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Practice for Qualifying Exams
Electricity and Magnetism

Problem Source: **CMU August 2001 Qualifying Exam**

(3) A solid glass sphere of mass M and radius R carries on its surface a uniformly distributed positive charge Q. It is spinning in outer space with angular speed ω.

(a) Since the sphere is spinning, there are accelerated charges. Explain briefly why the spinning sphere does not emit electromagnetic radiation.

Radiation is emitted when the acceleration has a component in the direction of velocity. The acceleration in the rotating sphere is perpendicular to the direction that the charges are moving in, and therefore no radiation is emitted in this case.

(b) Next, an external magnetic field of magnitude B is established and the sphere moves in pure precession with the spin axis at an angle α to the magnetic field. Derive how much time it takes to precess through one complete turn.

First I will find the magnetic moment of the sphere:

$$\vec{\mu} = \frac{1}{2}\int \vec{x} \times \vec{J}(\vec{x}) d^3x$$

Thus the magnetic moment of a loop with uniform current I is

$$\vec{\mu} = \frac{1}{2}\hat{n}\int (rI) r d\theta = I\pi r^2$$

Now integrating a stack of loops to obtain a sphere, I have:

$$\vec{\mu} = \hat{n}\int_0^\pi \left[I(\phi)\pi r(\phi)^2\right] R\sin\phi d\phi$$

$$r(\phi) = R\sin\phi$$

$$I(\phi) = \frac{Q}{4\pi R^2}(\omega R\sin\phi)$$

$$\vec{\mu} = \hat{n}\frac{Q\omega R^2}{4}\int_0^\pi \sin^4\phi d\phi$$

$$= \hat{n}\frac{Q\omega R^2}{4}\left[\frac{3\pi}{8}\right]$$

Now I may use this to determine the torque on the sphere:

$$\tau = \left|\vec{\mu} \times \vec{B}\right| = |\mu||B|\sin\alpha = \frac{3\pi}{32} Q\omega R^2 B\sin\alpha$$

the torque will be in the direction perpendicular to both the direction of rotation and the magnetic field.

Now: $\vec{\tau} = \frac{d\vec{L}}{dt}$ $\quad \vec{L} = I_s \vec{\omega}$

Note that since the torque is always perpendicular to the angular momentum, the angular momentum will precess about the magnetic field and never actually change.

Finally, I must find the period of precession from this. In order to do so, I consider the total amount which the angular momentum must change in one precession: since $|L| = I_s \omega$, it will trace a circle of perimeter $2\pi(|L| \sin\alpha) = 2\pi I_s \omega \sin\alpha$. Then, the time for one precession is

$$period = \frac{2\pi I_s \omega \sin\alpha}{\tau} = \frac{64}{3\pi Q R^2 B} I_s$$

where

$$I_s = \frac{M}{\frac{4}{3}\pi R^3} \int_0^R \int_0^\pi \int_0^{2\pi} (r\sin\phi)^2 r^2 \sin\phi \, d\theta \, d\phi \, dr = \frac{MR^5}{\frac{4}{3}\pi R^3} \left(\frac{2\pi}{5}\right) \int_0^\pi (\sin\phi)^2 \sin\phi \, d\phi = \frac{MR^5}{\frac{4}{3}\pi R^3} \left(\frac{2\pi}{5}\right)\left(\frac{4}{3}\right) = \frac{2}{5} MR^2$$

