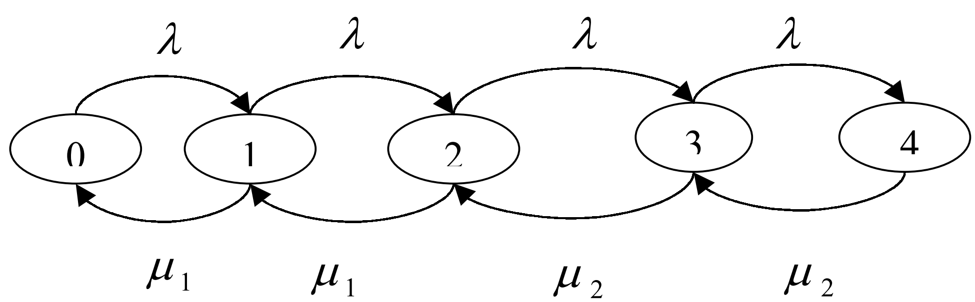
In-class problems

1. Customers arrive at a single-service facility at a Poisson rate of 40 per hour. When two or fewer customers are present, a single attendant operates the facility, and the service time for each customer is exponentially distributed with a mean value of 2 minutes. However, when there are three or more customers at the facility, the attendant is joined by an assistant and, working together, they reduce the mean service time to one minute. Assuming a system capacity of four customers,

   $\lambda = 40$/hour

   $1/\mu_1 = 2$ minutes $= 1/30$ hour $\Rightarrow \mu_1 = 30$ /hour

   $1/\mu_2 = 1$ minute $= 1/60$ hour $\Rightarrow \mu_2 = 60$ /hour



Then, the balance equations are the following:

$\lambda\ \pi_0 = \pi_1\ \mu_1$

$\lambda\ \pi_0 + \pi_2\ \mu_1 = \pi_1\ (\mu_1 + \lambda)$

$\lambda\ \pi_1 + \pi_2\ \mu_2 = \pi_2\ (\mu_1 + \lambda)$

$\lambda\ \pi_2 + \pi_4\ \mu_2 = \pi_3\ (\mu_2 + \lambda)$

$\lambda\ \pi_3 = \pi_4\ \mu_2$

And we know what the generic solution to this system of equations is:

Define

$d_1 = (\lambda)/(\mu_1) = 1.333 \Rightarrow \pi_1 = d_1\pi_0 = 1.333\ \pi_0$

$d_2 = (\lambda^2)/(\mu_1^2) = 1.778 \Rightarrow \pi_2 = d_2\pi_0 = 1.778\ \pi_0$

$d_3 = (\lambda^3)/(\mu_1^2\mu_2) = 1.185 \Rightarrow \pi_3 = d_3\pi_0 = 1.185\ \pi_0$

$d_4 = (\lambda^4)/(\mu_1^2\mu_2^2) = 0.79 \Rightarrow \pi_4 = d_4\pi_0 = 0.79\ \pi_0$

Since $\pi_0+\pi_1+\pi_2+\pi_3+\pi_4=1$, $\pi_0=1/(1+1.333+1.778+1.185+0.79)=1/6.086=0.164$.

Then,

$\pi_1=(1.333)*(0.164)=0.2186$
$\pi_2=(1.778)*(0.164)=0.292$
$\pi_3=(1.185)*(0.164)=0.194$
$\pi_4=(0.790)*(0.164)=0.1296$

(a) what proportion of time are both servers free?

π_0=0.164

(b) each man is to receive a salary proportional to the amount of time he is actually at work servicing customers, the rate being the same for both. If together they earn $100 per day, how should this money be split?

Suppose s_1=salary of the major operator and s_2=salary of the assistant:

$s_1+s_2 = 100$ and $s_1/s_2 = (1-\pi_0)/(\pi_3+\pi_4) = 0.836/0.3236 = 2.58 \Rightarrow s_1 = (2.58)s_2$

Then, $(2.58)*s_2+ s_2 = s_2*(1+2.58) = s_2*(3.58) = 100 \Rightarrow$ **s_2=100/3.58 = 27.90**

s_1 = 100-27.9 = 72.1

2. Consider a sequential-service system consisting of two servers, A and B. The arriving customers will enter this system only if server A is free. If a customer does enter, then he is immediately served by server A. When his service by A is completed, he then goes to B if B is free, or if B is busy, he leaves the system. Upon completion of service at server B, the customer departs. Assuming that the (Poisson) arrival rate is two customers an hour, and that A and B serve at respective (exponential) rates of four and two customers an hour.

