Recitation Notes February 9
Ben Sauerwine

Quiz Solution:

Part 1: Given two large charged plates with surface charge densities $\pm\sigma$ with a relatively small separation d, find the potential difference between the two plates in volts.

Using page 590, I see that for a large plate and small separation, I have $E = \frac{\sigma}{2\varepsilon_0}$. Now the electric field between the two is $2E = \frac{\sigma}{\varepsilon_0}$, and $\Delta V = -\int \vec{E} \cdot d\vec{l}$, for $\Delta V = -d\frac{\sigma}{\varepsilon_0}$ (the minus sign may be different, depending on your choice of starting point).

Part 2: Suppose that the distance d between the plates is increased. Now examine the formula $E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\left(R^2 + z^2\right)^{\frac{1}{2}}}\right]$. If I move the plates further apart, will my potential difference increase, decrease, or stay the same? You do not need to write or evaluate the integral. Just indicate your reasoning.

Notice that if I moved the plates further apart, the same electric field exists from the plates as existed at short range. By superposition, then, adding more field—even if it's small—can only increase the potential difference.

RQ 16.1) Electric potential energy has units Joules, electric potential has units Joules per Coulomb.

RQ 16.2) Electric field has units Newtons per Coulomb.

RQ 16.4) Electric potential energy from a distribution obeys the superposition principle just as electric field does: $\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}$ from a point charge. Just add these results up. I will do a more advanced example later.

RQ 16.8)
(a) While the electric field inside a metal in static equilibrium is always zero, the electric potential needs not be. In other words, with respect to some point at infinity, you may have had to travel against some electric field to get inside the metal.

- **(b)** Just because electric potential is large, electric field does not have to be. Take the example from part a: The electric field inside is zero, but if you put lots of charge on the conductor that evenly distributed itself on the surface you'd still have to do a lot of work to get inside!

(c) If you get very close to a lonely negative point charge, the potential does indeed diverge to negative infinity. However, if you had something to overpower it that diverged to positive infinity more quickly, such as the potential from a positive line of charge, you could end up in the very large positive region!

(d) $\Delta V = -EL$ is a reasonable approximation anywhere where the field is varying slowly. The question is, what is a valid electric field E, since the direction it points is critical. Suppose you took E to be the component of the electric field in the direction that I'm planning on traveling. In that case, this would indeed allow for a good small approximation in a slowly varying field area: unfortunately, near a point charge does not qualify!

(e) $\Delta V = -(E_j - E_i)L$ is not a very good approximation: suppose field didn't change at all. This would predict a difference of zero! A better approximation would be $\Delta V = -\frac{E_f + E_i}{2} L$, where these E fields point in the direction that I'm traveling. Here I have taken an "average" field.

Quiz from last week, Potential Energy Style:

Given a parabola $y = x^2$ with line charge density σ and a point $(0, d)$, find the potential energy and show that last week's quiz result was correct for electric field.

I use the superposition principle: $V = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dq}{r}$. Again, $r = \sqrt{(-x)^2 + (d - x^2)^2}$ and $dq = \sigma dl = \sigma\sqrt{1-(2x)^2}\,dx$, so $V = \frac{\sigma}{4\pi\varepsilon_0}\int_{-L}^{L}\sqrt{\frac{1-4x^2}{x^2+(d-x)^2}}dx$. I can't actually integrate this! However, I know that $\vec{E} = -\vec{\nabla}V = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle V$, and since I know by symmetry that the electric field points only in the y direction and since d plays the role of y, that I have

$$\vec{E} = -\frac{\sigma}{4\pi\varepsilon_0}\hat{y}\frac{\partial}{\partial d}\int_{-L}^{L}\sqrt{\frac{1-4x^2}{x^2+(d-x)^2}}dx$$

$$= -\frac{\sigma}{4\pi\varepsilon_0}\hat{y}\int_{-L}^{L}\frac{-1}{2}\frac{2(d-x)\sqrt{1-4x^2}\,dx}{\left(x^2+(d-x)^2\right)^{\frac{3}{2}}} = \frac{\sigma}{4\pi\varepsilon_0}\hat{y}\int_{-L}^{L}\frac{(d-x)\sqrt{1-4x^2}\,dx}{\left(x^2+(d-x)^2\right)^{\frac{3}{2}}}$$

Just like last week!

The thing to notice is that I was allowed to "commute" the derivative through the integral sign since the integral doesn't depend at all on d, only x.

Integrals of Electric Field

Suppose I have a charge bullet of length L, which is essentially the volume produced by rotating a hyperbola $y = x^2$ about the y axis in three dimensions. I'd like to know the potential energy a distance down the y axis d.

My bullet has total charge Q, uniformly distributed.

First, I need to know the volume of the bullet. I integrate little disks along the length:

$$\int_0^L \pi(x)^2\,dy = \int_0^L \pi\left(\sqrt{y}\right)^2 dy = \frac{\pi}{2}L^2$$

Now I know my charge density is $\frac{Q}{\frac{\pi}{2}L^2}$ in units $\frac{C}{m^3}$. Let me consult trusty page 590 in the book for an equation for the electric field for a charged disk:

$E = \frac{\frac{Q}{A}}{2\varepsilon_0}\left[1 - \frac{y}{\sqrt{R^2 + y^2}}\right]$. Now, to evaluate the integral of the electric field for all of my slices, I need several pieces:

$\frac{Q}{A}, y, R$.

Certainly, the distance down the y axis to my point from a given disk is $d + y$.

Also, it's clear that the radius of my disk is $R = x \quad y = x^2 \quad R = \sqrt{y}$.

Now I need the charge per unit area of my disk. This is a bit more tricky, and here's the caveat: I have the net charge and also the volume of the object, but $\frac{Q}{A}$ is in units $\frac{C}{m^2}$, not $\frac{C}{m^3}$. I can certainly not just multiply this by something of units meters (what would I choose?) and expect an answer. There are several ways to approach this dilemma. One way is to write the charge contained in a thin slice of thickness dy:

$$dQ = \frac{Q}{V}A\,dy = \frac{Q}{\frac{\pi}{2}L^2}\pi y\,dy$$

$$A = \pi y \quad V = \frac{\pi}{2}L^2 \quad \frac{dQ}{A} = \frac{\frac{Q}{V}\pi y\,dy}{\pi y} = \frac{Q}{\frac{\pi}{2}L^2}dy$$

Be careful, though: suppose you thought you could build a disk out of a circle charge lines arranged like spokes on a wheel, so you multiply the charge line field by the number of radians in a circle. In this way, you "favor" the center of the circle more than the outside! This is why I advocate being very careful about your integrals. Now I can rewrite my overall integral:

$$E = \int_0^L \frac{\frac{dQ}{A}}{2\varepsilon_0}\left[1 - \frac{y}{\sqrt{R^2 + y^2}}\right] = \frac{Q}{\pi\varepsilon_0 L^2}\int_0^L \left[1 - \frac{(y+d)}{\sqrt{y + (y+d)^2}}\right] y dy$$