IE 315
Spring 2001
Midterm Examination - **Solution Key**
B. Deler

This examination is open book and notes. You will have **1 hour and 10 minutes**. Show all work, answer only the questions asked, and if you need additional space use the back of the page. **You only have to give a numerical answer if you are specifically asked for one**. Good luck!

1. In a factory, the engineers described the time to failure of a machine (in hours) as a random variable X having a distribution with the following probability density function:

$$f_X(x) = \begin{cases} 0, & x < 1 \\ \dfrac{3x^2}{124}, & 1 \leq x \leq 5 \\ 0, & x > 5 \end{cases}$$

   (a) (5 points) What is the cumulative distribution function for this random variable? Remember that $F_X(x) = \int_{-\infty}^{x} f_X(a)da$:

   If $x < 1$, then $F_X(x) = \int_{-\infty}^{x}(0)da = 0$.
   If $1 \leq x \leq 5$, then $F_X(x) = \int_{-\infty}^{1}(0)da + \int_{1}^{x}(\frac{3a^2}{124})da = \frac{x^3-1}{124}$.
   If $x > 5$, then $F_X(x) = \int_{-\infty}^{1}(0)da + \int_{1}^{5}(\frac{3a^2}{124})da + \int_{5}^{x}(0)da = 1$.

   Thus,

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \dfrac{x^3-1}{124}, & 1 \leq x \leq 5 \\ 1, & x > 5 \end{cases}$$

   (b) (5 points) What is the probability that time to failure will be more than 2 hours but less than 4 hours? A numerical answer is expected here.

$$\begin{aligned} Pr\{2 < X < 4\} &= Pr\{X \le 4\} - Pr\{X \le 2\} \\ &= F_X(4) - F_X(2) \\ &= \frac{4^3 - 1}{124} - \frac{2^3 - 1}{124} \\ &= 0.45161 \end{aligned}$$

(c) (10 points) Give an expression for the inverse cdf for this random variable.

$U = F_X(x) = \frac{x^3-1}{124} \rightarrow (124)(U) + 1 = x^3$.

Since $1 \le x \le 5$, $x = ((124)(U) + 1)^{\frac{1}{3}}$.

(d) (10 points) Write a complete algorithm that includes the function `random()` to generate values of X. What value does your algorithm generate if `random()` returns the value 0.2? A numerical answer is expected here.

(1) $U = random()$
(2) Return $x = ((124)(U) + 1)^{\frac{1}{3}}$
Thus, $x = ((124)(0.2) + 1)^{\frac{1}{3}} = 2.95488$.

2. Arrivals of customers to the Peet's coffee & tea place were recorded from 9 am to 10 am on two consecutive days. The arrival times are given below:

Day 1	what happened	Day 2	what happened
9:00	start observing	9:00	start observing
9:08	call arrived	9:02	call arrived
9:17	call arrived	9:15	call arrived
9:18	call arrived	9:21	call arrived
9:20	call arrived	9:38	call arrived
9:41	call arrived	9:49	call arrived
9:48	call arrived	10:00	stopped observing
9:50	call arrived		
9:51	call arrived		
10:00	stopped observing		

Suppose that this arrival process is well-modeled as a Poisson process.

(a) (10 points) Estimate the arrival rate of calls and give a standard error for your estimate. A numerical answer is expected here.

Day 1 $\rightarrow$ 8 arrivals in 1 hour
Day 2 $\rightarrow$ 5 arrivals in 1 hour

$\hat{\lambda} = \frac{8+5}{1+1} = \frac{13}{2} = 6.5$

$\hat{se} = \sqrt{\frac{\hat{\lambda}}{1+1}} = \sqrt{\frac{6.5}{1+1}} = 1.80278$

(b) (10 points) Using your estimates above, what is the probability that the time between arrival of customers to the coffee place will be greater than 5 minutes? A numerical answer is expected here.

Let G be the random variable representing the time between arrivals of customers. Since this is a Poisson process, $G \sim exp(6.5)$:

$$Pr\{G > 5/60\} = Pr\left\{G > \frac{1}{12}\right\}$$

$$
\begin{aligned}
&= 1 - Pr\left\{G \leq \frac{1}{12}\right\} \\
&= 1 - \left(1 - e^{-(6.5)(\frac{1}{12})}\right) \\
&= 0.58178.
\end{aligned}
$$

(c) (5 points) Do you think that a Poisson arrival process is likely to be a good model for this system? Why or why not?

There is a large number of potential customers that arrive at the Peet's place for the morning coffee and we expect them to act independently. If we focus on a small time interval such as 8-10 am, then the arrival rate can be found to be steady. Then, a Poisson arrival process is likely to be a good model for this system.

3. ON-LINE INFO is a company that provides computerized data-base searches to customers over telephone lines. The requests from customers can be classified into those that require printing a document and those that do not require printing. Requests that require printing arrive at a rate of 400 per hour; those that do not require printing arrive at a rate of 1000 per hour. Suppose that both arrival processes are well modeled as independent Poisson processes.

   (a) (5 points) A Poisson process has stationary increments. What does *stationary increments* mean in this situation?

   The requests from customers in a time interval of length $\triangle$t depends only on the length of the interval and not when the actual clock times correspond to.

(b) All incoming requests are initially handled by a single front-end computer.

i. (5 points) What is the probability that this computer receives more than 2000 requests between 1 pm and 2:30 pm?

