Ben Sauerwine Exam 3 Review--Physics 2 Spring 2007

Study Plan, in order of importance:

- **Do and understand all problems on exam 3 from 2003.**
- **Make sure you understand any mistakes you made on quizzes 2 and 3.**
- **Recommended recitation and homework problems. These are problems that are likely to appear on exam 3, and their solutions are available on blackboard.**
    - **19.X.26 (Ex. 19.33) (Conceptual problem about batteries and voltmeters)**
    - **20.P.49 (Prob 20.7) (Lorentz Force)**
    - **20.P.53 (Prob 20.11) (Hall Effect). 20.P.60 is another good example, which was done in recitation and is also done below.**
    - **Problem 1 from Homework 12M (Motional EMF and Torque)**
    - **Problems 5 and 6 from exam 3 are great reviews of Gauss's and Ampere's Law.**

Outline of topics covered in this guide:

- **Hall Effect**
    - **General procedure**
- **Capacitance of an arbitrary device**
- **Motional EMF**
    - **Lorentz force**
    - **Alternate derivation**
    - **General procedure**
- **Kirchoff's Laws**
    - **General procedure**
    - **With capacitor**
    - **Differential equation**
- **Gaussian Cylinder**
    - **Technical details not covered in recitation**

Other important topics, not covered in this guide:

- **Ampere's Law**
- **Magnetic moments and torque**
- **Magnetic field from a current**
- **Voltmeters and ammeters**
- **Surface charge gradients on conductors**
- **Electric field in dielectrics**

Never forget:

- $\vec{E} = 4$**, for example, is always wrong. Vectors never equal scalars!**
- $\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$ $a^2 + b^2 \neq (a+b)^2$ $|\vec{A} + \vec{B}| \neq |\vec{A}| + |\vec{B}|$**.**

The Hall Effect

See Chabay and Sherwood Problem 20.P.60 for another example. This is available online at http://www.andrew.cmu.edu/user/bsauerwi/Phys2/halleffectb5.pdf

Procedure:

1. Determine the direction of conventional current in the bar, using a voltmeter connected across the device. If the voltage reads positive, then the electric field in the device points from the point where the positive probe is connected to the point where the negative probe is connected.
2. Infer that based on this, either a positive charge carrier is flowing in the direction of conventional current, or a negative charge carrier is flowing against the direction of conventional current.
3. Using a sensitive voltmeter connected across the bar in the direction perpendicular to current flow and magnetic field in the region, measure this voltage.
4. Consider the top and bottom of the bar as plates of a parallel capacitor in the steady state. Infer that an electric field must then point, in the steady state, from the end of the sensitive voltmeter that reads high to the end of the meter that reads low.
5. Infer that since this is in steady state, the magnetic force must exactly cancel the electric force on a charge carrier. Specifically, $E_{Hall} = \frac{\Delta V_{\perp}}{height} = v|B|$
6. Now one can determine the mobility of the charges (which is a linear velocity response to an electric field, e.g. $\frac{m/s}{N/C}$, so that $u = \frac{v}{\Delta V_{in \quad direction \quad of \quad current}}$.
7. Knowing the steady-state current flowing through the bar, now one can determine the charge carrier density: $i = nAv$, e.g., $\frac{I}{q} = nAv$ or $\frac{I}{qAv} = n$, where A is the cross-sectional area of the bar (perpendicular to the direction of steady-state current flow). Note that you cannot determine the charge carrier charge using this experiment! You have to assume it.
8. Note that you can determine resistance of a device using an analog of the microscopic current formula:

$$i = nAuE$$

$$\frac{I}{q} = \frac{nAu}{L}[EL]$$

$$I = \frac{qnAu}{L}[\Delta V]$$

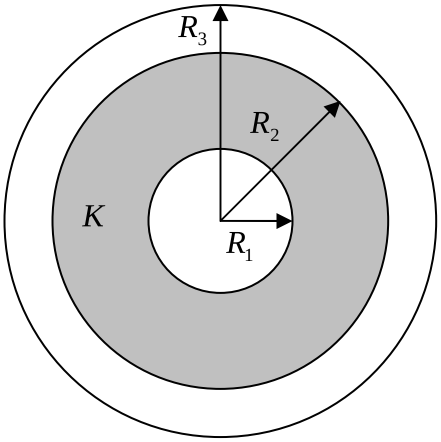
$$\left[\frac{L}{qnAu}\right]I = [\Delta V]$$

comparing this to $\Delta V = IR$, I may conclude that $R = \frac{L}{qnAu}$.

