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Practice for Qualifying Exams

Thanks to Ben B. for this solution.

Near absolute temperature T_0, the tension F in a rubber band is given by:

$$F = aT^2(L - L_0)$$

where L is the stretched length, L_0 is the unstretched length, and a is a constant. When $L = L_0$, the heat capacity at constant length is:

$$C_L = bT$$

(a) Find the entropy $S(L,T)$ for temperatures T near T_0 in terms of its value at T_0 and L_0.

$$P \approx F(T,L)$$
$$S \approx S(T,L)$$
$$dS = \left(\frac{\partial S}{\partial T}\right)_L dT + \left(\frac{\partial S}{\partial L}\right)_T dL$$
$$\left(\frac{\partial S}{\partial T}\right)_L = \frac{C_L}{T}$$
Maxwell :
$$dA = \left(\frac{\partial F}{\partial T}\right)_L dT + \left(\frac{\partial F}{\partial L}\right)_T dL = -SdT - PdL = -SdT - FdL$$
$$0 = -\left(\frac{\partial S}{\partial L}\right)_T - \left(\frac{\partial F}{\partial T}\right)_L$$
$$\left(\frac{\partial S}{\partial L}\right)_T = \left(\frac{\partial F}{\partial T}\right)_L$$
Then,
$$\left(\frac{\partial F}{\partial T}\right)_L = 2aT(L - L_0)$$
$$dS = \frac{C_L}{T} dT + 2aT(L - L_0)dL$$
Integrate,
$$S = S(T_0, L_0) + bT + aT(L^2 - 2LL_0)$$

(b) If one starts at T_0 and L_0 and stretches the insulated rubber band quasi-statically to L, what will the temperature T be of the rubber band?

$$T = \frac{C_L}{\left(\frac{\partial S}{\partial T}\right)_L}$$

$$T = \frac{b}{b + a\left(L^2 - 2LL_0\right)}$$

(c) Calculate the heat capacity at constant tension $C_F\left(L, T\right)$.

$$C_F \equiv T\left(\frac{\partial S}{\partial T}\right)_F$$

$$F = aT^2\left(L - L_0\right)$$

$$F^2 = a^2T^4\left(L^2 - 2LL_0 + L_0^2\right)$$

$$\frac{F^2}{aT^3} - aTL_0^2 = aT\left(L^2 - 2LL_0\right)$$

$$S = S\left(T_0, L_0\right) + bT + aT\left(L^2 - 2LL_0\right) = S\left(T_0, L_0\right) + bT + \frac{F^2}{aT^3} - aTL_0^2$$

$$C_F = T\left(\frac{\partial S}{\partial T}\right)_F = T\left[b - \frac{4F^2}{aT^4} - aL_0^2\right]$$