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Practice for Qualifying Exams

Problem Source: **CMU Qualification Exam Day 1 (August 2001)**

(1) Double Pendulum: Consider small planar oscillations of a double pendulum consisting of two metal bars, the upper bar of mass 2m and length D, and the lower one of mass m and length 2D. Each bar has uniform mass density and negligible width and thickness. Let θ_1 denote the angle of the upper bar from the vertical and θ_2 denote the angle of the lower bar from the vertical. All frictional effects are negligible. The moment of inertia about an axis perpendicular to and passing through the center of a long thin rod of mass M and length L is $\frac{1}{12}ML^2$.

(a) Determine the characteristic angular frequencies ω_1 and ω_2 of the normal modes of small oscillation. Ensure that $\omega_1 < \omega_2$

Note that then the moment of inertia about a fixed endpoint is $I_{end} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2$

For the upper pendulum portion, then:

$$K_{upper} = \frac{1}{2} I_{1,end}\dot{\theta}_1^2$$

$$V_{upper} = (2m)g\frac{D}{2}(1-\cos\theta_1)$$

For the lower pendulum portion, I will consider things a bit differently. Take the center-of-mass motion:

$$K_{CM,lower} = \frac{1}{2}m\left[\left(\dot{\theta}_1(D)\cos\theta_1 + \dot{\theta}_2\left(\frac{2D}{2}\right)\cos\theta_2\right)^2 + \left(\dot{\theta}_1(D)\sin\theta_1 + \dot{\theta}_2\left(\frac{2D}{2}\right)\sin\theta_2\right)^2\right]$$

$$= \frac{1}{2}m\begin{bmatrix}\dot{\theta}_1{}^2 D^2\cos^2\theta_1 + \dot{\theta}_2{}^2 D^2\cos^2\theta_2 + 2\dot{\theta}_1\dot{\theta}_2 D^2\cos\theta_1\cos\theta_2 \\ + \dot{\theta}_1{}^2 D^2\sin^2\theta_1 + \dot{\theta}_2{}^2 D^2\sin^2\theta_2 + 2\dot{\theta}_1\dot{\theta}_2 D^2\sin\theta_1\sin\theta_2\end{bmatrix}$$

$$= \frac{1}{2}mD^2\left[\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\right]$$

$$= \frac{1}{2}mD^2\left[\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

$$K_{rot,lower} = \frac{1}{2}I_{2,CM}\dot{\theta}_2{}^2$$

And similarly with the potential:

$$V_{lower} = mg\left(D(1-\cos\theta_1) + 2D(1-\cos\theta_2)\right)$$

Now I can write:

$$L = K - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1}$$

And:

$$L = \frac{1}{2} I_{1,end}\dot{\theta}_1^2 + \frac{1}{2} mD^2\left[\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + \frac{1}{2} I_{2,CM}\dot{\theta}_2^2$$

$$-(2m)g\frac{D}{2}(1-\cos\theta_1) - mg\left(D(1-\cos\theta_1) + 2D(1-\cos\theta_2)\right)$$

Dropping consts,

$$L = \frac{1}{2}\left(I_{1,end} + mD^2\right)\dot{\theta}_1^2 + \frac{1}{2}\left(I_{2,CM} + mD^2\right)\dot{\theta}_2{}^2 + mD^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + 2mgD(\cos\theta_1 + \cos\theta_2)$$

Now I will expand the Lagrangian in the small-oscillation limit:

$$\cos(\theta_1 - \theta_2) \approx 1 \quad \cos\theta \approx 1 - \frac{\theta^2}{2}$$

Then,

$$L \approx \frac{1}{2}\left(I_{1,end} + mD^2\right)\dot{\theta}_1^2 + \frac{1}{2}\left(I_{2,CM} + mD^2\right)\dot{\theta}_2{}^2 + mD^2\dot{\theta}_1\dot{\theta}_2 + 2mgD\left(2 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2}\right)$$

Now I'll find the equations of motion based on this Lagrangian:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2}$$

$$\frac{d}{dt}\left(\left(I_{2,CM} + mD^2\right)\dot{\theta}_2 + mD^2\dot{\theta}_1\right) = -2mgD\theta_2$$

$$\left(I_{2,CM} + mD^2\right)\ddot{\theta}_2 + mD^2\ddot{\theta}_1 = -2mgD\theta_2$$