Capacitance of an arbitrary device

You might remember this example from recitation, but it's a very simple thing to determine the capacitance of an arbitrary device!

Consider two concentric conducting spheres with a dielectric slab of constant K between them as shown:



Find the capacitance between the inner and outer spheres.

Note that capacitance $C = \frac{Q}{\Delta V}$. All I have to do, then, is find the magnitude of ΔV between the shells if I charge them with equal and opposite charges $\pm Q$.

Choosing to put the charge $+Q$ on the inner shell, then, I recall that this manifests itself in the region $R_1 < r < R_2$ as a point charge at the center, or

$$\vec{E}(R_1 < r < R_2) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

Now all I have to do is integrate:

$$\Delta V_{R_2 \to R_1} = -\int_{R_2}^{R_1} \vec{E} \cdot d\vec{l}$$

$$d\vec{l} = \hat{r}dr$$

$$\Delta V_{R_2 \to R_1} = -\int_{R_2}^{R_1} \frac{Q}{4\pi\varepsilon_0 r^2} dr = -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{R_1} + \frac{1}{R_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

Since $\frac{1}{R_1} > \frac{1}{R_2}$, this verifies my expectation that potential should go up as I get closer to the positive charge.

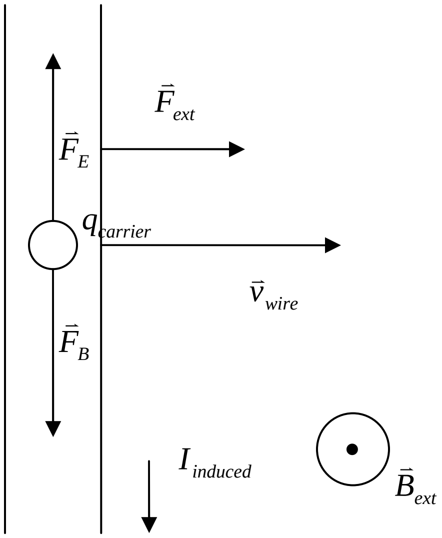
Finally, I write $C = \frac{Q}{\Delta V}$, so that $C = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]} = 4\pi\varepsilon_0\left[\frac{1}{R_1} - \frac{1}{R_2}\right]^{-1}$ for this device.

Motional EMF

Please note that this explanation is not a substitute for complete understanding of section 20.5 in Chabay and Sherwood, and is not required knowledge for anything in the class. This is merely an alternate formulation of motional EMF that I feel clarifies the concept.

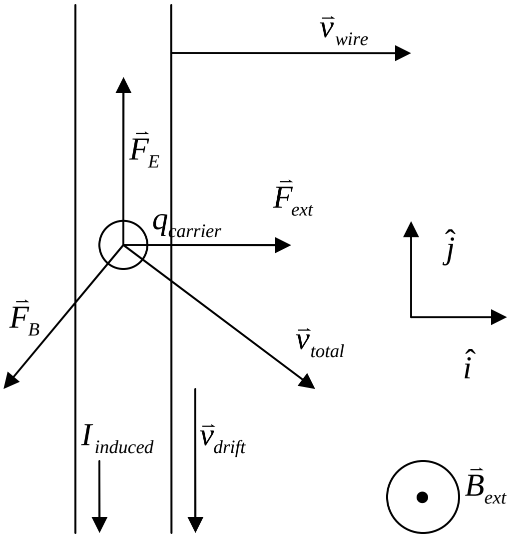
Motional EMF is a difficult concept, in part because it can be a bit misleading. For one thing, the EMF does not come from the motion, per se. The EMF comes from charges that have built up in the steady state as a response to the charge carriers being pushed through the electric field—that is, the EMF is being created from a *force*, not a *motion*. So which force is creating it?

