Physics II: 33–107

Math Review
Spring 2002

This review of some math topics which we will use in this course was originally prepared by Professors Kraemer and Meyer. In addition to this handout, you are responsible for the material in Appendix B of your text book. You will be expected to know this supplementary material at all times during this course.

A Quadratic Equation

The quadratic equation has the form:

$$y = Ax^2 + Bx + C = 0$$

where A, B and C are constants, and we want the values of x which makes the equation 0. This equation has two solutions, given as:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

As an *example*, consider the quadratic equation:

$$\begin{aligned} 3x^2 - 5x - 2 = 0 \quad &\Rightarrow \quad A = 3, B = -5, C = -2 \\ &\Rightarrow \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-2)}}{(2)(3)} \\ &\Rightarrow \quad x = \frac{5 \pm \sqrt{49}}{6} = \frac{5+7}{6} = 2 \;\; or \;\; \frac{5-7}{6} = -\frac{1}{3} \end{aligned}$$

B Powers

We will use the power of ten notation often for large and small numbers, (10^{-27} or 10^{38}).
Remember: $10^x + 10^y \neq 10^{(x+y)}$,
Example: $10^6 + 10^3 = 1,000,000 + 1,000 = 1,001,000$, which is **not** $10^9 = 1,000,000,000$.
However,

$$10^x 10^y = 10^{(x+y)} \rightarrow (10)^3(10)^6 = (10)^9.$$

Also,

$$\frac{10^x}{10^y} = 10^{(x-y)} = \frac{1}{10^{(y-x)}}$$

Example: $10^7/10^9 = 10^{-2} = 1/10^2$.

C Trigonometry

You should know how the trigonometric functions are defined with respect to a right triangle.

$\sin\theta = \frac{b}{c} = \cos\alpha$

$\cos\theta = \frac{a}{c} = \sin\alpha$

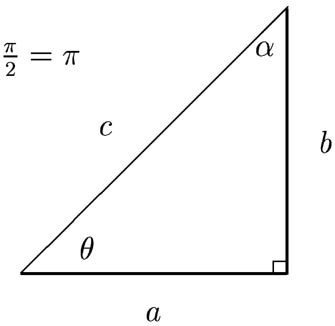
$\tan\theta = \frac{b}{a} = \cot\alpha$

$a^2 + b^2 = c^2$

$c^2\cos^2\theta + c^2\sin^2\theta = c^2$ or

$\cos^2\theta + \sin^2\theta = 1$

$\alpha + \theta + \frac{\pi}{2} = \pi$



You should be familiar with a *3,4,5 triangle*. In this triangle, $a = 3$, $b = 4$ and $c = 5$, and the angles are 37° and 53°. Which one is which?
We have that $\sin\alpha = \frac{3}{5} = 0.6$. Now think of the *sine* curve. Then $\sin 30^\circ = \frac{1}{2}$, which is close to 0.6. We can then conclude that $\alpha = 37^\circ$, and therefore $\theta = 53^\circ$.

You should also know that 2π radians equals 360°, so π radians equals 180°. So, for example:

$$30^\circ = \frac{30}{180}\pi = \frac{\pi}{6} radians$$

How can you get $\sin\theta$ if you don't have a calculator? You can use the series identity:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

This is actually what your calculator is doing. As an example, take $\theta = 30^\circ$. First you must convert to radians:

$$\frac{2\pi\ radians}{360^\circ} = \frac{x\ radians}{30^\circ} \Rightarrow \quad x = \frac{\pi}{6}$$

We can then use this to compute $\sin 30$:

$$\begin{aligned} \sin 30^\circ &= \frac{\pi}{6} - \frac{1}{6}\left(\frac{\pi}{6}\right)^3 + \frac{1}{120}\left(\frac{\pi}{6}\right)^5 + \cdots \\ \sin 30^\circ &= 0.5236 - 0.0239 + 0.0003 = 0.5000 \end{aligned}$$

Also, the expansion for $\cos\theta$ is

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

You should try to obtain $\cos 30^\circ = 0.866$ from this expansion.