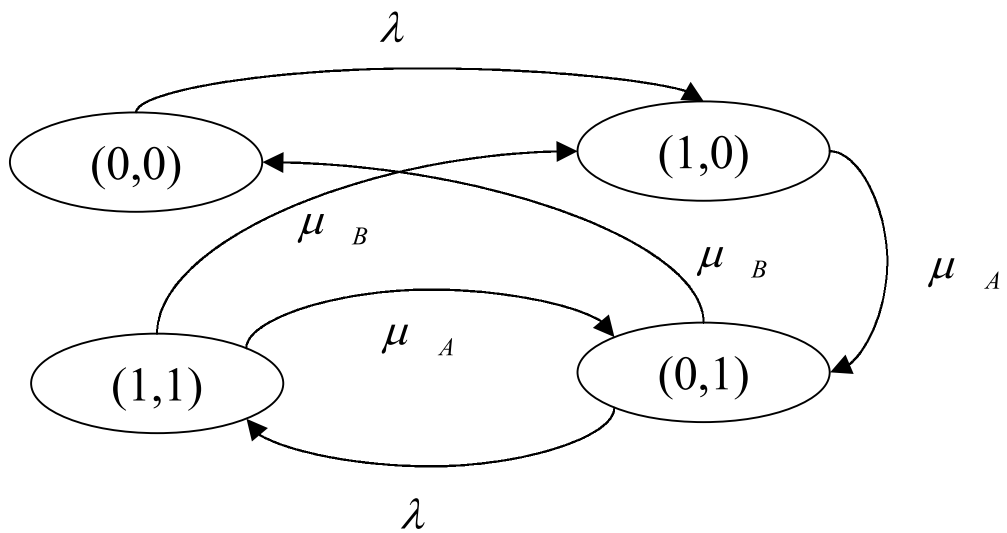
$\lambda = 2$ / hour

$\mu_A = 4$ / hour

$\mu_B = 4$ / hour

State definition = (0/1 if server A is idle/busy, 0/1 if server B is idle/busy)

State space = {(0,0), (0,1), (1,0), (1,1)}



Then, the balance equations are the following:

$\pi_{(0,1)}\,\mu_A = \pi_{(0,0)}\,\lambda \Rightarrow 4\,\pi_{(0,1)} = 2\,\pi_{(0,0)} \Rightarrow 2\,\pi_{(0,1)} = \pi_{(0,0)}$

$\pi_{(0,1)}\,\mu_B + \pi_{(0,0)}\,\lambda = \pi_{(1,0)}\,\mu_A \Rightarrow 4\,\pi_{(0,1)} + 2\,\pi_{(0,0)} = 4\,\pi_{(1,0)} \Rightarrow 2\,\pi_{(0,1)} + \pi_{(0,0)} = 2\,\pi_{(1,0)}$

$(\pi_{(1,0)} + \pi_{(1,1)})\,\mu_A = \pi_{(0,1)}\,(\mu_B + \lambda) \Rightarrow 2\,(\pi_{(1,0)} + \pi_{(1,1)}) = 3\,\pi_{(0,1)}$ (after dividing by 2)

$\pi_{(0,1)}\ \lambda = \pi_{(1,1)}\ (\mu_A + \mu_B) \Rightarrow 2\ \pi_{(0,1)} = 8\ \pi_{(1,1)} \Rightarrow \pi_{(0,1)} = 4\ \pi_{(1,1)}$.

Next, let's try to write everything in terms of $\pi_{(0,0)}$ (or any other state probability):

$\pi_{(0,1)} = 0.5 * \pi_{(0,0)}$

$\pi_{(1,1)} = (0.25)*(0.5)* \pi_{(0,0)} = 0.125 * \pi_{(0,0)}$

$\pi_{(1,0)} = 3*0.5 * \pi_{(0,0)} * 0.5 - 0.125* \pi_{(0,0)} = 0.625* \pi_{(0,0)}$

Since $\pi_{(0,0)} + \pi_{(1,0)} + \pi_{(0,1)} + \pi_{(1,1)} = 1$, $\pi_{(0,0)} = 1/(1+0.5+0.125+0.625) = 1/(2.25) =0.44$.

Then,

$\pi_{(0,1)} = 0.5*0.44 = 0.22$

$\pi_{(1,1)} = 0.125*0.44 = 0.055$

$\pi_{(1,0)} = 0.625*0.44 = 0.275$

(a) what proportion of customers enter the system?

$\pi_{(0,0)} + \pi_{(0,1)} = 0.44+0.22 =$ **0.66**

(b) what proportion of entering customers receive service from B?