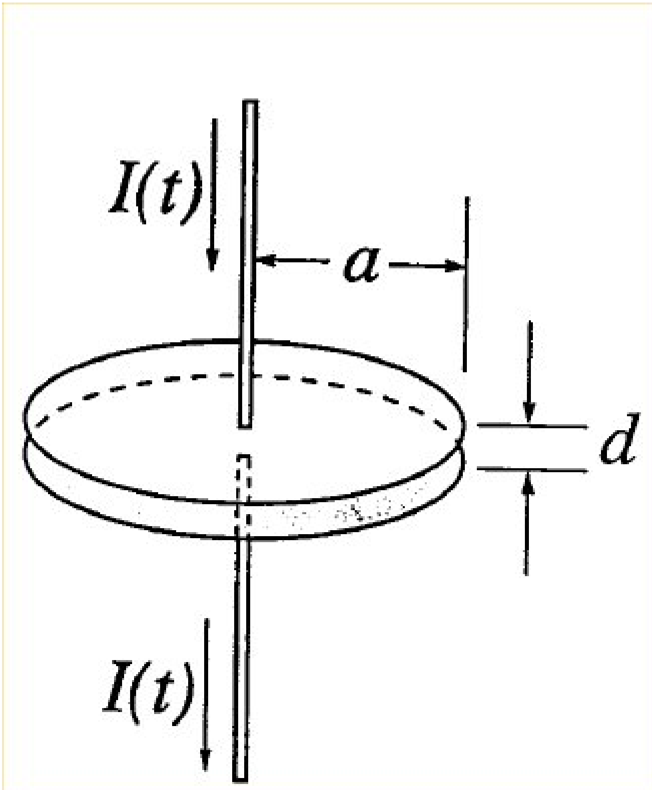
$$period = \frac{128M}{15\pi QB}$$

Note that this breaks down due to division by zero if $\omega = 0$.

Problem Source: **CMU August 2001 Qualifying Exam**

(4) Alternating-Circuit Capacitor

An ideal circular parallel-plate capacitor of radius a and plate separation d << a is connected to an alternating-current generator by axial leads, as shown in the sketch below. The current in the wire is $I(t) = I_0 \sin(\omega t)$, where $\omega << \frac{c}{a}$ and c is the speed of light. Assume that the axial leads lie along the z-axis with the origin at the center of the capacitor.



(a) **The electric and magnetic fields inside the capacitor between the plates have the form, neglecting fringe effects,**

$$E(\rho,\phi,z) = E_0 f_E\left(\frac{\omega\rho}{c}\right)\cos(\omega t)\hat{k}$$

$$B(\rho,\phi,z) = -\frac{E_0}{c} f_B\left(\frac{\omega\rho}{c}\right)\sin(\omega t)\hat{e}_\phi$$

where ρ,ϕ,z **are standard cylindrical coordinates and** E_0 **is defined such that** $f_E(0)=1$**. Note that the electric field at the center of the capacitor is given by** $E_0\cos(\omega t)\hat{k}$**. The functions** $f_E(x)$ **and** $f_B(x)$**, where** $x=\frac{\omega\rho}{c}$**, can be determined using the integral forms of Maxwell's equations:**

$$\varepsilon_0\int_{S_c}\vec{E}\cdot d\vec{a} = \int_V \rho dV \qquad \oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_{S_0}\vec{B}\cdot d\vec{a}$$

$$\varepsilon_0\int_{S_c}\vec{B}\cdot d\vec{a} = 0 \qquad \frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{l} = \int_{S_o}\vec{J}\cdot d\vec{a} + \varepsilon_0\frac{\partial}{\partial t}\int_{S_0}\vec{E}\cdot d\vec{a}$$

Where the closed surface S_c encloses volume V, closed loop C is the boundary of the open surface S_o, ρ is a charge density, and $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Use these relations with suitable loops, surfaces, and volumes to show that

$$f_E(x) = 1 - \int_0^x dx' \, f_B(x')$$

$$x f_B(x) = \int_0^x dx' \, x' \, f_E(x')$$

Next, show that $f_E(x) = J_0(x)$ and $f_B(x) = J_1(x)$, where $J_n(x)$ are the Bessel functions of the first kind satisfying Bessel's differential equation

$$x^2 \frac{d^2}{dx^2} y(x) + x \frac{d}{dx} y(x) + \left(x^2 - n^2\right) y(x) = 0.$$

Using

$$\frac{1}{\mu_0} \oint_C \vec{B} \cdot d\vec{l} = \int_{S_o} \vec{J} \cdot d\vec{a} + \varepsilon_0 \frac{\partial}{\partial t} \int_{S_0} \vec{E} \cdot d\vec{a}$$

I define a contour to be a closed, circular loop between the capacitor plates and centered on the z-axis and oriented such that the positive z-axis is the surface normal.