You should always be able to draw the sine or cosine curves for one cycle, $(0^\circ \rightarrow 360^\circ)$, (see below).

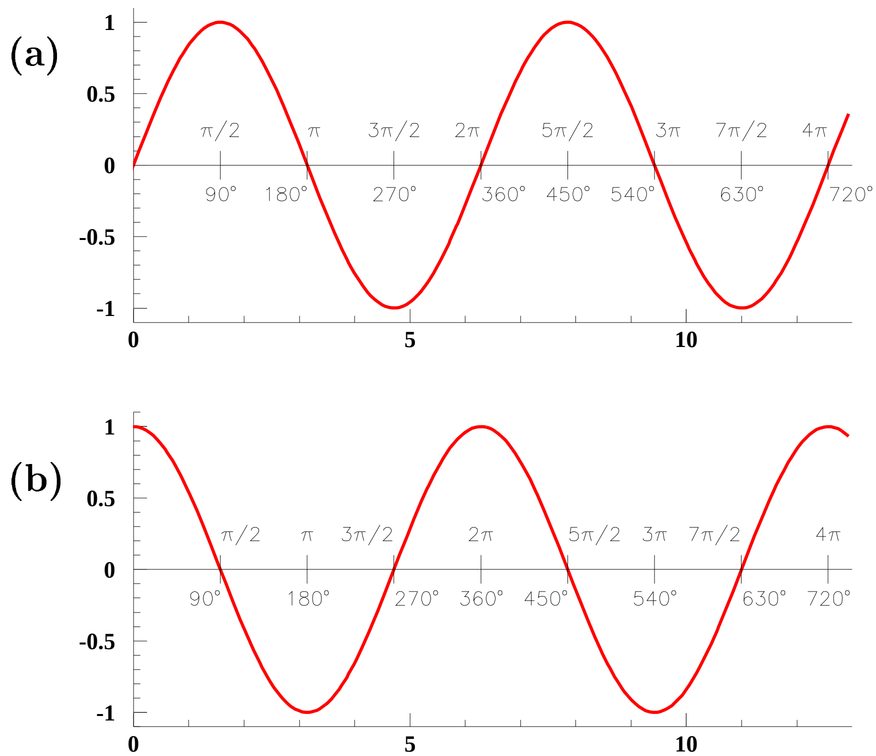


Figure 1: **(a)** Sine Curve. **(b)** Cosine Curve.

D Binomial Expansion

The binomial expansion is given as:

$$(a+b)^n = \frac{a^n}{0!} + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \cdots$$

where ! is the *factorial* symbol. $0! = 1$, $1! = 1$, $2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$, and so forth. This is very useful in the form $(1+x)^n$ where x is less than 1.

Example: What is $\sqrt{1.02}$?

$$\begin{aligned}
\sqrt{1.02} &= (1+0.02)^{1/2} \\
&= \frac{1^{1/2}}{0!} + \frac{\frac{1}{2}(1)^{-1/2}(0.02)}{1!} + \frac{\frac{1}{2}(1)^{-3/2}(0.02)^2}{2!} \\
&= 1 + 0.01 - \frac{1}{8}(0.0004) + \cdots \\
&= 1.01 \\
\text{I.}e. \qquad & (1.01)^2 = 1.02 \Rightarrow \textit{actually it is } 1.0201.
\end{aligned}$$

Example: Expand $(1 + \frac{z^2}{r^2})^{-\frac{1}{2}}$ for the case of $r \gg z$, we can do this quite easily using the binomial expansion.