In recitation and lecture, we showed you this picture. In a test situation, this is the force balance you should recall when you equate $-q\vec{E} = q\vec{v} \times \vec{B}$, the right-hand side being the Lorentz force on a charge. (note that the charge of the charge carrier drops out of this equation) for the carriers:



So what's wrong with this picture? If there's no current flowing through this wire and the external force is zero, nothing is wrong. This balance of forces makes perfect sense. However, if there is an induced current in this wire, there is one very serious problem that appears: the magnetic field is doing work! It is clearly pushing this charge in the direction of conventional current, as the carrier is moving in the direction of the current, which is opposite the electric field!

Let's draw another more honest picture, then.



Now I notice a few things: In this picture, the magnetic force is definitely orthogonal to the velocity of the charge carrier. Further, I see clearly that the external force is doing the work, and also that the electric field is doing some work. The magnetic field is doing none. Let me derive the justification that we use, with $E = vB$ which we use from this.

In a typical problem, $\vec{v}_{wire}$ and $\vec{F}_{ext}$ are given, as is $\vec{B}_{ext}$.

In the steady state, I know that the velocity of the charge carrier is constant, so that the net force on the charge carrier is zero. This means that $\vec{F}_{ext} + \vec{F}_E + \vec{F}_B = 0$.

Since $\vec{F}_{ext} \perp \vec{F}_E$, this means that then in the x-dimension, $\vec{F}_{ext} \cdot \hat{i} = -\vec{F}_B \cdot \hat{i}$.

Further, I know that the Lorentz force $\vec{F}_B = q_{carrier}\vec{v}_{total} \times \vec{B}_{ext}$. From linearity of products, however, I may break up my cross product into components:

$$\begin{aligned}
&\vec{F}_B = q_{carrier}\vec{v}_{total} \times \vec{B}_{ext} \\
&= q_{carrier}\left[v_{wire}\hat{i} - v_{drift}\hat{j}\right] \times B_{ext}\hat{k} \\
&= q_{carrier}B_{ext}\left[-v_{wire}\hat{j} - v_{drift}\hat{i}\right]
\end{aligned}$$

However, from force balance, I know that $\vec{F}_B \cdot \hat{j} = -\vec{F}_E \cdot \hat{j}$. From the line above, then, this means that

$\vec{F}_B \cdot \hat{j} = -q_{carrier}B_{ext}v_{wire} = -F_E = -q_{carrier}E$.

Or, $q_{carrier}B_{ext}v_{wire} = q_{carrier}E$, e.g. $vB = E$ in magnitude.

There is another important note to make here: in this situation, then, the drift velocity is **not** driven by our microscopic current rule $i = nAuE = nAv$! Our equation above indicates that in the motional EMF regime that

$$\vec{F}_B \cdot \hat{i} = -q_{carrier} B_{ext} v_{drift} = -F_{ext}$$

$$v_{drift} = \frac{F_{ext}}{q_{carrier} B_{ext}}$$

contrary to the rules of the microscopic current regime! This explains how the current in the motional EMF situation flows opposite the electric field!

Calculating the power exerted by each force, then, I get:

$$P_{external} = \vec{F}_{ext} \cdot \vec{v}_{total} = F_{ext} v_{wire}$$

$$P_E = \vec{F}_E \cdot \vec{v}_{total} = -F_E \cdot v_{drift} = -q_{carrier} v_{wire} B_{ext} \frac{F_{ext}}{q_{carrier} B_{ext}} = -F_{ext} v_{wire}$$

$$P_B = \vec{F}_B \cdot \vec{v}_{total} = 0$$

so that the external power equals the power dissipated in the circuit by the electric field, and the work done by the magnetic field equals zero, just as expected!

So in general, here is how to approach a motional EMF problem:

1. Using the right-hand rule, determine the direction of the Lorentz force on a positive charge carrier in sections moving perpendicular to the magnetic field which have their length in the direction of this force. In this situation, then, the conventional current flows through the wire in the direction indicated by the right-hand rule, and the electric field points in the opposite direction.
2. Using $vB = E$ and $\Delta V = EL$, determine the battery-like voltage across each section of wire.
3. Using the loop rule and the resistance of the circuit, determine the current in the device, representing the moving parts affected by motional EMF as batteries.
4. Now, use the Lorentz force in the form $\vec{F} = I\vec{L} \times \vec{B}$ for a straight wire or $\vec{\tau} = \vec{\mu} \times \vec{B}$ along with force balance $\sum forces = \vec{0}$ for constant velocity to solve for any remaining unknown.

