Supplemental notes on Electric Potential
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You have of course seen the formula $\Delta V = -\int_i^f \vec{E} \cdot d\vec{l}$, and superposition independently of that: e.g., if I have several charge distributions and I want the electric potential at a point because of the distributions, I may simply add the potentials from each distribution alone, ignoring the others. For an example of this, see problem 16.26, which I have done for you in the March 23 recitation notes: I found the potential from a sphere and a capacitor independently of one another, and simply added them up.

However, the more general and perhaps preferred way of doing this in advanced physics is to divide all of space into countless little point charges and then treat the situation as a superposition of many little point particles. Of course, you know that the potential from a point charge is $\frac{q}{4\pi\varepsilon_0 r}$. Thus, to write this in terms that are probably too advanced for you, I have $V(\vec{r}) = \int d^3x \frac{q(\vec{x})}{4\pi\varepsilon_0 |\vec{x} - \vec{r}|}$. *Do not remember this formula—it is too advanced, and you won't need it. What you might well need is the procedure I use in the sample problems later in these notes. Just remember what the integral means!* Let me go through the parts of it, and what they mean:

$V(\vec{r})$: The electric potential at a point $\vec{r}$: in other words, this is a map from the points in space to a single number representing its potential.

$\int d^3x$: This indicates that I am integrating over three dimensions. I am asking a question about point $\vec{r}$, but in order to find the potential here I need to examine all points in space $\vec{x}$.

$\frac{q(\vec{x})}{4\pi\varepsilon_0 |\vec{x} - \vec{r}|}$: This is jus the potential from a point charge $\frac{q}{4\pi\varepsilon_0 r}$ that you know, but rewritten. As I scan along all points $\vec{x}$, $q(\vec{x})$ tells me how much charge lies at this point (in the examples you do, this is usually zero except for a very specific region). The value $|\vec{x} - \vec{r}|$, as you know, is essentially the Pythagorean theorem: it gives me the distance from the point I'm "scanning", $\vec{x}$, to the point that I want to know the potential of, $\vec{r}$. This is a common problem that we see: students want to find a specific value for potential, when we are in fact asking for it as a function of a parameter. In this case, that parameter is $\vec{r}$.

An example where superposition is useful:

Charged Semicircle

Suppose that I have a semicircle with net charge Q, and I want to know the potential at the center. The formula $\Delta V = -\int_i^f \vec{E}\cdot d\vec{l}$ is unfit for this problem because I have no formula for electric field along the line from the center of the semicircle to infinity! I will essentially split the line into an array of tiny point charges to solve the problem.

Instead, let me superpose many little point charges dQ arranged along the semicircle. Switching to polar coordinates, I have:

$$V(0) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{dq}{4\pi\varepsilon_0 r}.$$

Dividing the uniformly charged line into tiny point charges (being careful to note that I am in polar coordinates), $dq = \frac{Q}{\pi B} r d\theta$ with separations meant to make distinct the polar differential and the line charge density.

From the symmetry of this problem, it is easy to see that $r = B$. Now I have

$$V(0) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{4\pi^2\varepsilon_0 B} = \frac{Q}{4\pi\varepsilon_0 B}$$

In summary, use superposition when:

- You have a number of simple distributions (see 16.26, worked on March 23—notes available online). It is often the case that there are a discrete number of things to add, rather than an integral to do. In that example I simply added the potentials from a sphere and a capacitor.
- You have a non-standard distribution of charge that can easily be converted into a set of smaller, standard distributions for which the potential is known. (Standard distributions are often given to you: examples are charged discs, point charges, rings of charge, and lines of charge. Chances are that this method was used to calculate the standard distribution.)
- Your potential is zero at infinity.
- The dependence of potential on position is relatively simple (for your purposes, you won't want to use this, for example, if position r is arbitrary: for example, if you are being asked for the potential along a line). In the case above, the position I wanted was the origin, so the dependence was very simple.

An example where your formula $\Delta V = -\int_i^f \vec{E} \cdot d\vec{l}$ is best:

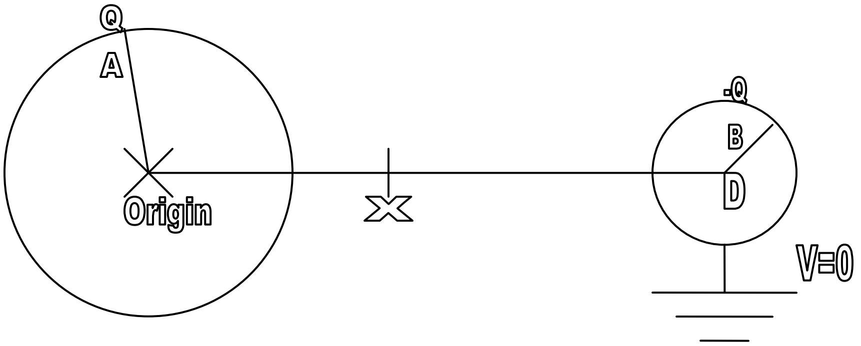
Two Charged Spheres

Suppose I want the potential outside a charged spherical shell of no thickness.

Imagine for a moment the prospect of integrating over the surface of the sphere the charge density along with the distance to a point. This procedure would be truly excruciating. However, we do know the form of the electric field due to the charged spherical shell: Namely, if the shell has radius B and charge Q and is located at the origin:

$$\vec{E}(r) = \begin{cases} 0 & r < B \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r > B \end{cases}$$

Also, in this case I will make an unusual definition: I want ground (notice the funny grounded symbol) one of my spheres so that the zero of potential is to be located at its surface. Then, to draw a picture I have



Now all that's left is for me to use my integral $\Delta V = -\int_i^f \vec{E} \cdot d\vec{l}$.

Here I will actually need to use superposition when I add the potentials from each sphere, but not in the integral form that may be new to you. Writing my two electric fields, I have:

$$\Delta V = -\int_B^x \left(\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} + \frac{-Q}{4\pi\varepsilon_0 (D-r)^2}\hat{r} \right) \cdot \hat{r} dr$$

$$= -\int_B^x \left(\frac{Q}{4\pi\varepsilon_0 r^2} + \frac{-Q}{4\pi\varepsilon_0 (D-r)^2} \right) dr$$

$$= \frac{Q}{4\pi\varepsilon_0 x} - \frac{Q}{4\pi\varepsilon_0 B} + \frac{-Q}{4\pi\varepsilon_0 (D-x)} - \frac{-Q}{4\pi\varepsilon_0 (D-B)}$$

To be fair, I could have seen this result by simply superposing the virtual point charges that we learned represent charged spheres and subtracting off the potential at the ground point and adding the electric fields of these two virtual points is superposition in the truest sense of the word, but I certainly integrating the charge density along the surface of the spheres would have been totally ludicrous.

In summary, use this formula when:

- You have a reference potential that is not zero at infinity. For example, there was a quiz problem in lecture where you were asked to allow the potential on one plate of a capacitor to be zero.
- You have a standard distribution of charge or a number of standard distributions for which electric field is known but potential is not given.
- You are being asked for the potential along a line or in a space as a function of position (in advanced classes, the superposition form or Gauss's law is usually

used to do this, but for your purposes you will typically want to use $\Delta V = -\int_{i}^{f} \vec{E} \cdot d\vec{l}$ to do these).