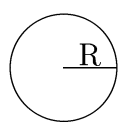
$$\begin{aligned}\left(1 + \frac{z^2}{r^2}\right)^{-\frac{1}{2}} &= (1)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(1)^{-\frac{3}{2}}\left(\frac{z^2}{r^2}\right) \\ &+ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-\frac{5}{2}}\left(\frac{z^2}{r^2}\right)^2\left(\frac{1}{2}\right) + \ldots \\ &= 1 - \frac{1}{2}\frac{z^2}{r^2} + \frac{3}{8}\frac{z^4}{r^4} + \ldots\end{aligned}$$

However, since $z^2/r^2 \ll 1$, we can safely ignore the $\frac{z^4}{r^4}$ term to obtain that:

$$\left(1 + \frac{z^2}{r^2}\right)^{-\frac{1}{2}} = 1 - \frac{1}{2}\frac{z^2}{r^2}$$

E Circumference, Area and Volume (L, L^2, L^3)

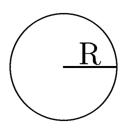
Circle:



$C = 2\pi R$

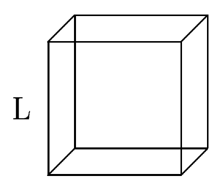
$A = \pi R^2$

Sphere:



$A = 4\pi R^2$
$V = \frac{4}{3}\pi R^3$

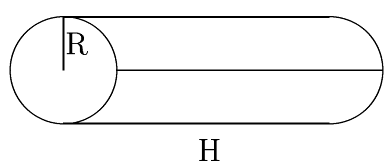
Cube:



$A = 6L^2$

$V = L^3$

Cylinder:



$A = \pi R^2 + \pi R^2 + 2\pi RH = 2\pi R(R + H)$

$V = \pi R^2 H$

F Logarithms

Define: If $y = A^x$, then $x \equiv \log_A(y)$. That is A is the **BASE** which is raised to the power x to give y. Then x is the power (or the logarithm) to which the base A is raised.

Example: $10^2 = 100$. This gives $A = 10$, (the base), $x = 2$, (the power) and $y = 100$, (the answer), or:

$$\log_{10}(100) = 2$$

Note: Knowledge of the BASE is crucial.

In mathematics, there is another common base, given by the symbol e, where $e \doteq 2.71828$. $log_e(3) = 1.0986$ means that $(2.71828)^{1.0986} = 3$. A common notation, especially on calculators is: $\log_e(x) = \ln(x) \Rightarrow$ ln means base e.

You must know the **BASE** for a logarithm to make any sense!

$$\begin{array}{rcl} \log_{10}(3) = 0.47 & \Leftrightarrow & \ln(3) = 1.0986 \\ 10^{0.47} = 3 & & (2.7128)^{1.0986} = 3 \end{array}$$

What is the number e? This is defined in terms of the infinite convergent series:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

If we examine the first 5 terms,

$$e \doteq 1 + 1 + 0.5 + 0.1667 + 0.0417 = 2.7084$$

this is actually not very good, we would need to take several more terms to get the accuracy that we have quoted.

Rules For Logarithms (any base)

1 $\log(ab) = \log(a) + \log(b)$.
Proof: Let $a = 10^n$, $b = 10^m$.
Then $a \cdot b = 10^{m+n}$, and $\log(ab) = \log(10^{n+m}) = n + m = \log(a) + \log(b)$.

2 $\log(\frac{a}{b}) = \log(a) - \log(b)$.

3 $\log(a^n) = n \cdot \log(a)$.

You need to know, and be able to use these rules. **Example:**

$$\log_{10}(380) = \log_{10}(3.8 \cdot 10^2) = \log_{10}(3.8) + \log_{10}(10^2) = 0.58 + 2.0 = 2.58$$

In other words, $10^{2.58} = 380$. Does this make any sense? We know that $10^2 = 100$ and that $10^3 = 1000$. As such, we would expect something between 2 and 3 for $\log_{10}(380)$.
Example:

$$\log_{10}(0.081) = \log_{10}(8.1) + \log_{10}(10^{-2}) = 0.908 - 2.0 = -1.092$$

In other words, $10^{-1.092} = 1/10^{1.092} = 0.081$. Does this make sense? We know that $\log_{10}(0.1) = -1$ and $\log_{10}(0.01) = -2$. We then expect the answer to lie between -1 and -2.