Now I have:

$$\frac{1}{\mu_0} \oint_C \vec{B} \cdot d\vec{l} = \int_{S_o} \vec{J} \cdot d\vec{a} + \varepsilon_0 \frac{\partial}{\partial t} \int_{S_0} \vec{E} \cdot d\vec{a}$$

$$E(\rho, \phi, z) = E_0 \, f_E\left(\frac{\omega \rho}{c}\right) \cos(\omega t) \hat{k}$$

$$B(\rho, \phi, z) = -\frac{E_0}{c} \, f_B\left(\frac{\omega \rho}{c}\right) \sin(\omega t) \hat{e}_\phi$$

$$\vec{J} = 0$$

for

$$\frac{1}{\mu_0} (2\pi\rho) \left(-\frac{E_0}{c} \, f_B\left(\frac{\omega \rho}{c}\right) \sin(\omega t) \right) = \varepsilon_0 \frac{\partial}{\partial t} \int_0^\rho 2\pi\rho' \left[E_0 \, f_E\left(\frac{\omega \rho'}{c}\right) \cos(\omega t) \right] d\rho'$$

and simplifying

$$\rho \left(f_B\left(\frac{\omega \rho}{c}\right) \right) = \int_0^\rho \frac{\omega \rho'}{c} \left[f_E\left(\frac{\omega \rho'}{c}\right) \right] d\rho'$$

let

$$\rho\left(f_B\left(\frac{\omega\rho}{c}\right)\right) = \int_0^{\rho} \frac{\omega\rho'}{c}\left[f_E\left(\frac{\omega\rho'}{c}\right)\right] d\rho'$$

$$x' = \frac{\omega\rho'}{c} \quad dx' = \frac{\omega}{c} dp'$$

$$x = \frac{\omega\rho}{c} \quad \rho = \frac{cx}{\omega}$$

$$x f_B(x) = \int_0^{x} x'\left[f_E(x')\right] dx'$$

Now using

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int_{S_0} \vec{B} \cdot d\vec{a}$$

I define a contour to be a rectangular region oriented vertically between the plates, with one end along the z-axis and the top edges of the contour on the capacitor plates. Let this contour be oriented so that its surface normal is in the $\hat{\phi}$ direction.

Now I have:

$$dE(0) - dE(\rho) = -\frac{\partial}{\partial t} d\int_0^{\rho} B(\rho') d\rho'$$

$$E(\rho, \phi, z) = E_0 f_E\left(\frac{\omega\rho}{c}\right)\cos(\omega t)\hat{k}$$

$$B(\rho, \phi, z) = -\frac{E_0}{c} f_B\left(\frac{\omega\rho}{c}\right)\sin(\omega t)\hat{e}_{\phi}$$

$$f_E(0) = 1$$

$$f_E(0)\cos(\omega t) - f_E\left(\frac{\omega\rho}{c}\right)\cos(\omega t) = \frac{\partial}{\partial t}\frac{1}{c}\int_0^{\rho} f_B\left(\frac{\omega\rho'}{c}\right)\sin(\omega t) d\rho'$$

$$1 - f_E\left(\frac{\omega\rho}{c}\right) = \frac{\omega}{c}\int_0^{\rho} f_B\left(\frac{\omega\rho'}{c}\right) d\rho'$$

Again using:

$$x' = \frac{\omega \rho'}{c} \quad dx' = \frac{\omega}{c} dp'$$

$$x = \frac{\omega \rho}{c} \quad \rho = \frac{cx}{\omega}$$

$$1 - f_E(x) = \int_0^x f_B(x) dx'$$

Just as expected.

