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Practice for Qualifying Exams

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Problem Source: **CMU August 2001 Qualifying Exam**

Pion Photoproduction

Consider the Pion photoreaction $\gamma + p = p + \pi^0$

(a) **If the initial proton is at rest in the laboratory, find the laboratory threshold gamma ray energy for this reaction to go.**

Use the energy four-vectors for this reaction. For all four-vectors in all frames, $\vec{v}^2 = m^2 c^2$.

For the incident gamma ray in the rest frame, assuming that it is moving in the x-direction, I have

$$\vec{\gamma} = \begin{bmatrix} \frac{E_\gamma}{c} \\ \frac{E_\gamma}{c} \\ 0 \\ 0 \end{bmatrix}$$

For the proton at rest, then, I have

$$\vec{p}_i = \begin{bmatrix} m_p c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming the minimum-energy interaction, I have for the resulting proton and pion that:

$$\vec{p}_f = \vec{p}_i = \begin{bmatrix} m_p c \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{\pi}^0 = \begin{bmatrix} m_\pi c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now considering:

$$\vec{\gamma} + \vec{p}_i = \vec{p}_f + \vec{\pi}^0$$

$$\left(\vec{\gamma} + \vec{p}_i\right)^2 = \left(\vec{p}_f + \vec{\pi}^0\right)^2$$

$$\vec{\gamma}^2 + \vec{p}_i^{\,2} + 2\vec{p}\vec{\gamma} = \vec{p}_f^{\,2} + \vec{\pi}^{0\,2} + 2\vec{p}\vec{\pi}^0$$

$$2\vec{p}_i\vec{\gamma} = \vec{\pi}^{0\,2} + 2\vec{p}\vec{\pi}^0$$

$$2E_\gamma m_p = m_\pi^{\ 2}c^2 + 2m_p m_\pi c^2$$

$$E_\gamma = \frac{m_\pi^{\ 2} + 2m_p m_\pi}{2m_p}c^2$$

Solving this (substituting $m_\pi c^2 = 135MeV \quad m_p c^2 = 938MeV$), I have

$$E_\gamma = \frac{135}{2\cdot 938}\cdot 135 + 135MeV = 144.7MeV$$

(b) Consider a head-on collision between a proton and a typical photon in the 3K cosmic background radiation. Find the minimum proton energy that will allow this pion Photoproduction reaction to go.

This time, take my cosmic background photon to be traveling in the x direction:

$$\vec{\gamma} = \begin{bmatrix} \frac{KT}{c} \\ \frac{KT}{c} \\ 0 \\ 0 \end{bmatrix}$$

And give my initial proton the vector

$$\vec{p}_i = \begin{bmatrix} \frac{E_p}{c} \\ P_p \\ 0 \\ 0 \end{bmatrix}$$, remembering that $\left(\frac{E_p}{c}\right)^2 - P_p^2 = m_p^2 c^2$

My final photon and pion will be, as before, stationary: I want the threshold energy.

$$\vec{p}_f = \begin{bmatrix} m_p c \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{\pi}^0 = \begin{bmatrix} m_\pi c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Proceeding as before:

$$\vec{\gamma} + \vec{p}_i = \vec{p}_f + \vec{\pi}^0$$

$$\left(\vec{\gamma} + \vec{p}_i\right)^2 = \left(\vec{p}_f + \vec{\pi}^0\right)^2$$

$$\vec{\gamma}^2 + \vec{p}_i^{\ 2} + 2\vec{p}\vec{\gamma} = \vec{p}_f^{\ 2} + \vec{\pi}^{0\,2} + 2\vec{p}\vec{\pi}^0$$

$$m_p^{\ 2}c^2 + 2\left[\frac{KT}{c}\frac{E_p}{c} - \frac{KT}{c}P_p\right] = m_p^{\ 2}c^2 + m_\pi^{\ 2}c^2 + 2m_p m_\pi c^2$$

