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Practice for Qualifying Exams

Given the Maxwell's Equations in differential form (SI units)

$$\vec{\nabla}\cdot\vec{D}=\rho \quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \quad \vec{\nabla}\cdot\vec{B}=0 \quad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

(a) Write down these Maxwell's Equations in integral form.

$$\int_S \vec{D}\cdot\hat{n}dS=\int_V \rho dV \quad \int_C \vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\int_S \vec{B}\cdot\hat{n}dS \quad \int_S \vec{B}\cdot\hat{n}dS=0 \quad \int_C \vec{H}\cdot d\vec{l}=\int_S \vec{J}\cdot\hat{n}dS+\frac{\partial}{\partial t}\int_S \vec{D}\cdot\hat{n}dS$$

(b)

(i) Which term in which of Maxwell's Equations denotes Maxwell's "displacement current"?

$$\int_C \vec{H}\cdot d\vec{l}=\int_S \vec{J}\cdot\hat{n}dS+\underline{\underline{\frac{\partial}{\partial t}\int_S \vec{D}\cdot\hat{n}dS}}$$

(ii) For that particular equation, show how the integral form follows from the differential form.

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

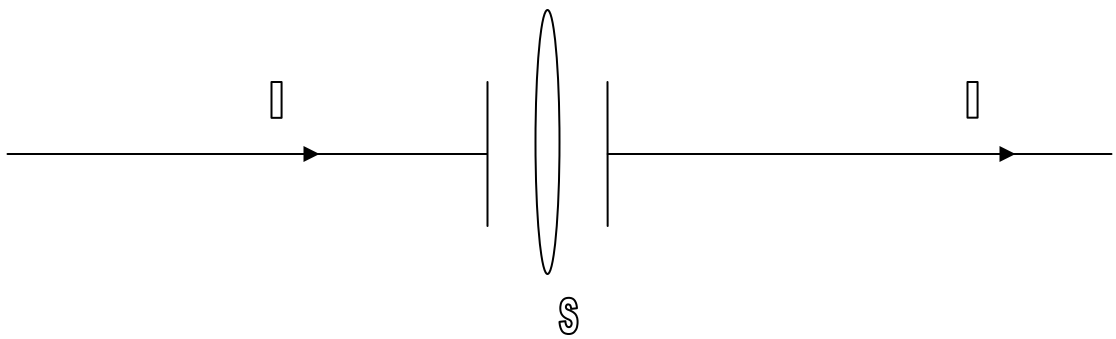
$$\int_S \vec{\nabla}\times\vec{H}\cdot\hat{n}dS=\int_S \vec{J}\cdot\hat{n}dS+\frac{\partial}{\partial t}\int_S \vec{D}\cdot\hat{n}dS$$

$$Stokes' \quad Theorem: \int_S \vec{\nabla}\times\vec{H}\cdot\hat{n}dS=\int_C \vec{H}\cdot d\vec{l}$$

$$\int_C \vec{H}\cdot d\vec{l}=\int_S \vec{J}\cdot\hat{n}dS+\frac{\partial}{\partial t}\int_S \vec{D}\cdot\hat{n}dS$$

(iii) Give an example (including a clearly labeled sketch) of a situation where omitting the displacement current term would lead to a glaring inconsistency. Explain clearly.

Consider a parallel-plate capacitor in the center of a very large, alternating-current wire.



It is easy to see that even neglecting the displacement current there will be some magnetic field on this symmetrically-placed surface from the current-carrying wires. However, neglecting the displacement current from Maxwell's Laws, one would be led to believe that there is no net magnetic field at all along this loop!

(c)

(i) Derive the wave equation for electromagnetic waves in a linear medium (characterized by ε and μ) in the absence of free charges and currents.

charge and current free:

$$\vec{\nabla}\cdot\vec{D}=0 \quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \quad \vec{\nabla}\cdot\vec{B}=0 \quad \vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$$

$$\vec{D}=\varepsilon\vec{E} \quad \vec{H}=\frac{1}{\mu}\vec{B}$$

$$Ampere: \frac{1}{\mu\varepsilon}\vec{\nabla}\times\vec{B}=\frac{\partial\vec{E}}{\partial t}\rightarrow\frac{1}{\mu\varepsilon}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}=\frac{\partial^2\vec{E}}{\partial t^2}$$

