrt{\frac{\omega^2}{T}\rho_0}f\right)$$

$$\sqrt{\frac{T}{\omega_n{}^2\rho_0}}\arctan\left[\frac{4}{m\omega_n}\sqrt{T\rho_0}\right]=f_n \quad CONSTRAINT\ (1)$$

$$\sqrt{\frac{\omega^2}{T}\rho_0}\left(\frac{L}{2}+f_n\right)=n\pi$$

$$\omega_{even,n}=\sqrt{\frac{T}{\rho_0}}n\pi\left(\frac{L}{2}+f_n\right)^{-1} \quad CONSTRAINT\ (2)$$

$$c_{s1}\rightarrow normalization \quad CONSTRAINT\ (3)$$

Thus I have three constants fixed by these constraints. Solving 1 and 2 together is not algebraically possible, but is possible in terms of an approximation.

(f) Find the lowest non-zero frequency and the solution $y(x)$ for the case where $y(x)$ is odd under the transformation $x \rightarrow -x$. Sketch this solution.

$$\omega=\sqrt{\frac{T}{\rho_0}}\frac{n\pi 2}{L}$$

$$\omega_{odd,1}=\sqrt{\frac{T}{\rho_0}}\frac{\pi 2}{L}$$

Of course, this frequency in time is not the same as the frequency in space: the solution in this case will be a simple odd sine wave, with zeroes at the edges L/2 and at the center zero.