CHANGE OF BASE

Suppose we have a number, y. We can express y in terms of base 10 or base e:

$$y = 10^z \quad or \quad y = e^x$$

$$\log_{10}(y) = z \quad and \quad \ln(y) = x$$

We also have that:

$$y = 10^z = e^x$$

This then tells us that

$$\ln(y) = \ln(e^x) = x = \ln(10^z) = z\ln(10) = z \cdot 2.303$$

¿From this we conclude that $x = 2.303 \cdot z$, or

$$\ln(y) = 2.303 \cdot \log_{10}(y)$$

If you have $\log_{10}(y)$, you multiply by 2.303 to get $\ln(y)$ in base e.

Example: Take $y = 1000$, what is $\ln(1000)$? We know that $\log_{10}(1000) = 3$. Therefore, $\ln(y) = 3 \cdot 2.303 = 6.909$. Of course, if we know $\ln(y)$, we can compute $\log(y)$ by dividing by 2.303, or equivalently:

$$\log_{10}(y) = \frac{1}{2.303}\ln(y) = 0.4343\ln(y)$$

G Functions of more than one variable, $f(x, y)$

Example: $f(x, y) = 3x^2 + 2xy + 4y^2 - 3x + 7$

Example: $f(x, y) = A\cos(ax + by)$

There are two possible derivatives of $f(x, y)$:

$\frac{\partial f}{\partial x}$, the partial derivative of f with respect to x. Treat y as a constant.

$\frac{\partial f}{\partial y}$, the partial derivative of f with respect to y. Treat x as a constant.

Example: $f(x, y) = A\cos(ax + by)$.

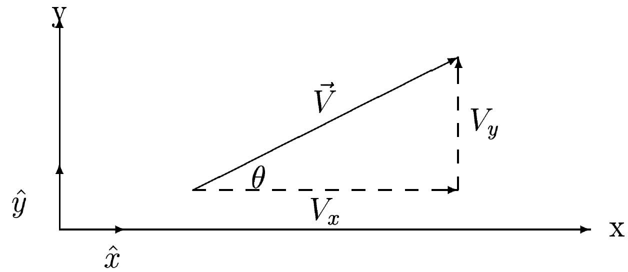
$$\frac{\partial f}{\partial x} = -Aa\sin(ax + by)$$

$$\frac{\partial f}{\partial y} = -Ab\sin(ax + by)$$

H Vectors (Two Dimensions)

$|\vec{V}|=$ *length* or *magnitude* of $\vec{V}$. This is a positive number.
$\hat{x}$ and $\hat{y}$ are *unit vectors*. The unit vector $\hat{V}$ is defined as: $\hat{V}=\vec{V}/|\vec{V}|$.

$$\vec{V}=(V_x,V_y)$$
$$\vec{V}=(V\cos\theta, V\sin\theta)$$
$$\vec{V}=V_x\hat{x}+V_y\hat{y}$$

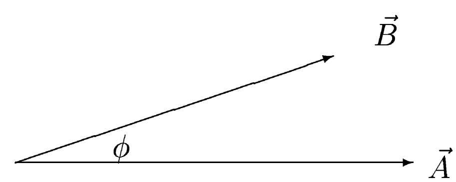


Example: $V=3\hat{x}+4\hat{y}$, so $|\vec{V}|=\sqrt{3^2+4^2}=+5$.
NOTE: The magnitude of a vector is always **positive**.
Therefore, $\hat{V}=(3\hat{x}+4\hat{y})/5=\frac{3}{5}\hat{x}+\frac{4}{5}\hat{y}$. This yields the length of $\hat{V}=\sqrt{(\frac{3}{5})^2+(\frac{4}{5})^2}=+1$.
The length of a unit vector is 1.
Example: Suppose $|\vec{V}|=5$ and $\theta=30°$, then $\vec{V}=(5\cos 30°, 5\sin 30°)=5\cos 30°\hat{x}+5\sin 30°\hat{y}$. We can then compute $\hat{V}$ as:

$$\hat{V}\equiv\frac{\vec{V}}{|\vec{V}|}=\frac{5\cos 30°\hat{x}+5\sin 30°\hat{y}}{5}$$
$$=\cos 30°\hat{x}+\sin 30°\hat{y}$$

and $|\hat{V}|=\sqrt{\cos^2 30°+\sin^2 30°}=1$. **The length of a unit vector is 1**.
Rules: $\vec{A}\pm\vec{B}=(A_x\pm B_x, A_y\pm B_y)$,
$\vec{A}/\vec{B}$ is not defined.
$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y=(|\vec{A}|)(|\vec{B}|)(\cos\phi)$ where ϕ is the angle between $\vec{A}$ and $\vec{B}$. This then yields:

