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Practice for Qualifying Exams

Thanks to Chip, Diana and Shiang Yong for their help with this solution.

Problem Source: **CMU August 2001 Qualifying Exam**

A white dwarf star is a dense, spherical start in which there is no longer any significant fusion generated power being produced. Pressure in the stellar material comes from degenerate electrons.

(a) Write an expression for the Fermi energy of a cold, non-relativistic electron gas.

Assume the electrons are confined to a box of volume V. Then, following the traditional procedure, allowed energies are:

$$E = \frac{\hbar^2 \pi^2}{2m_e V^{\frac{2}{3}}}\left(n_x^2 + n_y^2 + n_z^2\right)$$

which, considering energy in terms of density of states, can be thought of in terms of lattice points inside a sphere. Then, allowing each orbital to be 2-degenerate and considering one octant of the sphere, I have

$$\#_{states} = \frac{1}{8} \cdot 2 \cdot \frac{4}{3}\pi k^3$$

$$D\left(\#_{states}\right) = \pi k^2 dk$$

However, considering this number of states, it is clear that at the Fermi energy,

$$N = \#_{states} = \frac{1}{8} \cdot 2 \cdot \frac{4}{3}\pi k^3$$

$$\left(\frac{3N}{\pi}\right)^{\frac{1}{3}} = k_f$$

$$\varepsilon_f = \frac{\hbar^2 \pi^2 k^2}{2m_e V^{\frac{2}{3}}} = \frac{\hbar^2 \pi^2}{2m_e}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}$$

(b) Write an expression showing the dependence of pressure on density P(n). Do not evaluate any dimensionless integrals in P(n).

Now in terms of the Grand Canonical Ensemble, I have:

$$\mu \approx \varepsilon_F$$

For a single orbital, then, the partition function becomes:

$$Z_\varepsilon = \sum_{n=0,1} \left[e^{-\beta(\varepsilon-\mu)}\right]^n = 1 + e^{-\beta(\varepsilon-\mu)}$$

$$in \quad total,$$

$$\mathcal{Z} = \prod_\varepsilon Z_\varepsilon$$

Further,

$$P = \frac{k_B T}{V} \ln \mathcal{Z}$$ *(The proof of this is not clear to me. Pathria p. 93)*

Letting $\mu \approx \varepsilon_F$, then, I have

$$P = \frac{k_B T}{V} \int_0^\infty \ln\left[1 + e^{-\beta(\varepsilon(k)-\mu)}\right] D(k)\,dk$$

$$\varepsilon(k) = \frac{\hbar^2 \pi^2 k^2}{2m_e V^{\frac{2}{3}}}$$

$$D(k) = \pi k^2 \quad (part \quad a)$$

$$\mu \approx \varepsilon_F$$

$$Exactly,$$

$$\langle N \rangle = \int_0^\infty \frac{e^{-\beta(\varepsilon(k)-\mu)}}{\left[1 + e^{-\beta(\varepsilon(k)-\mu)}\right]} D(k)\,dk$$

can be used to solve for the exact chemical potential μ for a particular temperature and density.

(c) Typical parameters for a white dwarf are—radius 10^7 meters, number of electrons 10^{57}, surface temperature 10,000 Kelvin. Assuming temperature and density are uniform throughout the star, calculate whether the assumption that the electrons are degenerate is justified.

Since in terms of units, $kg = \frac{J \cdot s^2}{m^2}$, I may take:

$$\varepsilon_f = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}} = \frac{\left(\frac{4.14 \cdot 10^{-21} MeV \cdot s}{2\pi}\right)\left(\frac{6.63 \cdot 10^{-34} J \cdot S}{2\pi}\right)}{2\left(9.1 \cdot 10^{-31} kg\right)} \left(\frac{3\pi^2 10^{57}}{\frac{3}{4}\pi\left(10^7\right)^3 m^3}\right)^{\frac{2}{3}} = 0.2 \quad MeV$$

Comparing the Fermi energy to kT, I have:

$$kT = 8.617 \cdot 10^{-11} \frac{MeV}{K} \cdot 10^{5} K = 8.617 \cdot 10^{-6} MeV$$

I see that kT is much, much smaller than the Fermi energy, and so it is reasonable to assume that the electrons are degenerate. In this case, the temperature effect is dominated by a relatively small number of electrons that become excited into levels higher than the Fermi level.

