IE 315
Spring 2001
Final Examination
B. Deler

This examination is open book and notes. You will have **2 hours**. Show all work, answer only the questions asked, and if you need additional space use the back of the page. **You only have to give a numerical answer if you are specifically asked for one**. Good luck!

1. The following exercises ask you to perform basic calculations. In each case a numerical answer is expected. **To obtain any partial credit you must show how you obtained the answer even if you used a calculator to actually compute it.**

   (a) For a Poisson arrival process $\{Y_t; t \geq 0\}$ with arrival rate $\lambda_Y = 2$ per hour, and a second independent Poisson process $\{X_t; t \geq 0\}$ with arrival rate $\lambda_X = 5$ per hour, compute the following:

      i. (2 points) $\Pr\{Y_9 - Y_7 = 5 | Y_7 = 3\}$

      ii. (2 points) $\Pr\{Y_9 = 12 | Y_7 = 10\}$

iii. (2 points) $\Pr\{Y_2 + X_2 = 6 | Y_2 = 2\}$

iv. (2 points) $E\{Y_2 + X_2\}$

(b) (6 points) For a Markov chain $\{S_n; n = 0, 1, \ldots\}$ with state space $M = \{1, 2\}$ and one-step transition matrix

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

compute the following:

i. (2 points) $\Pr\{S_5 = 2 | S_4 = 2, S_3 = 1\}$

ii. (2 points) $\Pr\{S_3 = 2 | S_1 = 1\}$

iii. (2 points) $\Pr\{S_2 = 1, S_1 \neq 1 | S_0 = 2\}$

(c) (6 points) Consider a single-server queueing process $\{Y_t; t \geq 0\}$ with state space $M = \{0, 1, 2, 3\}$ and arrival rate given by

$$\lambda_i = \begin{cases} 1/2, & i = 0, 1, 2, \\ 0, & i = 3, 4, \ldots \end{cases}$$

It takes the server a fixed time to serve a customer with rate

$$\mu_i = \begin{cases} 2, & i = 1, 2, 3, \\ 0, & i = 4, 5, \ldots \end{cases}$$

Assuming a Markovian queueing model of this system, compute the following:

i. (4 points) Long-run probability of having only one customer in the system:

ii. (4 points) Rate that an arriving customer can enter the system:

iii. (4 points) Average time a customer spends in the queue of the system:

iv. (4 points) The service time for this single-server system is not actually well-approximated by an exponential distribution. Compute the correction factor needed to calculate the average time a customer spends in the queue of the system (w_q) more accurately; a numerical answer is expected here (but you do not have to compute w_q).

2. For each system described below, state what type of model might be capable of answering the question asked. Your choices are **Poisson arrival process; discrete-time Markov chain; or queueing process**. You may write P, MC or Q for short. **If more than one model seems appropriate, select the one that would be the easiest to parameterize and use to answer the question**.

   (a) (5 points) The production of printed circuit boards (PCBs) consists of a sequence of manufacturing operations, each operation followed by a test, tests followed by rework for the PCBs that fail the tests, and retesting for reworked PCBs. When PCBs fail a retest they are discarded. If the manufacturer releases a lot of N orders into the system they would like to know the distribution of the number of PCBs that will be reworked, that ultimately pass all tests, and that are discarded.

   (b) (5 points) The Office of Unemployed CAS Students takes information on student skills and tries to match them with employers. Students come to the office, take a form from a box, fill out the form and return it to another box. The office would like to know how many tables they should put out for students filling out forms.

   (c) (5 points) Milton Bradley toy company designs simple children's games in which a player rolls a die, moves their playing piece around a board and follows the directions on the board. For a new game, Milton Bradley would like to determine how many rolls are required, on average, to complete the game.

   (d) (5 points) One way to determine the number of fish in a lake is to catch fish, tag them, and count how many fish have to be caught until one is caught that has already been tagged. The Wildlife Service needs a model to predict the number of fish that must be caught until this event occurs.

(e) (5 points) Each time a web page is accessed it is called a "hit." Software X Company has a web page that advertises and sells their software. Not every hit results in a sale. Software X would like to model the distribution of the number of sales each day via the web page.

3. Consider a shoeshine shop consisting of two chairs. Suppose that an entering customer first will go to chair 1. When his work is completed in chair 1, he will go to either chair 2 if that chair is empty or else wait in chair 1 until chair 2 becomes empty. Suppose that a potential customer will enter this shop as long as chair 1 is empty. (Thus, for instance, a potential customer might enter even if there is a customer in chair 2).

   If we suppose that potential customers arrive in accordance with a Poisson process at rate 1 per hour, and that the service times for the two chairs are independent and have respective exponential rates of 2 per hour and 1 per hour, *then*

   (a) (10 points) develop a Markovian queueing model (give a transition rate diagram) for the shoeshine shop that is capable of answering the questions below. Be sure to define your state space and give the balance equations, but do not attempt to solve them. **Hint:** Consider a person in chair 1 whose work is completed, but waits in chair 1 until chair 2 becomes empty. This person, though not being served, is blocking potential arrivals from entering the system.

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In terms of the long-run probabilities, what is

(b) (5 points) the proportion of potential customers entering the system?

(c) (5 points) the mean number of customers in the system?

(d) (5 points) the average amount of time that an entering customer spends in the system?

4. An industrial engineer observes a machine once every 3 months. The machine is either functional (U) or down (D). A sequence observed over 3 years was:

U U U D U D D U U D U U

The IE believes that this process can be modeled as a Markov chain.

(a) (6 points) Define the state space and time index for such a Markov chain model.

(b) (6 points) Estimate the one-step transition matrix for the model from the given data. Carefully show your calculations; a numerical answer is expected here.

(c) (6 points) What would you compute to determine the long-run fraction of time that the machine is down? Be specific and **set up all calculations (although you do not need to complete them).**

(d) (6 points) If the machine is currently functional, what would you compute to determine the probability that the machine is first observed to be down exactly 1 year from now? Be specific and **set up all calculations (although you do not need to complete them).**

(e) (6 points) What would you compute to determine the average length of time the machine remains down when it goes down? Be specific and **set up all calculations (although you do not need to complete them).**