Ben Sauerwine
Practice for Qualifying Exams

Problem Source: **CMU August 2002 Qualifying Exam**

In a toy model of a nuclear reactor, the conservation law for the number of neutrons is described by:

$$\frac{d}{dt}\int_V u\,dV = -\int_S \vec{j}\cdot\hat{n}\,dS + \int_V ru\,dV \qquad \textbf{(1)}$$

where V is the volume of the reactor, S is the surface, and $\hat{n}$ a normal unit vector. The neutrons are described by a number density u and a flux vector $\vec{j}$. The constant r is a growth rate describing the rate of increase of neutrons (more neutrons result in more nuclear fissions which result in more neutrons). An equation for the flow of neutrons has the form

$$\vec{j} = -v\vec{\nabla}u \qquad \textbf{(2)}$$

where v, a constant, is a diffusion coefficient.

(a) Use the divergence theorem to derive the differential form of the conservation law (equation 1). Combine this with equation 2 to give an equation for the diffusion of neutrons.

The divergence theorem states that $\int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_S \vec{F}\cdot\hat{n}\,dS$. Thus, I get from (1),

$$\frac{d}{dt}\int_V u\,dV = -\int_V \vec{\nabla}\cdot\vec{j}\,dV + \int_V ru\,dV$$

$$\frac{d}{dt}u = -\vec{\nabla}\cdot\vec{j} + ru$$

$$\frac{d}{dt}u = v\vec{\nabla}\cdot\vec{\nabla}u + ru$$

$$\frac{d}{dt}u = v\nabla^2 u + ru$$

(b) Now assume the reactor is linear and one-dimensional. What is the resulting form of the diffusion equation? The one-dimensional reactor is of size L and has a perfect reflector of electrons (such as a lead-208 shield) at x = 0 and a perfect absorber of electrons (such as cadmium) at x = L. If the number density of neutrons at position x and time t is given by $u(x,t)$, what are the boundary conditions $\frac{\partial u(0,t)}{\partial x}$ *and* $u(L,t)$?

In one dimension, this equation takes the form $\frac{d}{dt}u(x,t) = -v\frac{\partial^2}{\partial x^2}u(x,t) + ru(x,t)$. At the lead barrier, there is certainly no net flux of neutrons since it is a perfect reflector so that $\frac{\partial u(0,t)}{\partial x} = 0$. At the surface of the perfect absorber, there are certainly no surviving neutrons so that $u(L,t) = 0$.

(c) Use separation of variables to find a solution (valid for any initial conditions) which satisfies the diffusion equation and the boundary conditions.

$$u(x)w'(t) = vu''(x)w(t) + ru(x)w(t)$$
$$\frac{w'(t)}{w(t)} = \frac{ru(x) + vu''(x)}{u(x)} = C$$

where these must equal a constant since these equations and their variables are now invisible to each other. Now:

$$w'(t) = Cw(t)$$
$$w(t) = e^{Ct}$$
$$and$$
$$ru(x) + vu''(x) = Cu(x)$$
$$u''(x) = \frac{C-r}{v}u(x)$$
$$u(x) = e^{\sqrt{\frac{C-r}{v}}x} + e^{-\sqrt{\frac{C-r}{v}}x}$$
$$or \quad for \quad simplicity,$$
$$r > c$$
$$u''(x) = -\left(\frac{r-C}{v}\right)u(x)$$
$$u(x) = A\sin\left[\sqrt{\frac{r-C}{v}}x\right] + B\cos\left[\sqrt{\frac{r-C}{v}}x\right]$$

C is then to be fixed by the boundary conditions. If $\frac{\partial u}{\partial x}(0,t) = 0$, then the only option is the cosine set of solutions (selected instead the exponential option above for this reason).

$$u(x) = B\cos\left[\sqrt{\frac{r-C}{v}}x\right]$$

The value of B cannot be fixed by these boundary conditions because it represents the overall amplitude of results, and so requires a separate condition corresponding the desired overall population of neutrons.

If $u(L,t)=0$, $u(x)=B\cos\left[\sqrt{\frac{r-C}{v}}x\right]$ and so

$$0=B\cos\left[\sqrt{\frac{r-C}{v}}L\right]$$

$$\sqrt{\frac{r-C}{v}}=\left(n+\frac{1}{2}\right)\frac{\pi}{L}$$

$$n\in Z$$

and so

$$w(t)=e^{Ct}$$

$$C=v\left(r-\left(n+\frac{1}{2}\right)^2\frac{\pi^2}{L^2}\right)$$

(d) Suppose that the neutron density at time t = 0 is given by $u(x,0)=A\cos\left(\frac{\pi}{2L}x\right)$. Only one value of L will prevent the reactor from either shutting down due to lack of electrons or exploding due to runaway. Determine this value.

As I've found above, the form of the time portion is an exponential that will either die off or blow up unless C is pure imaginary. However, since r is real anything but zero seems unreasonable. Here, n = 0 and

$$w(t)=e^{Ct}$$

$$C=v\left(r-\left(n+\frac{1}{2}\right)^2\frac{\pi^2}{L^2}\right)$$

$$0=v\left(r-\left(\frac{1}{2}\right)^2\frac{\pi^2}{L^2}\right)$$

$$L=\frac{\pi}{2\sqrt{r}}$$