Kirchoff's Laws:

For a representative example, go over problem 1 from quiz 3.

General Procedure:

1. Make any simplifications to the circuit that you can as allowable in the problem. If you're being asked for the current through a particular region, you can probably collapse other regions using the resistor series or parallel rule in order to reduce the circuit to a minimum number of distinct currents. Remember, on an exam, you probably won't be expected to solve a system of more than three equations simultaneously.
2. In each distinct branch of the circuit, label a current and direction. The direction is very important, as it tells you what sign to take on your resistor ΔV. If at the end you get a negative value for current, you know that the current in that region flows in the opposite direction as the direction you drew.
3. Use the loop and node rules to construct the same number of linearly independent equations as you have currents drawn. The easiest way to do this is, rather than using some rigid formula, to just choose loops and nodes that appear to get a good sampling of all regions (each current should appear at least once). If you don't have linearly independent equations at the end, you will find that you can't solve for one current and you end with results like 1=1. If you have capacitors, you must consider the charge on them and treat them like batteries with $\Delta V = \frac{Q}{C}$. At time ∞, you must solve for ΔV so that the current flowing through the capacitor is zero.
4. Solve these equations and interpret the resulting currents in the diagram you originally drew.

You may be asked to find the charge on a capacitor at an arbitrary time. In order to do this, you must consider this differential equation involving the current flowing through the capacitor:

$$I = \frac{dQ(t)}{dt}$$

However,

$$I = \frac{\Delta V}{R}$$

But,

$$\Delta V = \Delta V_{applied} - \Delta V_{capacitor} = \Delta V_{batteries} - \frac{Q(t)}{C}$$

Substituting this into the initial equation,

$$I = \frac{\Delta V_{applied}}{R} - \frac{Q(t)}{RC} = \frac{dQ(t)}{dt}$$

Next, I need to know the initial state of the capacitor. Typically, we have an uncharged capacitor at time zero so that $Q(0) = 0$.

Now, my system of equations is:

$$\frac{\Delta V_{applied}}{R} - \frac{Q(t)}{RC} = \frac{dQ(t)}{dt}$$

$$Q(0) = 0$$

If you aren't familiar with how to solve a differential equation, the next part won't mean much to you. In this case, you'll have to memorize the result.

First I need to find a general solution of the homogeneous equation (without the constant)

$$-\frac{Q_{Homogeneous}(t)}{RC} = \frac{dQ_{Homogeneous}(t)}{dt}.$$

This is a very common equation, and I see that it is satisfied by

$$Q_{Homogeneous}(t) = Xe^{-\frac{t}{RC}}$$

where X is some constant.

Now I need a particular solution to

$$\frac{\Delta V_{applied}}{R} - \frac{Q(t)}{RC} = \frac{dQ(t)}{dt}$$

This is clearly satisfied if I take:

$$\frac{\Delta V_{applied}}{R} - \left[C\Delta V_{applied} - Q_{Homogenous}(t)\right]\frac{1}{RC} = \frac{dQ_{Homogenous}(t)}{dt}$$

So that the particular solution is

$$Q(t) = C\Delta V_{applied} - Q_{Homogenous}(t) = C\Delta V_{applied} - Xe^{-\frac{t}{RC}}$$

Finally, solving for X using $Q(0) = 0$, I see that

$$Q(0) = \left[C\Delta V_{applied} - X\right]\frac{1}{RC} = 0$$

$$X = C\Delta V_{applied}$$

So that my charge at any time is given by:

$$Q(t) = C\Delta V_{applied}\left[1 - e^{\frac{-t}{RC}}\right]$$

and my current through the capacitor by:

$$I(t) = \frac{dQ(t)}{dt} = \frac{\Delta V_{applied}}{R} e^{\frac{-t}{RC}}$$

Now I see that in the simple case of a single loop, in order to find the charge or current through a capacitor at any time, I need only find the applied voltage across its terminals (e.g., from the batteries), and the resistance of the circuit.