Finally, I take:

$$f_E(x) = 1 - \int_0^x dx' \, f_B(x')$$

$$x f_B(x) = \int_0^x dx' \, x' \, f_E(x')$$

$$for$$

$$f_E'(x) = -f_B(x)$$

$$f_E''(x) = -f_B'(x)$$

$$f_B(x) + x f_B'(x) = x f_E(x)$$

$$2 f_B'(x) + x f_B''(x) = f_E(x) + x f_E'(x)$$

$$combining,$$

$$[-f_E'(x)] + x[-f_E''(x)] = x f_E(x)$$

$$x f_E(x) + f_E'(x) + x f_E''(x) = 0$$

$$x^2 f_E(x) + x f_E'(x) + x^2 f_E''(x) = 0$$

$$f_E(x) \to J_0(x)$$

$$and$$

$$2 f_B'(x) + x f_B''(x) = \frac{1}{x}\left(f_B(x) + x f_B'(x) \right) + x\left(- f_B(x)\right)$$

$$2x f_B'(x) + x^2 f_B''(x) = f_B(x) + x f_B'(x) - x^2 f_B(x')$$

$$x^2 f_B(x) - f_B(x) + x f_B'(x) + x^2 f_B''(x) = 0$$

$$\left(x^2 - 1\right) f_B(x) + x f_B'(x) + x^2 f_B''(x) = 0$$

$$f_B(x) \to J_1(x)$$

b) Show that neglecting fringe effects, the coefficient E_0 is given by

$$E_0 = \frac{I_0}{2\pi \varepsilon_0 a c J_1\left(\frac{\omega a}{c}\right)}$$

You will need the following property of Bessel functions:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

Consider:

$$\varepsilon_0 \int_{S_c} \vec{E} \cdot d\vec{a} = \int_V \rho dV$$

Ignore all fringe effects and define a pillbox-shaped surface to encompass one of the two capacitor plates, with the circular ends parallel to the capacitor plate.

Now it is easy to see that:

$$I(t) = I_0 \sin(\omega t) = \frac{\partial \rho_{total,1 \quad plate}(t)}{\partial t}$$

$$\rho_{total,1 \quad plate}(t) = -\frac{I_0}{\omega}\cos(\omega t)$$

$$\varepsilon_0 \int_{S_c} \vec{E} \cdot d\vec{a} = \int_V \rho dV = \rho_{total,1 \quad plate}(t) = -\frac{I_0}{\omega}\cos(\omega t)$$

$$= \varepsilon_0 \int_0^{2\pi}\int_0^{\rho} \left[E_0 f_E\left(\frac{\omega \rho'}{c}\right)\cos(\omega t)\hat{k} \cdot \left(-\hat{k}\right)\right] \rho' d\rho' d\theta$$

$$= -2\pi\varepsilon_0 E_0 \cos(\omega t) \int_0^a J_0\left(\frac{\omega \rho'}{c}\right)\rho' d\rho'$$

$$u = \frac{\omega \rho'}{c} \qquad du = \frac{\omega \rho'}{c} d\rho'$$

$$= -2\pi\varepsilon_0 E_0 \cos(\omega t)\frac{c^2}{\omega^2}\int_{\rho'=0}^{\rho'=a} J_0(u) u du$$

$$= -2\pi\varepsilon_0 E_0 \cos(\omega t)\frac{c^2}{\omega^2}\left(J_1\left(\frac{\omega a}{c}\right)\frac{\omega a}{c}\right) = -2\pi\varepsilon_0 E_0 \cos(\omega t)\frac{ca}{\omega}\left(J_1\left(\frac{\omega a}{c}\right)\right)$$