Let $\{Y_{1,t}; t \geq 0\}$ be a Poisson arrival process representing the arrival requests that require printing; $\lambda_1 = 400/hour$.

Let $\{Y_{2,t}; t \geq 0\}$ be a Poisson arrival process representing the arrival requests that do not require printing; $\lambda_2 = 1000/hour$.

Then, $Y_t = Y_{1,t} + Y_{2,t}$ is Poisson with rate $\lambda = \lambda_1 + \lambda_2 = 1400/hour$.

$$Pr\{Y_{1.5} > 2000\} = \sum_{j=2001}^{\infty} \frac{e^{-(1400)(1.5)}(1400(1.5))^j}{j!} \approx 0.985.$$

ii. (5 points) What is the expected time for the 2000[th] request to arrive at the front-end computer? A numerical answer is expected here.

Let T_{2000} be the time when 2000[th] request arrives at the front-end computer.

$T_{2000} = \sum_{i=1}^{2000} G_i \sim Erlang(2000, 1400)$ as $G_i \sim exponential(1400)$. Thus, $E(T_{2000}) = \frac{2000}{1400} = 1.42857$.

(c) (10 points) Suppose that the front-end computer distributes the requests randomly between two processing computers (called A and B) that fulfill the requests, in such a way that each of the processing computers is equally likely to receive a new request. Assuming that the time for the front-end computer to do this is essentially 0, what is the probability that both of the processing computers receive more than 1000 requests between 1 pm and 2:30 pm?

Let $\{Y_{A,t}; t \geq 0\}$ and $\{Y_{B,t}; t \geq 0\}$ represent the arrivals to computers A and B, respectively.

> $Y_{A,t}$ is Poisson with $\lambda_A = \frac{1}{2}(\lambda) = 700/hour$,
> $Y_{B,t}$ is Poisson with $\lambda_B = \frac{1}{2}(\lambda) = 700/hour$, and they are independent.

Therefore,

$$
\begin{aligned}
Pr\{Y_{A,1.5} > 1000, Y_{B,1.5} > 1000\} &= Pr\{Y_{A,1.5} > 1000\} Pr\{Y_{B,1.5} > 1000\} \\
&= \left\{ \sum_{j=1001}^{\infty} \frac{e^{-(700)(1.5)}((700)(1.5))^j}{j!} \right\}^2 \approx 0.877
\end{aligned}
$$

4. (20 points) For the following situation, formulate a stochastic process model by defining the state of your system, S_n, the input random variables, and the system event-functions (i.e., the logic for each system event):

   An airline reservation division has a call center with 2 operators, which are indexed as operator 1 and operator 2. When a call arrives at the call center, one of the following three cases can take place: (i) if both of the operators are busy, then callers wait in a hold queue. However, the queue can only hold 5 callers in addition to the callers talking to the operators; if the queue is full, then callers receive a busy signal and are turned away. Calls are answered first-come-first-served: When one of the operators becomes available, then the next call waiting in the hold queue is answered by that available operator. (ii) If both of the operators are available, then call is answered by operator 1. (iii) If one of the operators is available, then the call is answered by that available operator.

The time gap between the arrival of calls is well modeled by an exponential distribution, while the time to answer a call is approximated by a Weibull distribution for operator 1 and the time to answer a call is approximated by an Erlang distribution for operator 2 .

Hint: Consider the departure process from operator 1 and operator 2 separately.

$\{S_n; n = 0, 1, 2, \ldots\}$ = number of servers in the system just after the n^{th} event.

$S_n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

$\{T_n; n = 0, 1, 2, \ldots\}$ = time of the n^{th} event.

Inputs:

X $\sim$ interarrival time with cdf F_X

$G_1 \sim$ service time for operator 1 with cdf F_{G_1}

$G_2 \sim$ service time for operator 2 with cdf F_{G_2}

Events:

Arrival of a customer, $e_1()$, with clock time C_1

Departure from operator 1, $e_2()$, with clock time C_2

Departure from operator 2, $e_3()$, with clock time C_3

e$_0$() for Initialization :

$S_0 \leftarrow 0$

$T_0 \leftarrow 0$

$C_1 \leftarrow F_X^{-1}(random())$

$C_2 \leftarrow \infty$

$C_3 \leftarrow \infty$

e$_1$() for Arrival :

if $\{S_n = 7\}$

$S_{n+1} \leftarrow S_n$

else if $\{S_n = 0\}$

$S_{n+1} \leftarrow S_n + 1$

$C_2 \rightarrow T_{n+1} + F_{G_1}^{-1}(random())$

else if $\{S_n = 1\}$

$S_{n+1} \leftarrow S_n + 1$

find i such that $C_i = \infty, i = 2, 3$

$C_i \rightarrow T_{n+1} + F_{G_i}^{-1}(random())$

else

$S_{n+1} \leftarrow S_n + 1$

endif

$C_1 \rightarrow T_{n+1} + F_X^{-1}(random())$

$\underline{\text{e}_i(), i = 2, 3 \text{ for Departure :}}$

$S_{n+1} \leftarrow S_n - 1$

if $\{S_{n+1} \geq 2\}$

$C_i \rightarrow T_{n+1} + F_{G_i}^{-1}(random())$

else

$C_i \leftarrow \infty$

endif