(d) For the same parameters calculate whether the assumption that the electrons are non-relativistic is justified.

Considering the Fermi energy of 0.2 MeV, this corresponds to a velocity of:

$$E_k = 0.2MeV = m_e c^2 \gamma - m_e c^2$$

$$\frac{0.511 + 0.2}{0.511} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$c\sqrt{1 - \left(\frac{0.511}{0.511 + 0.2}\right)^2} = v$$

$$0.695c = v$$

So I see that at just the Fermi energy the electrons are traveling at greater than half the speed of light. These are certainly relativistic particles.

(e) Estimate the heat capacity of the star.

$$C_V = \left(\frac{dU}{dT}\right)_{N,V}$$

$$U = \langle E \rangle = \frac{\sum_i \varepsilon_i Z_i}{\text{Ƶ}} = -\frac{\partial \ln \text{Ƶ}}{\partial \beta}$$

$$C_V = \left(\frac{dU}{dT}\right)_{N,V} = \left(\frac{dU}{d\beta}\right)_{N,V} \left(\frac{d\beta}{dT}\right)_{N,V} = \frac{k}{\beta^2} \left(\frac{d^2 \ln \text{Ƶ}}{d\beta^2}\right)_{N,V}$$

$$= \frac{k_B}{\beta^2} \frac{d^2}{d\beta^2} \int_0^\infty \ln\left[1 + e^{-\beta(\varepsilon(k)-\mu)}\right] D(k) dk = \frac{k_B}{\beta^2} \frac{d}{d\beta} \int_0^\infty \frac{(\varepsilon(k) - \mu) e^{-\beta(\varepsilon(k)-\mu)}}{1 + e^{-\beta(\varepsilon(k)-\mu)}} D(k) dk$$

$$= \frac{k_B}{\beta^2} \int_0^\infty \left[\frac{(\varepsilon(k) - \mu)^2 e^{-2\beta(\varepsilon(k)-\mu)}}{\left(1 + e^{-\beta(\varepsilon(k)-\mu)}\right)^2} - \frac{(\varepsilon(k) - \mu)^2 e^{-\beta(\varepsilon(k)-\mu)}}{1 + e^{-\beta(\varepsilon(k)-\mu)}}\right] D(k) dk$$

I cannot complete this integral, nor do I understand how to use the Sommerfield expansion to obtain the heat capacity as in [Sekerka]. I need this estimate to complete the problem. Further, the Stefan-Boltzmann constant is not given on the exam.

(f) Assuming the star acts as a black body, how much power does it radiate?

In terms of the Stefan-Boltzmann constant, power radiated per unit area is $\frac{W}{m^2} = \sigma T^4$. Multiplied by the area of the star, I have $W = \left(4\pi R^2\right)\sigma T^4$.

(g) Estimate a cooling time for the star. Compare the cooling time to the age of the universe.

Using the heat capacity from part (e), $\left(\frac{\partial T}{\partial t}\right)_V = \left(\frac{\partial T}{\partial Q}\right)_V \left(\frac{\partial Q}{\partial t}\right)_V = \frac{4\pi\sigma R^2 T^4}{C_V}$. Then, the total temperature change is

$$T'(t) = \frac{4\pi\sigma R^2 T(t)^4}{C_V}$$

$$\frac{T'(t)}{T(t)^4} = \frac{4\pi\sigma R^2}{C_V}$$

$$\frac{3}{T(0)^3} - \frac{3}{T(t)^3} = \frac{4\pi\sigma R^2}{C_V} t$$

$$\frac{C_V}{4\pi\sigma R^2}\left[\frac{3}{T(0)^3} - \frac{3}{T(t)^3}\right] = t$$

which gives the cooling time to a particular temperature. I might choose the current cosmic background temperature, 3K.

The age of the universe is 13.7 billion years.