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Practice for Qualifying Exam

Problem Source: **CMU August 2003 Qualifying Exam**

**The Maxwell equations in SI units are given by:**

$$\vec{\nabla}\cdot\vec{D}=\rho \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

**Below the plane z = 0 (region 1) is a vacuum with permittivity $\varepsilon_0$ and permeability $\mu_0$. Above the plane z = 0 (region 2) is a material with permittivity $\varepsilon = k\varepsilon_0$ and permeability $\mu = m\mu_0$. There is no free charge at the boundary and there is no surface current at the boundary.**

**(a) Use Maxwell's equations to find the speed of propagation of a wave in each of the regions.**

First, note that $\vec{D}=\varepsilon_{rgn}\vec{E}$ and that $\vec{H}=\frac{1}{\mu_{rgn}}\vec{B}$. Taking Ampere's law, I have:

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{J}=\vec{0}$$

$$\vec{\nabla}\times\vec{B}=\mu_{rgn}\varepsilon_{rgn}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}=\mu_{rgn}\varepsilon_{rgn}\frac{\partial^2\vec{E}}{\partial t^2}$$

Substituting from Faraday's law, I have:

$$-\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\mu_{rgn}\varepsilon_{rgn}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$-\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{\nabla}^2\vec{E}\right)=\mu_{rgn}\varepsilon_{rgn}\frac{\partial^2\vec{E}}{\partial t^2}$$

Using Coulomb's Law, then, I have:

$$\vec{\nabla}^2\vec{E}=\mu_{rgn}\varepsilon_{rgn}\frac{\partial^2\vec{E}}{\partial t^2}$$

Now I see that this has the form of the three-dimensional wave equation:

$$\vec{\nabla}^2 f=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \text{ so that } v=\frac{1}{\sqrt{\mu_{rgn}\varepsilon_{rgn}}}.$$

**(b) Use the Maxwell's equations to find the wave impedance $Z = \frac{|E|}{|H|}$ in both regions.**

Let the electric field be of the form: $\vec{E} = E_0 \hat{z} \cos(kx - \omega t)$. From Faraday's Law, I see that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial x} E_z \hat{y} = kE_0 \hat{y} \sin(kx - \omega t) = -\frac{\partial \vec{B}}{\partial t}$$

$$-\int \frac{k}{\omega} E_0 \hat{y} \sin(kx - \omega t) = -\int \frac{k}{\omega} E_0 \hat{y} \cos(kx - \omega t)$$

$$\frac{k}{\omega} = \frac{s}{m} = \frac{1}{v_{rgn}}$$

Now certainly, $\vec{H} = \frac{1}{\mu_{rgn}} \vec{B}$ and $|H| = \frac{1}{\mu_{rgn}} |B|$ so that

$$\frac{|E|}{|H|} = \mu_{rgn} \left( \frac{E_0}{B_0} \right) = \mu_{rgn}(v) = \mu_{rgn} \frac{1}{\sqrt{\varepsilon_{rgn} \mu_{rgn}}} = \sqrt{\frac{\mu_{rgn}}{\varepsilon_{rgn}}}$$

**(c) Show that the following boundary conditions apply:**

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 0 \quad (\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0$$
$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \quad (\vec{H}_2 - \vec{H}_1) \times \hat{n} = 0$$

Since this is charge-free space, Coulomb's Law says that $\vec{\nabla} \cdot \vec{D} = \rho = 0$ everywhere. Considering a very thin Gaussian pillbox enclosing the surface, this implies that $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 0$ since otherwise a charge density $\vec{D}_2 - \vec{D}_1 = \sigma$ would be present.

An unnamed Maxwell's Law says that $\vec{\nabla} \cdot \vec{B} = 0$, and so considering the same Gaussian pillbox then $(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$ since otherwise a net magnetic flux would be present.

Now from Faraday's Law I have that $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$. Considering a very narrow Amperian loop with length l parallel to the surface, I then have $\vec{l} \cdot \vec{E}_2 - \vec{l} \cdot \vec{E}_1 = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{A}$, and the magnetic field will vanish in the limit of a very thin loop implying then that $(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0$.

Next, from Ampere's Law I expect $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \frac{\partial \vec{D}}{\partial t}$ in a current-free region. Again, I define an Amperian loop parallel to the surface so that

$\vec{l} \cdot \vec{H}_2 - \vec{l} \cdot \vec{H}_1 = \frac{\partial}{\partial t}\int_S \vec{D} \cdot d\vec{A}$. Again, in the limit of a very thin loop the integral will vanish, leaving $\left(\vec{H}_2 - \vec{H}_1\right) \times \hat{n} = 0$.