Further, using the other equation of motion I get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1}$$

$$\frac{d}{dt}\left(\left(I_{1,end} + mD^2\right)\dot{\theta}_1 + mD^2\dot{\theta}_2\right) = -2mgD\theta_1$$

$$\left(I_{1,end} + mD^2\right)\ddot{\theta}_1 + mD^2\ddot{\theta}_2 = -2mgD\theta_1$$

First, I want to simplify a bit: in this case,

$$I_{1,end} + mD^2 = \frac{1}{12}(2m)D^2 + (2m)\frac{D^2}{4} + mD^2 = \frac{5}{3}mD^2$$

$$I_{2,CM} + mD^2 = \frac{1}{12}(m)(2D)^2 + mD^2 = \frac{4}{3}mD^2$$

Using these, I see that:

$$\frac{4}{3}mD^2\ddot{\theta}_2 + mD^2\ddot{\theta}_1 = -2mgD\theta_2$$

$$\frac{5}{3}mD^2\ddot{\theta}_1 + mD^2\ddot{\theta}_2 = -2mgD\theta_1$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -2m\frac{g}{D}\begin{bmatrix} \left(\frac{5}{3}-\frac{3}{4}\right)^{-1} & \left(\frac{5}{3}-\frac{3}{4}\right)^{-1}\left(-\frac{3}{4}\right) \\ \left(\frac{4}{3}-\frac{3}{5}\right)^{-1}\left(-\frac{3}{5}\right) & \left(\frac{4}{3}-\frac{3}{5}\right)^{-1} \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

The normal frequencies are then the eigenvalues of this matrix, in the form: $\left|\sqrt{eigenvalue}\right|$ since the derivative is taken twice.

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -2m\frac{g}{D}\begin{bmatrix} \frac{12}{11} & \frac{12}{11}\left(-\frac{3}{4}\right) \\ \frac{15}{11}\left(-\frac{3}{5}\right) & \frac{15}{11} \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = -\frac{2}{11}m\frac{g}{D}\begin{bmatrix} 12 & -9 \\ -9 & 15 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$C = -\frac{2}{11}m\frac{g}{D}$$

$$(12C-\lambda)(15C-\lambda) - 81C^2 = 0$$

$$\lambda^2 - 27C\lambda + 99C^2 = 0$$

$$\lambda_{\pm} = \left(-\frac{2}{11}m\frac{g}{D}\right)\frac{27 \pm \sqrt{27^2 - 4\cdot 99}}{2} = \left(-\frac{6}{11}m\frac{g}{D}\right)\frac{9 \pm \sqrt{37}}{2}$$

$$\omega_{1,2} = \left|\sqrt{\lambda_{-,+}}\right|$$

In this case, ω_1 will be the minus combination and ω_2 will be the plus combination, in order to minimize and maximize the respective coefficients.

(b) Determine the normal coordinates η_1 and η_2 corresponding to ω_1 and ω_2, respectively. Express these in terms of θ_1 and θ_2, ignoring overall constants.

These coordinates then correspond to the eigenvectors that go with these eigenvalues. The easiest way to find these will be to simply find the eigenvectors of

$\begin{bmatrix} 12 & -9 \\ -9 & 15 \end{bmatrix}, or \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix}$, which will be the same eigenvectors within a constant factor.

In order to find them, I take:

$$\begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} \rightarrow (4-\lambda)(5-\lambda)-9=0 \rightarrow \lambda^2 - 9\lambda + 11 = 0 \rightarrow \lambda_{\pm} = \frac{9 \pm \sqrt{81-44}}{2} = \frac{9 \pm \sqrt{37}}{2}$$

And now the eigenvectors are within a constant factor the null spaces of

$\begin{bmatrix} 4-\lambda_+ & -3 \\ -3 & 5-\lambda_+ \end{bmatrix}$ *and* $\begin{bmatrix} 4-\lambda_- & -3 \\ -3 & 5-\lambda_- \end{bmatrix}$ (with implicit multiples of $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$). These occur where:

$$\begin{bmatrix} (4-\lambda_+)c_1 - 3c_2 = 0 \\ -3c_1 + (5-\lambda_+)c_2 = 0 \end{bmatrix} \quad and \quad \begin{bmatrix} (4-\lambda_-)c_1 - 3c_2 = 0 \\ -3c_1 + (5-\lambda_-)c_2 = 0 \end{bmatrix}$$

respectively. Thus, $c_1 = \frac{3}{(4-\lambda_{\pm})} c_2$ and so $\eta_{\pm} = \theta_1 + \frac{3}{(4-\lambda_{\pm})}\theta_2$ within a constant factor.