And combining this with the earlier portion from the integral of charge density, I have:

$$-\frac{I_0}{\omega}\cos(\omega t) = -2\pi\varepsilon_0 E_0 \cos(\omega t)\frac{ca}{\omega}\left(J_1\left(\frac{\omega a}{c}\right)\right)$$

$$I_0 = 2\pi\varepsilon_0 E_0 ca\left(J_1\left(\frac{\omega a}{c}\right)\right)$$

$$\frac{I_0}{2\pi\varepsilon_0 ca J_1\left(\frac{\omega a}{c}\right)} = E_0$$

Just as expected.

c) **Use the small-x approximations** $J_0(x) \approx 1 - \frac{1}{4}x^2, J_1(x) \approx \frac{1}{2}x - \frac{1}{16}x^3$ **to calculate the capacitance C and the inductance L of the equivalent series LC circuit to leading order in** $\frac{\omega a}{c}$**. Note that with the above conventions for Maxwell's equations, the energy density of** $\vec{E}$ **is** $\frac{1}{2}\varepsilon_0 \vec{E}^2$ **and the energy density of** $\vec{B}$ **is** $\frac{1}{2\mu_0}\vec{B}^2$**. Given L and C, estimate the resonant frequency of the system in terms of** $\frac{c}{a}$**.**

$$E(\rho,\phi,z) = E_0 J_0\left(\frac{\omega\rho}{c}\right)\cos(\omega t)\hat{k}$$

$$B(\rho,\phi,z) = -\frac{E_0}{c} J_1\left(\frac{\omega\rho}{c}\right)\sin(\omega t)\hat{e}_\phi$$

Now integrating to find the energy of the electric and magnetic fields in the capacitor, I have:

$$\int E^2(\rho,\phi,z)d^3x \approx E_0^{\ 2}\cos^2(\omega t)(2\pi d)\int_0^a \left(1-\frac{1}{4}\left(\frac{\omega\rho}{c}\right)^2\right)^2 \rho d\rho$$

$$= E_0^{\ 2}\cos^2(\omega t)(2\pi d)\int_0^a \left(1-\frac{1}{2}\left(\frac{\omega\rho}{c}\right)^2+\frac{1}{16}\left(\frac{\omega\rho}{c}\right)^4\right)\rho d\rho$$

$$= E_0^{\ 2}\cos^2(\omega t)(2\pi d)\left[\frac{a^2}{2}-\frac{1}{2}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4}+\frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}\right]$$

And for magnetic field:

$$\int B^2(\rho,\phi,z)d^3x \approx \left(\frac{E_0}{c}\right)^2\sin^2(\omega t)(2\pi d)\int_0^a \left(\frac{1}{2}\frac{\omega\rho}{c}-\frac{1}{16}\left(\frac{\omega\rho}{c}\right)^3\right)^2 \rho d\rho$$

$$= \left(\frac{E_0}{c}\right)^2\sin^2(\omega t)(2\pi d)\int_0^a \left(\frac{1}{4}\left(\frac{\omega\rho}{c}\right)^2-\frac{1}{16}\left(\frac{\omega\rho}{c}\right)^4+\frac{1}{256}\left(\frac{\omega\rho}{c}\right)^6\right)\rho d\rho$$

$$= \left(\frac{E_0}{c}\right)^2\sin^2(\omega t)(2\pi d)\left[\frac{1}{4}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4}-\frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}+\frac{1}{256}\left(\frac{\omega}{c}\right)^6\frac{a^8}{8}\right]$$