**An electromagnetic wave of intensity $I_0$ travels in the z-direction in region 1. Te wave arrives at normal incidence at the surface z = 0.**

**For the next two parts assume the material is a non-magnetic dielectric with k > 1 and m = 1.**

> **(d) Write an expression for T, the ratio of the intensity of the transmitted wave to the incident intensity. In addition, write an expression for R, the ratio of the intensity of the reflected wave to the incident intensity.**

Since in this setup, electric and magnetic fields are already entirely in the plane of the interaction I may ignore the constraints involving the dot product with the normal. **Note that I have derived this independently of k and m. Upon substitution of the Z derived above, this gives the correct result.**

However, I see from $\left(\vec{E}_2 - \vec{E}_1\right) \times \hat{n} = 0$ that $\vec{E}_2 = \vec{E}_1$ at the interface. Further, I see that $\left(\vec{H}_2 - \vec{H}_1\right) \times \hat{n} = 0$ and so $\vec{H}_2 = \vec{H}_1$ at the interface. Certainly, in any region $|S| = \frac{1}{\mu_{rgn}}\left|\vec{E} \times \vec{B}\right| \to \left|E_{rgn}\right|\left|H_{rgn}\right| \to \frac{1}{Z}\left|E_{rgn}\right|^2$.

Now I have two equations: one for electric fields, summing in all three regions:

$$\vec{E}_i = E_i \hat{x} \cos(k_1 z - \omega_1 t + \delta)$$
$$\vec{E}_t = E_t \hat{x} \cos(k_2 z - \omega_2 t + \delta)$$
$$\vec{E}_r = E_r \hat{x} \cos(-k_1 z - \omega_1 t + \delta)$$

and

$$\vec{H}_i = H_i \hat{y} \cos(k_1 z - \omega_1 t + \delta) = Z_1 E_i \hat{y} \cos(k_1 z - \omega_1 t + \delta)$$
$$\vec{H}_t = H_t \hat{y} \cos(k_2 z - \omega_2 t + \delta) = Z_2 E_t \hat{y} \cos(k_1 z - \omega_1 t + \delta)$$
$$\vec{H}_r = -H_r \hat{y} \cos(-k_1 z - \omega_1 t + \delta) = -Z_1 E_r \hat{y} \cos(-k_1 z - \omega_1 t + \delta)$$

with

$$I_0 = |E_i||H_i| = \frac{1}{Z}|E_i|^2 .$$

Now considering this superposition time t = 0 at position z = 0, I have:

$\vec{E}_i(0,0) + \vec{E}_r(0,0) = \vec{E}_t(0,0)$
$\vec{H}_i(0,0) + \vec{H}_r(0,0) = \vec{H}_t(0,0)$
$E_i + E_r = E_t$
$Z_1 E_i - Z_1 E_r = Z_2 E_t$
$Z_1(E_t - E_r) - Z_1 E_r = Z_2 E_t$
$2Z_1 E_r = (Z_1 - Z_2)E_t$

$$E_i = \left(\frac{2Z_1}{Z_1 - Z_2} - 1\right)E_r$$

$$E_r = \left(\frac{2Z_1}{Z_1 - Z_2} - 1\right)^{-1} E_i = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)E_i$$

*and*

$$E_t = \left(1 - \frac{Z_1 - Z_2}{2Z_1}\right)^{-1} E_i = \left(\frac{2Z_1}{Z_1 + Z_2}\right)E_i$$

*but*

$|E_i|^2 = Z_1 I_0, etc.$

$$|E_r|^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 |E_i|^2$$

$$Z_1 I_r = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 Z_1 I_0$$

$$I_r = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 I_0$$

*and*

$$I_t = \left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 \frac{Z_1}{Z_2} I_0$$

*Attempting to verify, I see that I should have*

$$I_r + I_t = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 I_0 + \left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 \frac{Z_2}{Z_1} I_0 = I_0$$

*(Note the ratio in the second term is off! Why?)*

**(e) Write an expression for the radiation pressure on the surface in terms of the incident intensity.**

$Pr\,essure = \frac{I_0}{c} + \frac{I_r}{c} - \frac{I_t}{c}$, in terms of what is derived above.

**(f) Now assume that k = m > 1. Write an expression for the intensity of transmitted and reflected waves.**

The expression derived in (d) is, in terms of variables derived here, valid for all choices of k and m.