$$Faraday: -\frac{1}{\mu\varepsilon}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\frac{\partial^2\vec{E}}{\partial t^2}\rightarrow-\frac{1}{\mu\varepsilon}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{\nabla}^2\vec{E}\right]=\frac{\partial^2\vec{E}}{\partial t^2}$$

$$Gauss: \frac{1}{\mu\varepsilon}\vec{\nabla}^2\vec{E}=\frac{\partial^2\vec{E}}{\partial t^2}$$

$$Ampere: \frac{1}{\mu\varepsilon}\vec{\nabla}\times\vec{B}=\frac{\partial\vec{E}}{\partial t}\rightarrow\frac{1}{\mu\varepsilon}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$Faraday: \frac{1}{\mu\varepsilon}\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{\nabla}^2\vec{B}\right)=-\frac{\partial^2}{\partial t^2}\vec{B}$$

$$Gauss/Magnetism: \frac{1}{\mu\varepsilon}\vec{\nabla}^2\vec{B}=\frac{\partial^2}{\partial t^2}\vec{B}$$

(ii) Write down, in complex notation, expressions describing the fields $\vec{E}$ and $\vec{B}$ of a plane electromagnetic wave propagating in a linear medium in the positive z direction.

$$\vec{E}=\vec{E}_0 e^{i(kz-\omega t)} \quad \vec{B}=\vec{B}_0 e^{i(kz-\omega t)}$$

To simplify this a bit further, take $\vec{E}_0 = E_0\hat{x}$, and then from Faraday's Law,

$$\vec{\nabla}\times\vec{E} = -ikE_0\hat{y}e^{i(kz-\omega t)} \quad \frac{\partial}{\partial t}\vec{B} = -i\omega\vec{B}_0 e^{i(kz-\omega t)}$$

$$Faraday: -ikE_0\hat{y} = -\left(-i\omega\vec{B}_0\right) \rightarrow \vec{B}_0 = -\frac{k}{\omega}E_0\hat{y}$$

And then I also have Ampere's Law:

$$\frac{1}{\mu\varepsilon}\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t}$$

$$\frac{1}{\mu\varepsilon}\left(-ikB_0\hat{x}\right) = i\omega E_0\hat{x}$$

$$\frac{1}{\mu\varepsilon} = \frac{\omega^2}{k^2}$$

So I see that I am free to pick an amplitude of electric field and a wavelength or frequency, and in terms of the free variables:

$$\vec{E} = E_0\hat{x}e^{i\left(kz-\frac{k}{\sqrt{\mu\varepsilon}}t\right)} \quad \vec{B} = -\frac{1}{\sqrt{\mu\varepsilon}}E_0\hat{y}e^{i\left(kz-\frac{k}{\sqrt{\mu\varepsilon}}t\right)}$$

(iii) How is the speed of propagation related to ε and μ?

The classical wave equation is:

$$\vec{\nabla}^2\vec{F} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec{F}$$

Then from the form of the differential equations in part (i), I see that this indicates that

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

(d) Write down all the boundary conditions which electric and magnetic fields must satisfy when crossing the boundary from one linear medium (characterized by ε_1, μ_1) to another (ε_2, μ_2)? Pick one and show how it derives from the Maxwell's Equations.

In a charge free and current-free region, I have

$$\int_S \vec{D}\cdot\hat{n}dS = 0 \quad \int_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\hat{n}dS \quad \int_S \vec{B}\cdot\hat{n}dS = 0 \quad \int_C \vec{H}\cdot d\vec{l} = \frac{\partial}{\partial t}\int_S \vec{D}\cdot\hat{n}dS$$

And from Gauss's laws, I see that I must have $\vec{D}_\perp^1 = \vec{D}_\perp^2 \quad \vec{B}_\perp^1 = \vec{B}_\perp^2$, since a Gaussian pillbox drawn across the surface would have to contain no net charge and no magnetic source or sink and so for the large faces on either side of the surface to have no net flux these must be equal. Next, draw an Amperian loop running parallel and very close to the surface, and allow it to clamp down very close to the surface. From Faraday's Law, I see that this clamps my magnetic field inside the contour down to zero and so $\vec{E}_\parallel^1 = \vec{E}_\parallel^2$.

Similarly, from Ampere's law in current-free space I have an analogous relation and $\vec{H}_\parallel^1 = \vec{H}_\parallel^2$.

(e) When a plane electromagnetic wave, initially propagating in a vacuum, enters a linear medium of refraction n > 1, which of the characteristic properties of the wave remain unchanged? (A) wavelength, (B) frequency, (C) speed.