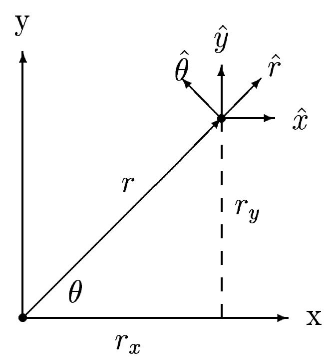
$$\cos\phi=\frac{A_xB_x+A_yB_y}{|\vec{A}||\vec{B}|}$$



You need to read and review pages 10 to 22 (inclusive) of your textbook. Do it **NOW**. Do not worry about the *Vector Product*, we will review this in class when we need it.

I Polar Coordinates

The coordinate of a point on a plane can be expressed in *Cartesian coordinates* as (r_x, r_y) where r_x is the distance of the point from the y axis, and r_y is the distance of the point from the x axis. We can also express the vector from the origin to the point (r_x, r_y) as $\vec{r} = r_x\hat{x} + r_y\hat{y}$. In many situations, it is more convenient to express the coordinate of a the point in terms of its distance from the origin r, and θ, the angle from the x–axis.



$$\vec{r} = (r_x, r_y) = (r, \theta)$$
$$\vec{r} = (r\cos\theta, r\sin\theta)$$
$$\vec{r} = r(\cos\theta\hat{x} + \sin\theta\hat{y})$$
$$r_x = r\cos\theta$$
$$r_y = r\sin\theta$$
$$r = \sqrt{r_x^2 + r_y^2}$$
$$\theta = \tan^{-1}\left(\frac{r_y}{r_x}\right)$$

The unit vector $\hat{x}$ always points along the x–axis and the unit vector $\hat{y}$ always points along the y–axis. What about the unit vector $\hat{r}$? We can compute the unit vector $\hat{r}$ as $\hat{r} = \vec{r}/\mid\vec{r}\mid$. Where $\mid\vec{r}\mid$ is the magnitude of $\vec{r}$:

$$\mid\vec{r}\mid = \sqrt{r_x^2 + r_y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{\cos^2\theta + \sin^2\theta} = r$$

This implies that:

$$\hat{r} = \vec{r}/r = r(\cos\theta\hat{x} + \sin\theta\hat{y})/r = \cos\theta\hat{x} + \sin\theta\hat{y}.$$

The unit vector $\hat{r}$ points away from the origin. Its direction also depends upon the position of the point. $\hat{r}$ refers to the direction away from the origin at the given point, it is not fixed like $\hat{x}$ or $\hat{y}$.

We can also define a second direction perpendicular to $\hat{r}$. We will call this direction $\hat{\theta}$, and because $\hat{r}$ depends on position, $\hat{\theta}$ also depends on position.

$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$$

We can see that $\hat{\theta}$ and $\hat{r}$ are perpendicular to each other by examining their dot product:

$$\begin{aligned}
\hat{r}\cdot\hat{\theta} &= (\cos\theta\hat{x} + \sin\theta\hat{y})\cdot(-\sin\theta\hat{x} + \cos\theta\hat{y}) \\
&= (\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta) \\
&= 0
\end{aligned}$$

J Vector Functions

The function $\vec{E}(\vec{r}) = (E_x(\vec{r}), E_y(\vec{r}), E_z(\vec{r})$ is a *vector function.* $\vec{r} = (x, y, z)$. $\vec{E}(\vec{r})$ is actually three functions.

- $E_x(\vec{r})$ for the x–component.
- $E_y(\vec{r})$ for the y–component.
- $E_z(\vec{r})$ for the z–component.