Now using the formulae for energy density, I have:

$$\frac{1}{2\mu_0}\int \vec{B}^2 d^3x = \frac{1}{2}LI^2$$

$$\left(\frac{E_0}{c}\right)^2\sin^2(\omega t)(2\pi d)\left[\frac{1}{4}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4}-\frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}+\frac{1}{256}\left(\frac{\omega}{c}\right)^6\frac{a^8}{8}\right] = \frac{1}{2}LI_0^{\ 2}\sin^2(\omega t)$$

$$2\left(\frac{E_0}{I_0 c}\right)^2(2\pi d)\left[\frac{1}{4}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4}-\frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}+\frac{1}{256}\left(\frac{\omega}{c}\right)^6\frac{a^8}{8}\right] = L$$

and taking leading order in $\frac{\omega a}{c}$ I have

$$\frac{\pi d}{4\omega^2}\left(\frac{E_0}{I_0}\right)^2\left(\frac{\omega a}{c}\right)^4 = L$$

where E_0 is defined above. Further,

$$\frac{1}{2}\varepsilon_0\int \vec{E}^2 d^3x = \frac{1}{2}\frac{Q^2}{C}$$

$$Q = \int I(t) = -\frac{I_0}{\omega}\cos(\omega t)$$

$$\frac{1}{2}\varepsilon_0 E_0^{\ 2}\cos^2(\omega t)(2\pi d)\left[\frac{a^2}{2} - \frac{1}{2}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4} + \frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}\right] = \frac{1}{2C}\left(\frac{I_0}{\omega}\right)^2\cos^2(\omega t)$$

$$\varepsilon_0\left(\frac{\omega E_0}{I_0}\right)^2(2\pi d)\left[\frac{a^2}{2} - \frac{1}{2}\left(\frac{\omega}{c}\right)^2\frac{a^4}{4} + \frac{1}{16}\left(\frac{\omega}{c}\right)^4\frac{a^6}{6}\right] = \frac{1}{C}$$

and to leading order:

$$\varepsilon_0 c^2\left(\frac{E_0}{I_0}\right)^2(2\pi d)\left(\frac{\omega a}{c}\right)^2 = \frac{1}{C}$$

Now I consider the natural frequency of an LC circuit:

$$\frac{1}{C}Q(t) + LQ''(t) = 0$$

$$-\frac{1}{LC}Q(t) = Q''(t)$$

so that solutions are clearly of the form

$$Q(t) = A\sin\left(\frac{1}{\sqrt{LC}}t\right) + B\cos\left(\frac{1}{\sqrt{LC}}t\right)$$

so that the frequency in cycles per second is $\frac{1}{2\pi}\frac{1}{\sqrt{LC}}$.

Then I have:

$$\frac{1}{2\pi}\sqrt{\frac{\varepsilon_0 c^2\left(\frac{E_0}{I_0}\right)^2(2\pi d)\left(\frac{\omega a}{c}\right)^2}{\frac{\pi d}{4\omega^2}\left(\frac{E_0}{I_0}\right)^2\left(\frac{\omega a}{c}\right)^4}} = \frac{c}{2\pi}\sqrt{8\varepsilon_0}\left(\frac{c}{a}\right) = \frac{1}{\pi}\sqrt{\frac{2}{\mu_0}}\left(\frac{c}{a}\right)$$

d) From a large distance away, the capacitor can be viewed simply as an oscillating electric dipole with $p_E(t) = \vec{p}_0\cos(\omega t)$. Calculate $\vec{p}_0$.

Neglecting time-retardation effects, I have $\vec{p} = q\vec{d}$. Taking

$$\rho_{total,1 \quad plate}(t) = -\frac{I_0}{\omega}\cos(\omega t)$$

$$\vec{p}_0 = -\vec{d}\,\frac{I_0}{\omega}$$

e) For a localized system of charges and currents varying sinusoidally in time, $\rho(\vec{r},t) = \rho(\vec{r})e^{-i\omega t}$, $\vec{J}(\vec{r},t) = \vec{J}(\vec{r})e^{-i\omega t}$, the vector potential far from the capacitor is given in the long-wavelength approximation and the Lorentz gauge by

$$\vec{A}(\vec{r},t) = \vec{A}(\vec{r})e^{-i\omega t} \qquad \vec{A}(\vec{r},t) = \frac{e^{\frac{i\omega r}{c}}}{4\pi\varepsilon_0 c^2 r}\int d^3r\vec{J}(\vec{r})$$

where $r = |\vec{r}|$. Show that for an oscillating electric dipole,

$$\vec{A}(\vec{r}) = -\frac{i\omega\vec{p}_0}{4\pi\varepsilon_0 c^2}\frac{e^{\frac{i\omega r}{c}}}{r}$$