$$Lemma: \quad P_p^2 = \left(\frac{E_p}{c}\right)^2 - m_p^2c^2 \quad P_p = \sqrt{\left(\frac{E_p}{c}\right)^2 - m_p^2c^2}$$

$$m_p^{\ 2}c^2 + 2\frac{KT}{c}\left[\frac{E_p}{c} - P_p\right] = m_p^{\ 2}c^2 + m_\pi^{\ 2}c^2 + 2m_p m_\pi c^2$$

$$\frac{E_p}{c} - P_p = \frac{c}{KT}\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]$$

$$\left(\frac{E_p}{c} - P_p\right)^2 = \left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2$$

$$\left(\frac{E_p}{c}\right)^2 + P_p^2 - 2\frac{E_p}{c}P_p = \left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2$$

$$-2\frac{E_p}{c}P_p = \left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2 - \left(\frac{E_p}{c}\right)^2 - \left[\left(\frac{E_p}{c}\right)^2 - m_p^2c^2\right]$$

$$-2\frac{E_p}{c}P_p = \left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2 - 2\left(\frac{E_p}{c}\right)^2 + m_p^2c^2$$

$$4\left(\frac{E_p}{c}\right)^2 P_p^{\ 2} = \left[\left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2 - 2\left(\frac{E_p}{c}\right)^2 + m_p^2c^2\right]^2$$

$$4\left(\frac{E_p}{c}\right)^2\left(\left(\frac{E_p}{c}\right)^2 - m_p^2c^2\right) = \left[\left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi^{\ 2}c^2 + m_p m_\pi c^2\right]^2 - 2\left(\frac{E_p}{c}\right)^2 + m_p^2c^2\right]^2$$

Continuing along this line, I have:

$$4\left(\frac{E_p}{c}\right)^2\left(\left(\frac{E_p}{c}\right)^2 - m_p^2c^2\right) = \left[\left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^2 - 2\left(\frac{E_p}{c}\right)^2 + m_p^2c^2\right]^2$$

After Simplification,

$$0 = \left(\frac{c}{KT}\right)^4\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^4 + 2m_p^2c^2\left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^2$$

$$-4\left(\frac{E_p}{c}\right)^2\left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^2 + m_p^4c^4$$

And,

$$4\left(\frac{E_p}{c}\right)^2 = \left(\frac{c}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^2 + 2m_p^2c^2 + m_p^4c^4\left[\frac{1}{2}m_\pi{}^2c^2 + m_pm_\pi c^2\right]^{-2}\left(\frac{c}{KT}\right)^{-2}$$

$$4\left(\frac{E_p}{c}\right)^2 = \left(\frac{1}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^2c^6 + 2m_p^2c^2 + m_p^4\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^{-2}\left(\frac{c}{KT}\right)^{-2}$$

$$E_p{}^2 = \frac{1}{4}\left(\frac{1}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^2c^8 + \frac{1}{2}m_p^2c^4 + \frac{1}{4}m_p^4\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^{-2}(KT)^2$$

A quick glance shows that units match. Evaluating this using

$m_\pi c^2 = 135 MeV \quad m_pc^2 = 938 MeV$

$K = 8.617\times10^{-11}\frac{eV}{K}$

Then,

$$E_p{}^2 = \frac{1}{4}\left(\frac{1}{KT}\right)^2\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^2c^8 + \frac{1}{2}m_p^2c^4 + \frac{1}{4}m_p^4\left[\frac{1}{2}m_\pi{}^2 + m_pm_\pi\right]^{-2}(KT)^2$$

$E_p = 2.62\times10^{14} MeV$

(c) Speculate briefly on the implications of your result from (b) for the energy spectrum of cosmic ray protons.

This result means that any intergalactic proton radiation of energy greater than $2.62\times10^{14} MeV = 42J$ would not reach us because this reaction would occur—but yet we observe cosmic rays this powerful incident from space!