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Recitation Notes and Quiz Answers
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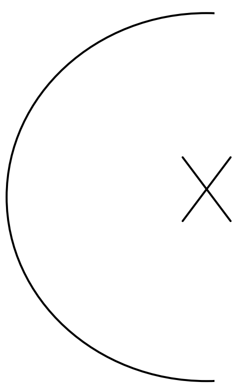
Recitation Examples:

These examples essentially exhibit the superposition principle for electric fields. A lot of relatively simple problems have integrals are either incredibly difficult to solve. I've done a few examples here. The essentials are as such:

You have some "basis distribution". The most basic case is that of a point charge alone, and it is from this that we build up the distribution of spheres, planes, and rods. We can mix these fields however we like as we add them up by integrating the infinitesimals. Here I'll do a few examples. Note that this is not a very effective way of finding electric fields! A better way is to find the electric potential first, then take the gradient of that. That will be covered in the next chapter.

Example 1: A Line of Charge

Here I have a semicircular arc of charge around a central point (from the book, this is problem 15.20). It has radius R and carries a charge –Q evenly distributed along it.



Consider any individual point in the arc. Using Coulomb's Law, I have:

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad q = \frac{-Q}{\pi R} \quad r = R$$

For a technical view of how to convert arc length in a more general case, please look at the quiz solution below.

Note that $dl = rd\theta$ in order to keep the lengths appropriate.
Now the last thing to consider is the superposition of these tiny vectors. The fields in the y direction must cancel out by symmetry. At a point at angle theta about the arc, however, the unit vector in the x direction is $-\cos(\theta)\hat{x}$. Now integrating, I have

$$\frac{1}{4\pi\varepsilon_0}\frac{-Q}{\pi R}\hat{x}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{-\cos\theta}{R^2}Rd\theta=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{x}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\theta d\theta=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{x}\sin\theta\Big|_{\theta=\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{x}\sin\theta\Big|_{\theta=\frac{3\pi}{2}}^{\frac{\pi}{2}}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{x}(-1-1)=\frac{-1}{4\pi\varepsilon_0}\frac{2Q}{\pi R^2}\hat{x}$$

Example 2: Solid Sphere of Charge

Now consider the situation of a solid sphere of charge. I'd like to integrate this without actually using Gauss's Law. We will use the method of adding up the spherical shells to find what the field must be inside and outside the sphere. Let the sphere have radius R and total charge Q. In that case, allowing each to contribute their net charge centered at the middle of the sphere, I have

Outside the sphere, I may simply add up the field from each of the shells. Since the total amount of charge on the shells is Q, I may simply use the field $\vec{E}=\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$. Now consider the field inside the sphere, for an even charge distribution.

I integrate again over the radius r, until I get to the surface. However, now I must consider the proportion of the net charge I have. I will get:
$q=Q\frac{r^3}{R^3}\hat{r}$. Now using the formula for the spheres, I have $\vec{E}=\frac{Q}{4\pi\varepsilon_0 R^3}r\hat{r}$.

The critical thing to note is that I am selecting a set of field sources to integrate over. Here are today's quiz answers:

Quiz Solution:

The problem was as follows: I want to find the electric field at a point above the lowest point of an even-indexed polynomial curve, with charge density q C/m where the curve is given by $y=x^{2n}$ and the point is at a distance d above the center at (0, d). The curve extends a distance of L left and right along the x-axis. There are three things to consider here:

First, I need to know the distance to this point in order to determine the strength of the field. This will be given by the Pythagorean Theorem: $\sqrt{\left(d-x^{2n}\right)^2+x^2}$. Here I have simply taken two vectors, one for initial position and one for destination, and subtracted and taken the norm just as before: $\left\| \langle 0, d \rangle - \langle x, x^{2n} \rangle \right\|$

Second, I need to know what component the electric field has here. By symmetry, the y component will cancel. Using normalization, the y component unit vector will be $\frac{\left(d - x^{2n}\right)}{\sqrt{\left(x^{2n} - d\right)^2 + x^2}}\hat{y}$ where I have simply divided the y component by the magnitude that I got from the Pythagorean Theorem earlier. Also acceptable and possibly more familiar to you would be $\frac{\langle -x, d - x^2 \rangle}{\left\| \langle 0, d \rangle - \langle x, x^{2n} \rangle \right\|} = \frac{\langle -x, d - x^2 \rangle}{\sqrt{\left(d - x^{2n}\right)^2 + x^2}}$, where I take only the y component due to symmetry.