At any point in space, $r_a = (x_a, y_a, z_a)$, there is a vector $\vec{E}(\vec{r}_a)$ given as $(E_x(\vec{r}_a), E_y(\vec{r}_a), E_z(\vec{r}_a))$.
Example: $E_x(\vec{r}) = a\sin(x)$, $E_y(\vec{r}) = ax + bz$ and $E_z(\vec{r}) = b\cos(y)$.
At the point $(0, 0, 0)$, $\vec{E} = (a\sin(0), 0a + 0b, b\cos(0)) = (0, 0, b)$.
At the point $(\frac{\pi}{2}, \frac{\pi}{2}, 1)$, $\vec{E} = (a\sin(\frac{\pi}{2}), \frac{\pi}{2}a + b, b\cos(\frac{\pi}{2})) = (a, \frac{a\pi}{2}, 0)$.
We can also define the derivatives of $\vec{E}(\vec{r})$ with respect to x, y and z.

$$\begin{aligned}
\frac{\partial \vec{E}}{\partial x} &= \left(\frac{\partial E_x}{\partial x}, \frac{\partial E_y}{\partial x}, \frac{\partial E_z}{\partial x}\right) \\
\frac{\partial \vec{E}}{\partial y} &= \left(\frac{\partial E_x}{\partial y}, \frac{\partial E_y}{\partial y}, \frac{\partial E_z}{\partial y}\right) \\
\frac{\partial \vec{E}}{\partial z} &= \left(\frac{\partial E_x}{\partial z}, \frac{\partial E_y}{\partial z}, \frac{\partial E_z}{\partial z}\right)
\end{aligned}$$

In our example above,

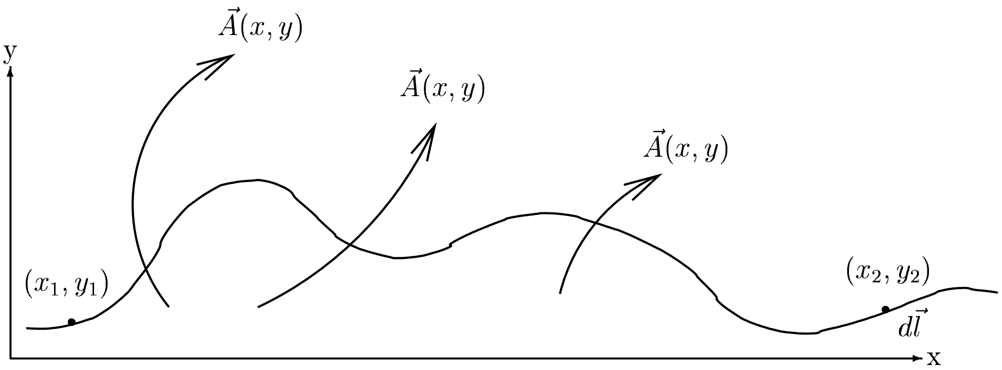
$$\begin{aligned}
\frac{\partial \vec{E}}{\partial x} &= (a\cos(x), a, 0) \\
\frac{\partial \vec{E}}{\partial y} &= (0, 0, -b\sin(y)) \\
\frac{\partial \vec{E}}{\partial z} &= (0, b, 0)
\end{aligned}$$

K Line Integrals

You are not expected to know this material at the start of the course. The needed material will be covered in the course. You will be responsible for this after the material has been taught.
If you have a vector field, $\vec{A}(x, y) = A_x(x, y)\hat{x} + A_y(x, y)\hat{y}$, we can define a line integral:

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \vec{A}(x, y) \cdot d\vec{l}$$

(Recall $W = \int_1^2 \vec{F} \cdot d\vec{l}$).



$$
\begin{aligned}
I &= \int_1^2 (A_x(x,y)\hat{x} + A_y(x,y)\hat{y}) \cdot (dx\hat{x} + dy\hat{y}) \\
&= \int_1^2 [A_x(x,y)dx + A_y(x,y)dy]
\end{aligned}
$$

In order to evaluate this, we need to know A_x and A_y as a function of position and to define the path of integration. Now let us a look at a one–dimensional line integral over a vector field you know about — Gravity. **Example:** We have a mass m falling from $r = R_1$ to $r = R_2$. How much work is done by gravity? The field is the **force field** due to the existence of gravity. $\mid \vec{F} \mid = mg$, $\vec{F} = -mg\hat{r}$.