Certainly, from part c-iii, I see that the speed changes. Based on part c-ii, I see that since wavelength and frequency are related through the photon's speed, at least one must change. Evidently, only the wavelength changes—objects appear to be the same color underwater, despite the fact that the wave has undeniably slowed down.

(f)

(i) Derive expressions for the electric and magnetic field of a plane wave in an ohmic conductor of good conductivity in the absence of free charges.

Here, $\vec{J} = \sigma\vec{E}$. Further, let $\rho = 0$ and in a conductor $\vec{D} \equiv 0$. Now

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \quad \vec{\nabla}\cdot\vec{B} = 0 \quad \frac{1}{\mu}\vec{\nabla}\times\vec{B} = \sigma\vec{E}$$

Starting with Ampere's Law, I have:

$$\frac{1}{\mu}\vec{\nabla}\times\vec{B} = \sigma\vec{E} \rightarrow \frac{1}{\mu}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \sigma\left(\vec{\nabla}\times\vec{E}\right)$$

$$Faraday: \frac{1}{\mu}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{\nabla}^2\vec{B}\right] = -\sigma\frac{\partial\vec{B}}{\partial t}$$

$$Gauss/Magnetism: \frac{1}{\mu}\vec{\nabla}^2\vec{B} = \sigma\frac{\partial\vec{B}}{\partial t}$$

and for electric field, I again start with Ampere's Law:

$$\frac{1}{\mu}\vec{\nabla}\times\vec{B} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \rightarrow \frac{1}{\mu}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = \sigma\frac{\partial\vec{E}}{\partial t}$$

$$Faraday: \frac{1}{\mu}\vec{\nabla}\times\left(-\vec{\nabla}\times\vec{E}\right) = \sigma\frac{\partial\vec{E}}{\partial t} \rightarrow -\frac{1}{\mu}\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{\nabla}^2\vec{E}\right) = \sigma\frac{\partial\vec{E}}{\partial t}$$

$$Gauss/Electric: \frac{1}{\mu}\vec{\nabla}^2\vec{E} = \sigma\frac{\partial\vec{E}}{\partial t}$$

I see that these differential equations have solutions of the same form, and can be solved through separation of variables, e.g.:

$$\frac{1}{\mu}\vec{\nabla}^2\vec{F} = \sigma\frac{\partial\vec{F}}{\partial t} = \vec{C}$$

However, for simplicity of solution I will assign a direction to this and then generalize.

$$\frac{1}{F(x)}\left[\frac{1}{\mu}\frac{\partial^2 F(x)}{\partial x^2}\right] = \frac{1}{F(t)}\sigma\frac{\partial F}{\partial t} = C$$

The solution in position x is then:

$F(x) = A_x e^{\sqrt{\mu C}x} + B_x e^{-\sqrt{\mu C}x}$

The solution in time is also very simple damped oscillator:

$F(t) = A_t e^{\frac{C}{\sigma}t}$

And so $E(x) \approx B(x) \approx F(x)F(t) \approx \left[A_x e^{\sqrt{\mu C}x} + B_x e^{-\sqrt{\mu C}x}\right]\left[A_t e^{\frac{C}{\sigma}t}\right]$

For some constant C. However, for this to be a propagating wave in time, the constant C must be imaginary. Thus, I have:

$\frac{C}{\sigma}t = i\omega t$

$C = i\omega\sigma$

These constants should then be fixed in the particular system and related as in part c-ii.

(ii) Point out some characteristic differences in the behavior of these fields compared to the fields in the vacuum.

These fields will die off over time as the imaginary constant C will create a real term in the exponentials from part f-i. This real term must be negative, or otherwise it would not by physically realizable.

(iii) Derive an expression for the "skin depth" of a good ohmic conductor.

As evidenced in the exponential from part f-i, the fields will not completely die out, but rather exponentially die to zero. The real part of $\sqrt{i\omega\mu\sigma}$ is $-\left|\mathrm{Re}\left[e^{i\frac{\pi}{4}}\sqrt{\omega\mu\sigma}\right]\right| = -\sqrt{\frac{\omega\mu\sigma}{2}}$. Then, I see that $\frac{1}{e}$ of the amplitude remains after $\sqrt{\frac{2}{\omega\mu\sigma}}$ seconds. This could be regarded as analogous to the "$\frac{1}{e}$-life" in terms of depth traveled into the conductor.