$\pi_{(0,1)} + \pi_{(1,1)} = 0.22 + 0.055=$ **0.275**

(c) what is the average number of customers in the system?

$l = 0* \pi_{(0,0)} + 1*(\pi_{(1,0)} + \pi_{(0,1)}) + 2* \pi_{(1,1)} = 0.275 + 0.22 + 2*0.055 =$ **0.605**

(d) what is the average amount of time that an entering customer spends in the system?

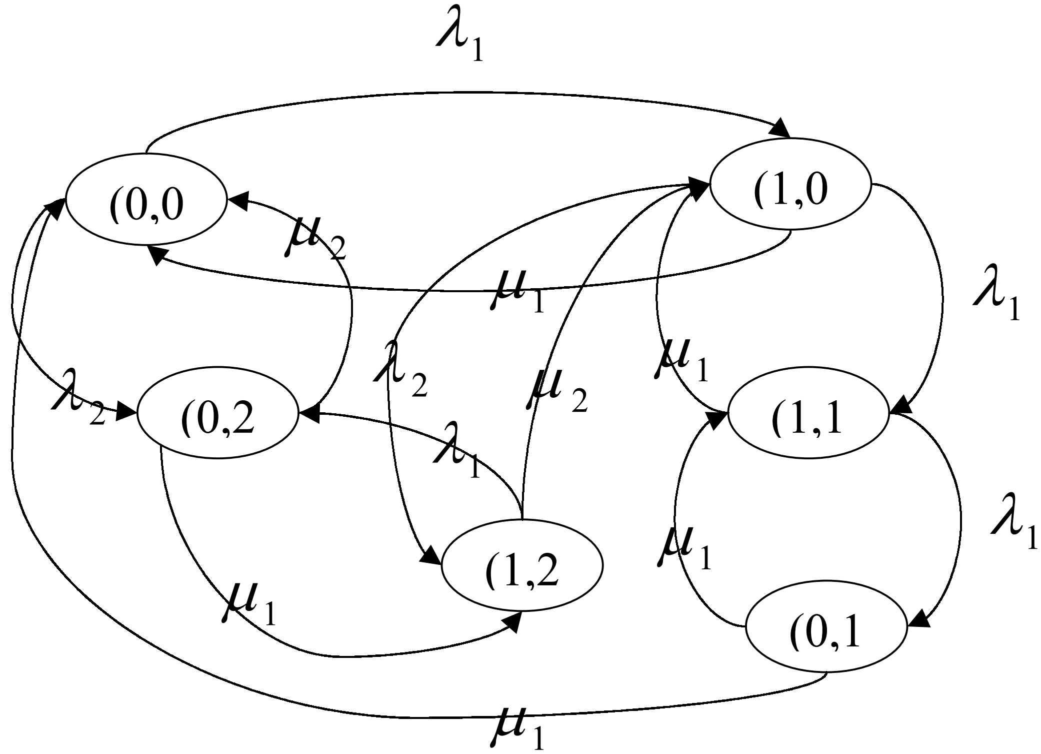
$w = l / [\lambda*(\pi_{(0,0)} + \pi_{(0,1)})] = 0.605 / [2*(0.44+0.22)] = 0.605/1.32 =$ **0.4583 hours**

3. There are two types of customers. Type i customers arrive in accordance with independent Poisson processes with respective rate λ_1 and λ_2. There are two servers. A type 1 arrival will enter service with server 1 if that service is free; if server 1 is busy and server 2 is free, then the type 1 arrival will enter service with server 2. If both servers are busy, then the type 1 arrival will go away. A type 2 customer can only be served by server 2; if server 2 is free when a type 2 customer arrives, then the customer enters service with that server. If server 2 is busy when a type 2 customer arrives, then that customer goes away. Once a customer is served by either server, he departs the system. Service times at server i are exponential with rate μ_i, i=1,2.

Suppose we want to find the average number of customers in the system.

(a) Define the states.

State definition = (type of customer on server 1, type of customer on server 2)
State space = {(0,0), (1,0), (0,1), (0,2), (1,1), (1,2)}



(b) Give the balance equations. Do not attempt to solve them.