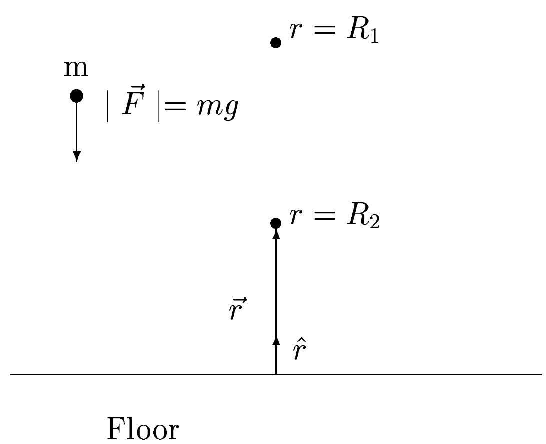
So, positive work was done by the gravitational force field.

$W = \int_{r=R_1}^{r=R_2} (-mg\hat{r}) \cdot d\vec{l}$

$W = \int_{R_1}^{R_2} (-mg\hat{r}) \cdot (dr\hat{r})$

$W = -mg \int_{R_1}^{R_2} dr$
$W = -mg \underbrace{[R_2 - R_1]}_{negative}$

$W = +mg \underbrace{[R_1 - R_2]}_{positive}$

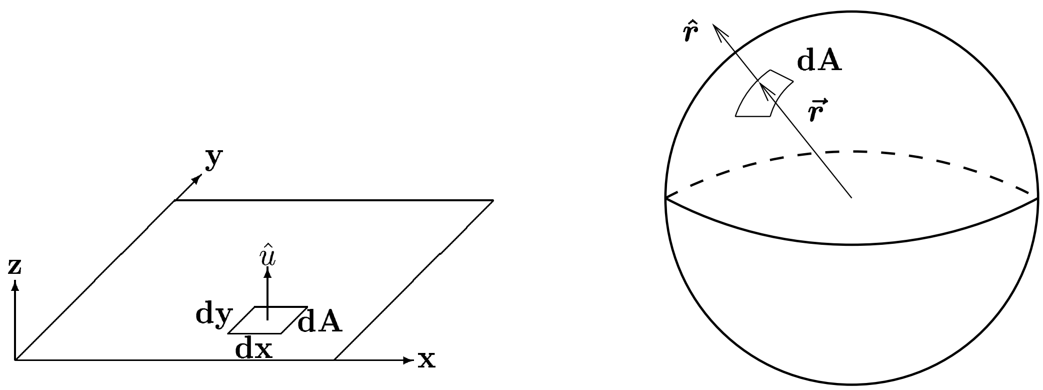


Note: $d\vec{l} = (+dr)\hat{r}$ in spite of the fact that the mass is moving toward smaller r. The limits on the integration take care of that. If you add an extra minus sign, you will get negative work done by gravity which is *levitation*.

L Surface Integrals

You are not expected to know this material at the start of the course. The needed material will be covered in the course. You will be responsible for this after the material has been taught.

Let us begin by examining a plane as shown on the left side of the figure below. The small element of area $dA = (dx)(dy)$ can be expressed as a vector: $d\vec{A} = dA\hat{u} = (dx)(dy)\hat{u}$ where $\hat{u}$ is a unit vector which is perpendicular to the plane. In the figure below, $\hat{u} = \hat{z}$.



We could also consider the sphere shown on the right hand side of the above figure. Here we can do the same thing, $d\vec{A} = dA\hat{r}$ where $\hat{r}$ is a unit vector along the radial direction, $\hat{r} = \vec{r}/\mid \vec{r} \mid$. Any time we have a surface, we can always define the area vector, $d\vec{A}$, as a vector whose magnitude is dA and whose direction is normal to the surface.