Using integration by parts, I see that:

$$\int d^3r\vec{J}(\vec{r})$$
$$u = \vec{J}(\vec{r}) \qquad dv = d^3r$$
$$du = \vec{\nabla}\cdot\vec{J}(\vec{r}) \qquad v = \vec{r}d^3r$$
$$= \vec{r}\vec{J}(\vec{r})\Big|_{-\infty}^{\infty} - \int \vec{r}\left(\vec{\nabla}\cdot\vec{J}(\vec{r})\right)d^3r = -\int \vec{r}\left(\vec{\nabla}\cdot\vec{J}(\vec{r})\right)d^3r$$
$$iw\vec{\rho} = \left(\vec{\nabla}\cdot\vec{J}\right) \quad continuity \quad equation$$
$$= -i\omega\int \vec{r}\rho(\vec{r})d^3r$$
$$(defn \quad of \quad dipole \quad moment)$$
$$= -i\omega\vec{p}_0$$

which upon substitution gives me exactly what I expect.

f) For the magnetic field far from the capacitor, let $\vec{B}(\vec{r},t) = \vec{B}(\vec{r})e^{-i\omega t}$, then use $\vec{B} = \vec{\nabla}\times\vec{A}$ to show that

$$\vec{B}(\vec{r}) = \frac{i\omega}{4\pi\varepsilon_0 c^2}\frac{\partial}{\partial r}\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\vec{p}_0\times\hat{e}_r$$

where $\hat{e}_r = \frac{\vec{r}}{r}$ is the unit vector directed radially away from the origin.

Determine the radiation field $\vec{B}_{rad}$ far from the capacitor and the time-

averaged power $\langle P \rangle$ radiated, both in the long wavelength approximation. Note that the Poynting vector is $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$.

Taking $\vec{\nabla} \times \vec{A}$, I see that only terms related to $\frac{\partial}{\partial r}$ will survive. Since the vector portion $\vec{p}_0$ is not a function of r, I may pull out the part that is related to r and allow the cross product to act between the r unit vector, which certainly may not survive this cross product since again r is the only variable appearing in the vector potential:

$$\vec{B}(\vec{r}) = -\frac{i\omega}{4\pi\varepsilon_0 c^2}\frac{\partial}{\partial r}\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)(\hat{e}_r \times \vec{p}_0) = \frac{i\omega}{4\pi\varepsilon_0 c^2}\frac{\partial}{\partial r}\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\vec{p}_0 \times \hat{e}_r$$

The radiation field is the portion of this that falls off as r^{-1}:

$$\vec{B}_{rad}(\vec{r}) = \frac{i\omega}{4\pi\varepsilon_0 c^2}\left(\frac{i\omega}{c}\right)\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\vec{p}_0 \times \hat{e}_r = -\frac{\omega^2(\vec{p}_0 \times \hat{e}_r)}{4\pi\varepsilon_0 c^3}\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)$$

Now the Poynting vector has units watts per square meter. By integrating over the surface $\frac{1}{cycle}\int_1 \int_{cycle} \vec{S} \cdot d\vec{S}$, then, I should be able to get power radiated.