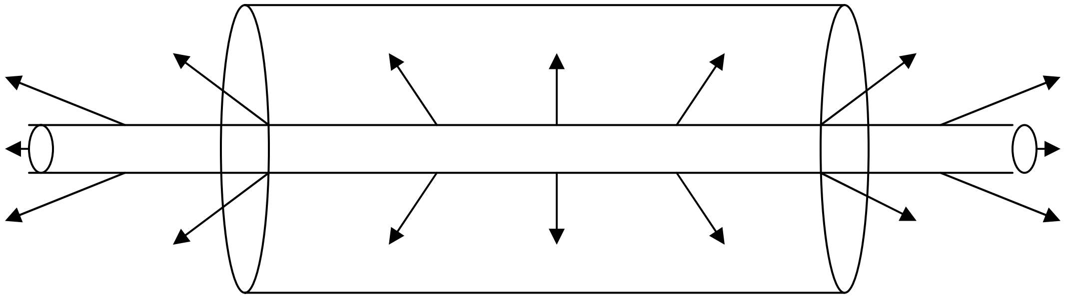
Gaussian Cylinder:

You should carefully study the procedure given in your recitation notes for solving a Gauss's Law problem. However, there are important technicalities that take place in a Gaussian Cylinder in our formulation:

Let me take as an example a problem with a single, long, uniformly charged cylinder of radius R, length L and total charge Q. I want to determine as best as I can the electric field from Gauss's Law. One of the steps from our Gauss's Law procedure tells me that I want to ensure that the electric field is either perpendicular to or vanishing on each surface. In that context, let me find a valid surface.

Clearly, I should choose a cylindrical surface since I have axial symmetry; where can I put my surface?

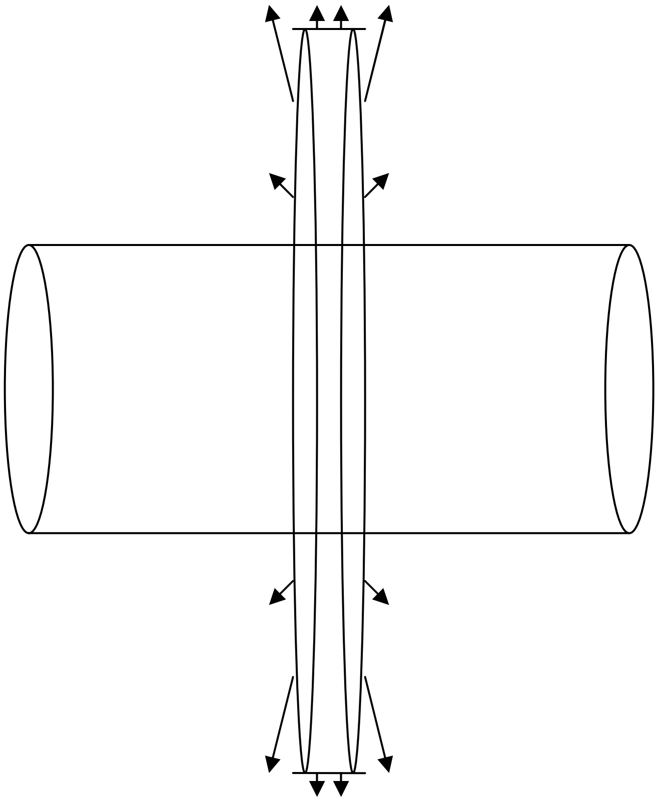
If I choose a very long Gaussian surface with the same central axis as my uniformly charged cylinder, does it seem likely that the electric fields on the end-caps will vanish, or that the electric field should be uniform across the surface?



Electric field is drawn at points along the Gaussian surface.

Above, it is clear that in this case, electric field is non-vanishing on the end caps and also that electric field is not constant along my Gaussian surface.

What about a Gaussian surface that is very thin and flat, oriented at the center?



Electric field is drawn at points along the Gaussian surface.