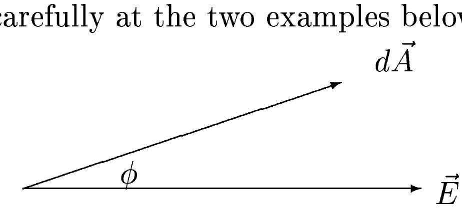
If we have a vector field, $\vec{E}(x, y, z)$, we will need to calculate

$$I = \int_{Surface} \vec{E} \cdot d\vec{A} = \mathit{The\ Flux\ of\ } \vec{E} \mathit{\ through\ the\ surface.}$$

We will need only simple cases and will explain them as we proceed. However, you need to recognize that:

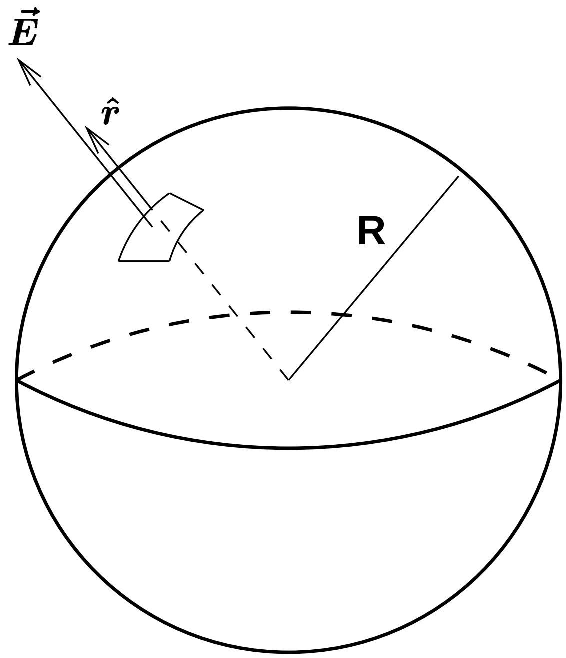
where ϕ is the angle between $\vec{E}$ and $d\vec{A}$. Look carefully at the two examples below:

$\int_{Surface} \vec{E} \cdot d\vec{A} = \int_S E dA \cos\phi$



Example: Take a sphere of radius R where the electric field vector $\vec{E}$ <u>at the surface</u> is $\vec{E} = E\hat{r}$ and $d\vec{A} = dA\hat{r}$. Now neither E nor dA is a function of the position on the surface of the sphere. This means that:

$$I = \int_S E\hat{r} \cdot dA\hat{r}$$



$$
\begin{aligned}
&= \int_S E dA(\hat{r}\cdot\hat{r}) \\
&= \int_S E dA \quad Where : \hat{r}\cdot\hat{r} = 1 \\
&= E\int_S dA \quad Where : E\, is\, a\, constant\, magnitude\, over\, the\, surface. \\
&= EA = E4\pi R^2
\end{aligned}
$$

Note: We will discuss the units of $\vec{E}$ in chapter 22.

Example: Take A as the square region in the x–y plane bounded by the points $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$, and $\vec{E}(x,y,z) = (0,0,xy)$. In this case, $d\vec{A}$ is $(dx)(dy)$ in the direction of the z–axis, $\hat{z}$. $d\vec{A} = dxdy\hat{z}$.

$$I = \int_A \vec{E} \cdot d\vec{A}$$

$$I = \int_0^1 \int_0^1 [0\hat{x} + 0\hat{y} + xy\hat{z}] \cdot [dxdy\hat{z}]$$

$$I = \int_0^1 \int_0^1 (xy) dxdy$$

$$I = \int_0^1 dx x(\tfrac{1}{2}y^2) \Big|_{y=0}^{y=1}$$

$$I = \int_0^1 dx x(\tfrac{1}{2})$$

$$I = \tfrac{1}{2}\tfrac{1}{2}x^2 \Big|_{x=0}^{x=1}$$

$$I = \tfrac{1}{4}$$