$(\pi_{(0,1)} + \pi_{(1,0)})*\mu_1 + \pi_{(0,2)} \mu_2 = \pi_{(0,0)} *(\lambda_1 + \lambda_2)$

$\pi_{(0,0)} \lambda_1 + \pi_{(1,2)} \mu_2 + \pi_{(1,1)} \mu_1 = \pi_{(1,0)} *(\lambda_1 + \lambda_2 + \mu_1)$

$(\pi_{(1,0)} + \pi_{(0,1)})*\lambda_1 = \pi_{(1,1)} *(2 \mu_1)$

$\pi_{(1,1)} \mu_1 = \pi_{(0,1)} *(\lambda_1 + \mu_1)$

$\pi_{(0,2)} *\lambda_1 + \pi_{(1,0)} *\lambda_2 = \pi_{(1,2)} *(\mu_1 + \mu_2)$

$\pi_{(0,0)} *\lambda_2 + \pi_{(1,2)} * \mu_1 = \pi_{(1,2)} *(\lambda_1 + \mu_2)$

In terms of long-run probabilities, what is

(c) the average number of customers in the system?

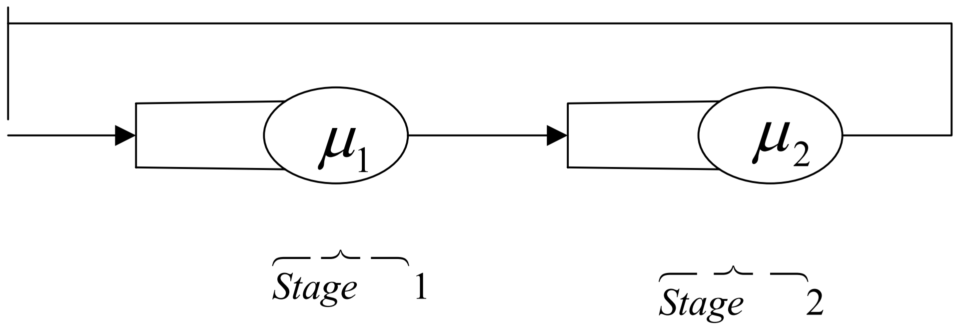
$l = 0*\pi_{(0,0)} + 1*(\pi_{(0,1)} + \pi_{(1,0)} + \pi_{(0,2)}) + 2*(\pi_{(1,1)} + \pi_{(1,2)})$

(d) the average time a customer spends in the system?

$w = l / [\, \lambda_1*(1 - \pi_{(1,1)} - \pi_{(1,2)}) + \lambda_2*(1 - \pi_{(1,1)} - \pi_{(1,2)} - \pi_{(0,1)} - \pi_{(0,2)}) \,]$

where $l = \pi_{(0,1)} + \pi_{(1,0)} + \pi_{(0,2)} + 2*(\pi_{(1,1)} + \pi_{(1,2)})$ from (c).

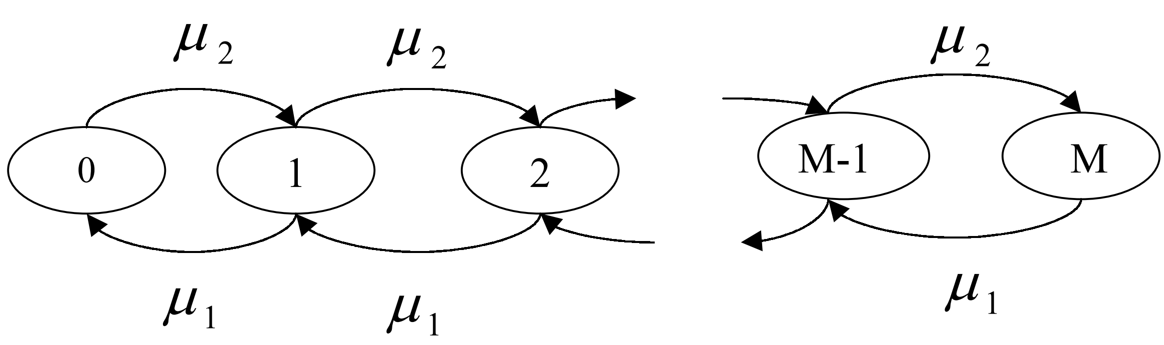
4. Consider a *cyclic queue* in which *M* customers circulate around through two queuing facilities as shown below.



Both servers are of the exponential type with rates μ_1 and μ_2, respectively. Suppose we are interested in the number of customers in each facility. Draw the state-transition-rate diagram.

Let's define the state of the system as the number of customers in the first queuing facility. For example, state 0 will correspond to having no customers in the first queuing facility but M customers in the second queuing facility; similarly, state 1 will correspond to having 1 customer in the first queuing facility and M-1 customers in the second queuing facility.

Then, the transition rate diagram turns out to be the following:



It turns out that this is exactly the state-transition-rate diagram of the finite storage system M/M/1/k with k = M, $\lambda = \mu_2$, and $\mu = \mu_1$.