Online Companion to

"Modeling and Generating Multivariate Time-Series Input Processes Using a Vector Autoregressive Technique"

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This document states and proves the distributional properties that make the Gaussian vector autoregressive process a perfect match to be used as a base process in the VARTA framework. Please note that these properties are typically stated without proof in time-series books. However, the proofs below are useful for completeness.

Distributional Properties of the VARTA Base Process

The objective is to establish Theorem 1 of Biller and Nelson [2002] for a stable p^{th}-order vector autoregressive process (the VAR_k(p) model) by using the fact that the $\text{VAR}_{kp}(1)$ model provides a state-space representation for the VAR_k(p) model.

Lemma 1 *Let $\mathbf{Z}_t$ denote a stable[1] p^{th}-order vector autoregressive process defined by*

$$\mathbf{Z}_t = \boldsymbol{\alpha}_1 \mathbf{Z}_{t-1} + \boldsymbol{\alpha}_2 \mathbf{Z}_{t-2} + \cdots + \boldsymbol{\alpha}_p \mathbf{Z}_{t-p} + \mathbf{u}_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

where $\mathbf{Z}_t = (Z_{1,t}, Z_{2,t}, \ldots, Z_{k,t})'$ is a $(k \times 1)$ random vector of the observations recorded at time t, the $\boldsymbol{\alpha}_i$, $i = 1, 2, \ldots, p$, are fixed $(k \times k)$ autoregressive coefficient matrices, $\mathbf{u}_t = (u_{1,t}, u_{2,t}, \cdots, u_{k,t})'$ is a k-dimensional white noise vector with a $(k \times k)$ positive definite covariance matrix $\boldsymbol{\Sigma}_u$, and

$$\text{E}[\mathbf{u}_t] = \mathbf{0}_{(k\times 1)} \ \textit{and} \ \text{E}[\mathbf{u}_t \mathbf{u}'_{t-h}] = \begin{cases} \boldsymbol{\Sigma}_u & \textit{if} \quad h = 0, \\ \mathbf{0}_{(k\times k)} & \textit{otherwise.} \end{cases}$$

Define the autocovariance structure of the $\mathbf{Z}_t$ process by

$$\boldsymbol{\Sigma}_{\mathbf{Z}} = \begin{pmatrix} \boldsymbol{\Sigma}_Z(0) & \boldsymbol{\Sigma}_Z(1) & \ldots & \boldsymbol{\Sigma}_Z(p-2) & \boldsymbol{\Sigma}_Z(p-1) \\ \boldsymbol{\Sigma}'_Z(1) & \boldsymbol{\Sigma}_Z(0) & \ldots & \boldsymbol{\Sigma}_Z(p-3) & \boldsymbol{\Sigma}_Z(p-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{\Sigma}'_Z(p-1) & \boldsymbol{\Sigma}'_Z(p-2) & \ldots & \boldsymbol{\Sigma}'_Z(1) & \boldsymbol{\Sigma}_Z(0) \end{pmatrix}_{(kp\times kp)},$$

where $\boldsymbol{\Sigma}_Z(h)$ corresponds to the second moment of the $\mathbf{Z}_t$ process at lag h, and assume that the autocovariance matrix $\boldsymbol{\Sigma}_{\mathbf{Z}}$ is positive definite.

If $\mathbf{u}_t$ is Gaussian white noise (that is, $\mathbf{u}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_u)$), then $\mathbf{Z}_t$ is a Gaussian process (that is, $\mathbf{Z}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_Z(0))$).

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[1] A stable vector autoregressive process $\mathbf{Z}_t$, $t = 0, \pm 1, \pm 2, \ldots$, is stationary [Lütkepohl 1993; Proposition 2.1].

Proof. The key to proving Lemma 1 is that any p^{th}-order vector autoregressive process can be written in a first-order vector autoregressive form. More precisely, if $\mathbf{Z}_t$ is a VAR$_k(p)$ model defined as in (1), then an equivalent kp-dimensional VAR(1) model,

$$\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \mathbf{U}_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

can be defined where

$$\mathbf{Y}_t = \begin{pmatrix} \mathbf{Z}_t \\ \mathbf{Z}_{t-1} \\ \mathbf{Z}_{t-2} \\ \vdots \\ \mathbf{Z}_{t-p+1} \end{pmatrix}_{kp\times 1} \quad \mathbf{A} = \begin{pmatrix} \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \ldots & \boldsymbol{\alpha}_{p-1} & \boldsymbol{\alpha}_p \\ \mathbf{I}_{k\times k} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k\times k} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_{k\times k} & \mathbf{0} \end{pmatrix}_{kp\times kp} \quad \mathbf{U}_t = \begin{pmatrix} \mathbf{u}_t \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}_{kp\times 1}.$$

We define $\boldsymbol{\Sigma}_{\mathbf{Y}}$ analogously to $\boldsymbol{\Sigma}_{\mathbf{Z}}$. By construction of the $\mathbf{Y}_t$ process, $\boldsymbol{\Sigma}_{\mathbf{Y}}(0) = \boldsymbol{\Sigma}_{\mathbf{Z}}$, and as $\mathbf{Y}_t$ is a first-order vector autoregressive process, $\boldsymbol{\Sigma}_{\mathbf{Y}}(0) = \boldsymbol{\Sigma}_{\mathbf{Y}}$. Thus, $\boldsymbol{\Sigma}_{\mathbf{Y}}$ is a positive definite matrix.