The final step is to consider my charge unit dl. Using the Pythagorean Theorem, I have

$$dl = \sqrt{dx^2 + dy^2}$$

$$y = x^{2n} \quad \frac{dy}{dx} = 2nx^{2n-1}$$

$$dl = \sqrt{dx^2 + \left[2nx^{2n-1}\right]^2 dx^2} = \sqrt{1 + \left[2nx^{2n-1}\right]^2}\,dx$$

Now I was given the charge density q, so I have combining the pieces into $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ and adding the integral symbol which acts as summation,

$$\frac{1}{4\pi\varepsilon_0}\hat{y}\int_{-L}^{L}\frac{\left(d - x^{2n}\right)\sqrt{1 + \left[2nx^{2n-1}\right]^2}}{\left[\left(x^{2n} - d\right)^2 + x^2\right]^{\frac{3}{2}}}\,dx.$$

For section b, let n = 1. For section c, let n = 2 to get the solution. This is a very difficult integral, and so I didn't actually want you to evaluate it. Just write it down. In fact, even Mathematica doesn't like this one.

One student actually wrote this in terms of y. This is a perfectly valid approach, and can be done by substituting y for x wherever possible using $y = \sqrt{x}$, with the caveat that you must do these two things also:

First, you must convert the limits on the integral. Since $y = \sqrt{x}$, you might typically think that I must take this from $-\sqrt{L}$ to $\sqrt{L}$, but recall that because the function turns around at x = 0, I must integrate two branches, the branch where x < 0 and the branch where x > 0. Because of the symmetry of the problem, this amounts to $2\int_0^{\sqrt{L}} ...dy$.

Example 3 (I didn't get to this in Section B) Charged Cone:

The important thing about this example is that you don't actually need to use Coulomb's Law for point charges: this is calculus, not Physics! Let's use calculus to build up the solution to the electric field from a volumetrically charged cone using the solution for a disc (see page 590 in your book for basic shapes):

First, I take the formula for electric field from a charged disk from page 590:

$$E = \frac{Q}{\pi R^2}\frac{1}{2\varepsilon_0}\left[1 - \frac{z}{\left(R^2 + z^2\right)^{\frac{3}{2}}}\right]$$

this indicates a magnitude, not a direction, but in this case it is directed away from the disk. I'm going to set up an integral to integrate over a cone. The first problem is that I need a consistent Q. The issue at hand is that I need to make the units on charge density versus the integrals work out. For a line, I need C/m. For a sheet, as in this case, I need coulombs per square meter. Let me work this out now:

To get total charge, I integrate over the charge density times pi times the radius squared, where the radius is alpha times this position as I integrate up the height H.

$$Q = \sigma\int_0^H \pi(\alpha r)^2\, dr$$

$$Q = \sigma\frac{H^3\pi\alpha^2}{3} \quad \frac{3Q}{H^3\pi\alpha^2} = \sigma$$

So now I have a consistent charge density on each disk. Now I notice that in this case, if the desired point is a distance d away from the tip of the cone I have

$$z = r + d$$

$$R = \alpha r$$

making these substitutions and noting that my little sigma plays the role of the $\frac{Q}{\pi R^2}$ in the original equation, I have:

$$E = \frac{3Q}{H^3\pi\alpha^2}\frac{1}{2\varepsilon_0}\int_0^H\left[1 - \frac{r+d}{\left((\alpha r)^2 + (r+d)^2\right)^{\frac{3}{2}}}\right]dr$$

Which is not easy to integrate, but again gives the form of a solution.