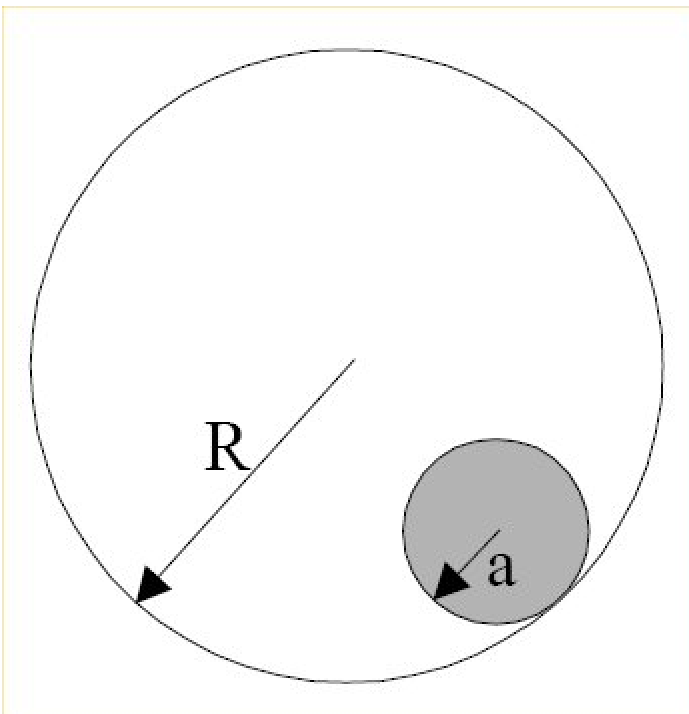
Ben Sauerwine
Practice for Qualifying Exams

Problem Source: **CMU Qualification Exam Day 2 (August 2002)**

(2) Consider a solid uniform sphere of radius a and mass m which rolls without slipping inside a fixed cylindrical pipe of radius R. Neglect friction, and let the sphere move only in the plane shown by the figure:



(a) Show that the moment of inertia of a uniform sphere is given by $I = \frac{2}{5}ma^2$**.**

$$\int_0^a\int_0^{2\pi}\int_0^{\pi}\left[\frac{m}{\frac{4}{3}\pi a^3}(r\sin\phi)^2\right]r^2\sin\phi d\phi d\theta dr$$

$$\to= \frac{m}{\frac{4}{3}\pi a^3}\int_0^a\int_0^{2\pi}\int_0^{\pi} r^4\sin^3\phi d\phi d\theta dr$$

$$\to= \frac{3}{4}\frac{m}{\pi a^3}\left(\frac{a^5}{5}\right)(2\pi)\int_0^{\pi}\sin^3\phi d\phi$$

$$use \quad \int_0^{\pi}\sin^3\phi d\phi = \int_0^{\pi}(1-\cos^2\phi)\sin\phi d\phi = 2+\frac{1}{3}\cos^3\phi\Big|_0^{\pi} = 2-\frac{2}{3} = \frac{4}{3}$$

$$\to= \frac{2}{5}ma^2$$

(b) Find the equation of motion for the sphere.

Now, taking the angle theta to be the angle between the center of the ball and the vertical, I have:

$$K = \frac{1}{2}I\left(\frac{R}{a}\dot{\theta}-\dot{\theta}\right)^2 + \frac{1}{2}(I+mR^2)\dot{\theta}^2$$

$$= \frac{1}{2}\left[I\left(\frac{R}{a}-1\right)^2 + I + mR^2\right]\dot{\theta}^2$$

$$V = mg(R-a)(1-\cos\theta)$$

$$L = K - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$$

$$\left[I\left(\frac{R}{a}-1\right)^2 + I + mR^2\right]\ddot{\theta} = mg(R-a)\sin\theta$$

(c) Find the period of small oscillations about the equilibrium position.

Expanding the sine term, I have

$$\left[I\left(\frac{R}{a}-1\right)^2 + I + mR^2\right]\ddot{\theta} \approx mg(R-a)\theta$$

This has solutions:

$$\theta = A\cos\left(\sqrt{\frac{mg(R-a)}{I\left(\frac{R}{a}-1\right)^2 + I + mR^2}}\,t\right)$$

and so the period for small oscillations is:

$$T = 2\pi\sqrt{\frac{I\left(\frac{R}{a}-1\right)^2 + I + mR^2}{mg(R-a)}}$$