In the case where the surface is very flat and very long, then, I see that by symmetry the electric fields near the center of the Gaussian cylinder almost cancel, but at great distance contribute a small flux to the end caps. The electric flux at the far end is, by symmetry, very nearly perpendicular as desired.

So what does this show me?

The first argument shows me that when I use my Gauss's Law for a cylinder, my result is going to be accurate only very near the plane which bisects the cylinder.

The second argument shows me that my Gauss's Law for a cylinder as I am about to do it (and you are supposed to on homework, quizzes and exams) is going to be bad at great distances because of the missing contributions from the end-caps. In fact, I will find that $E(r) \propto \frac{1}{r}$. This is actually a major problem!

If I integrate this electric field, I find that $\Delta V_{\infty \to r} = -\int \vec{E} \cdot d\vec{l} \propto \int_{\infty}^{r} \frac{1}{r} dr = \ln \infty - \ln r = \infty$.

This is a very, very serious problem! I see that my potential of my charged cylinder with reference to a great distance is infinity, despite the fact that I have a finite amount of charge. To see this another way, imagine that I am extremely far away from my cylinder.

I look at it from so far, in fact, that my cylinder looks like a tiny point, with electric field $\vec{E}_{very \quad far \quad cylinder} \approx \vec{E}_{po\text{int}} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$. This $E \propto \dfrac{1}{r^2}$ is in very sharp contrast to the $E \propto \dfrac{1}{r}$ which I will get here. One gives me a finite potential at infinity ("normalizable") versus the other which will become infinite at inifinity, ("non-normalizable"). So, my solution for a finitely long cylinder is valid only relatively close to the cylinder. Where did this discrepancy arise? One explanation is that we have ignored the contribution of the end-caps. The other explanation is that we implicitly take a long-cylinder limit, so that what we find is actually the electric field due to an infinitely large cylinder of charge per unit length $\dfrac{Q}{L}$.

So now let's solve the problem. I choose a small Gaussian cylinder of length d, and center it in the middle, along the axis of the large cylinder. Now, in the center as drawn in the picture above, the end-caps very nearly have cancellation of electric field and I may ignore them, and the side of the cylinder has a radial electric field very nearly perpendicular to it. Thus, I have:

$$\oint_{Cyl} \vec{E}\cdot\hat{n}dA = \oint_{sides} \vec{E}(r)\cdot\hat{n}dA = \oint_{sides} E(r)dA = E(r)\oint_{sides} dA = E(r)2\pi rl$$

I consider the charge inside for two regions:

$r < R$:

$$\frac{ch\arg e}{volume \quad cylinder}\cdot volume \quad gauss = \frac{Q}{\pi R^2 L}\cdot \pi r^2 l$$

$r > R$:

$$\frac{ch\arg e}{volume \quad cylinder}\cdot volume \quad inside = \frac{Q}{\pi R^2 L}\cdot \pi R^2 l = \frac{Q}{L}l$$

All that's left is to put these two pieces together in the two regions:

$r < R$:

$$\oint_{Cyl} \vec{E}(r)\cdot\hat{n}dA = E(r)2\pi rl = \frac{Q_{enc}(r)}{\varepsilon_0} = \frac{Q}{L}\frac{r^2}{R^2}l$$

$$E(r)2\pi rl = \frac{Q}{L}\frac{r^2}{R^2}l$$

$$E(r) = \frac{Q}{2\pi L}\frac{r}{R^2}$$

Note that now I have to interpret what the direction means: earlier, I dotted this with the normal, so positive $E(r)$ indicates electric field along the normal, and negative indicates electric field pointing opposite the normal. In the outer region, I have:

$$r > R:$$

$$\oint_{Cyl} \vec{E}(r) \cdot \hat{n} dA = E(r) 2\pi r l = \frac{Q_{enc}(r)}{\varepsilon_0} = \frac{Q}{L} l$$

$$E(r) = \frac{Q}{2\pi r L}$$

This outer region has the troublesome $\frac{1}{r}$ dependence, as I promised earlier. This must certainly break down for finitely long cylinders at an extreme distance, and become something like $\frac{1}{r^2}$, as discussed earlier!