Ceratinly, from Maxwell's equations, $\frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \vec{\nabla} \times \vec{B}$ in the far zone where there is no stray charge or current. Based on this I have:

$$\frac{-i\omega}{c}\vec{E} = \vec{\nabla} \times \vec{B}$$

$$\vec{E}_{rad} = -\left(\frac{1}{4\pi\varepsilon_0 c}\right)\left(\frac{i\omega}{c}\right)^2\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\hat{e}_r \times (\vec{p}_0 \times \hat{e}_r)$$

Now:

$$\vec{S} = \frac{1}{\mu_0}\left[-\left(\frac{1}{4\pi\varepsilon_0 c}\right)\left(\frac{i\omega}{c}\right)^2\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\cdot\left(-\frac{\omega^2}{4\pi\varepsilon_0 c^3}\left(\frac{e^{\frac{i\omega r}{c}}}{r}\right)\right)\right][(\hat{e}_r \times (\vec{p}_0 \times \hat{e}_r)) \times (\vec{p}_0 \times \hat{e}_r)]$$

$$\vec{S}(t) = \frac{1}{4\pi\varepsilon_0}\left[-\frac{\omega^4}{4\pi c^4}\right]\frac{e^{2\frac{i\omega r}{c}}}{r^2}e^{-2i\omega t}[(\hat{e}_r \times (\vec{p}_0 \times \hat{e}_r)) \times (\vec{p}_0 \times \hat{e}_r)]$$

Simplifying this with the triple cross product formula $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, I have

$$\vec{S}(t) = \frac{1}{4\pi\varepsilon_0}\left[-\frac{\omega^4}{4\pi c^4}\right]\frac{e^{2\frac{i\omega r}{c}}}{r^2}e^{-2i\omega t}\left[\left(\vec{p}_0 - \left(\vec{p}_0\cdot\hat{e}_r\right)\hat{e}_r\right)\times\left(\vec{p}_0\times\hat{e}_r\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\frac{\omega^4}{4\pi c^4}\right]\frac{e^{2\frac{i\omega r}{c}}}{r^2}e^{-2i\omega t}\left[\left(\vec{p}_0\cdot\hat{e}_r\right)\vec{p}_0 - \left(\vec{p}_0\cdot\hat{e}_r\right)\vec{p}_0 + \left(\vec{p}_0\cdot\hat{e}_r\right)^2\hat{e}_r\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\frac{\omega^4}{4\pi c^4}\right]\frac{e^{2i\omega r}}{r^2}e^{-2i\omega t}\left(\vec{p}_0\cdot\hat{e}_r\right)^2\hat{e}_r$$

Letting the capacitor vary sinusoidally, the integral over one cycle time-wise becomes:

$$\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t)dt = \int_0^{\frac{2\pi}{\omega}}\frac{1-\cos(2\omega t)}{2}dt = \frac{1}{2}\left(\frac{2\pi}{\omega}\right) - \frac{1}{2}\sin(4\pi) = \frac{\pi}{\omega}$$

Performing the spatial integration, now I have:

$$\langle P\rangle = \int\frac{1}{4\pi\varepsilon_0}\left[-\frac{\omega^4}{4\pi c^4}\right]\frac{e^{2\frac{i\omega r}{c}}}{r^2}\left(-\frac{\pi}{\omega}\right)\left(\vec{p}_0\cdot\hat{e}_r\right)^2\hat{e}_r r^2\sin\theta\cdot d\vec{S}$$

Certainly, then, I may cancel and make some substitutions, fixing the sphere's surface at a particular radius where $\frac{2\omega r}{c} = 2\pi k$ (proposing I observe this as the sphere by which flux passes.

$$\langle P\rangle = \int\frac{1}{4\pi\varepsilon_0}\left[\frac{\omega^3}{4c^4}\right]\left|\vec{p}_0\right|^2\cos^2\theta\sin\theta\cdot d\vec{S}$$

$$\int d\phi = 2\pi$$

$$\int_0^{\pi}\cos^2\theta\sin\theta d\theta = -\frac{\cos^3\pi}{3} + \frac{\cos^3 0}{3} = \frac{2}{3}$$

for

$$\langle P\rangle = \frac{\omega^3}{12c^4}\left|\vec{p}_0\right|^2$$

(Internet sources disagree with this—what's gone wrong?)