Starting at $t = 1$, we get

$$\begin{aligned} \mathbf{Y}_1 &= \mathbf{A}\mathbf{Y}_0 + \mathbf{U}_1 \\ \mathbf{Y}_2 &= \mathbf{A}\mathbf{Y}_1 + \mathbf{U}_2 = \mathbf{A}\left(\mathbf{A}\mathbf{Y}_0 + \mathbf{U}_1\right) + \mathbf{U}_2 = \mathbf{A}^2\,\mathbf{Y}_0 + \mathbf{A}\mathbf{U}_1 + \mathbf{U}_2 \\ &\vdots \\ \mathbf{Y}_t &= \mathbf{A}^t\mathbf{Y}_0 + \sum_{i=0}^{t-1}\mathbf{A}^i\mathbf{U}_{t-i}. \end{aligned} \tag{3}$$

Hence, the vectors $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_t$ are determined by $\mathbf{Y}_0, \mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_t$. Further, the joint distribution of $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_t$ is determined by the joint distribution of $\mathbf{Y}_0, \mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_t$.

In (1), it is assumed that the process has been started in the infinite past. Therefore, from (3) we have

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{A}\mathbf{Y}_{t-1} + \mathbf{U}_t \\ &= \mathbf{A}^{j+1}\mathbf{Y}_{t-j-1} + \sum_{i=0}^{j}\mathbf{A}^i\mathbf{U}_{t-i}, \quad \text{for any } \ j \geq 1. \end{aligned}$$

Since the process (2) is stable, its reverse characteristic polynomial has no roots in or on the complex unit circle. By Rule 7 of Appendix A.6 in Lütkepohl [1993], this is equivalent to the condition that all eigenvalues of $\mathbf{A}$ have modulus less than 1. This makes the sequence $\{\mathbf{A}^i;\ i = 0,1,2,\ldots\}$ absolutely summable (Lütkepohl [1993]; Appendix A, Section A.9.1). Also notice that $\mathrm{E}[\mathbf{U}_t\mathbf{U}_t'] \leq \mathbf{M}$ for some finite constant matrix $\mathbf{M}$. Hence, the infinite sum $\sum_{i=1}^{\infty}\mathbf{A}^i\mathbf{U}_{t-i}$, which is the infinite-order moving average representation of the corresponding $\mathrm{VAR}_{kp}(1)$ model, exists in mean square (Lütkepohl [1993]; Appendix C, Proposition C.7). Furthermore, $\mathbf{A}^{j+1}$ converges to zero rapidly as $j \to \infty$ and since the process $\mathbf{Y}_t$ is stable, thus, stationary [Lütkepohl 1993], the distribution of $\mathbf{Y}_t$ is the same for all t. Therefore, it follows from Slutsky's Theorem that the term $\mathbf{A}^{j+1}\mathbf{Y}_{t-j-1}$ vanishes in the limit [Lehmann 1998]. Hence, if all eigenvalues of $\mathbf{A}$ have modulus less than 1, then[2]

$$\lim_{j\to\infty}\left(\mathbf{A}^{j+1}\mathbf{Y}_{t-j-1} + \sum_{i=0}^{j}\mathbf{A}^i\mathbf{U}_{t-i}\right) \stackrel{\text{q.m.}}{=} \sum_{i=0}^{\infty}\mathbf{A}^i\mathbf{U}_{t-i}, \quad t = 0, \pm1, \pm2, \ldots.$$

Since convergence in quadratic mean implies convergence in distribution by Proposition C.1 of Lütkepohl [1993], we also have

$$\mathbf{Y}_t \stackrel{\text{d}}{=} \sum_{i=0}^{\infty}\mathbf{A}^i\mathbf{U}_{t-i}, \quad t = 0, \pm1, \pm2, \ldots. \tag{4}$$

Thus, by saying that $\mathbf{Y}_t$ is the $\mathrm{VAR}_{kp}(1)$ process (2), we mean that $\mathbf{Y}_t$ is the well-defined stochastic process given by (4). Notice that the distributions and joint distributions of the $\mathbf{Y}_t$ are uniquely determined by the distributions of the $\mathbf{U}_t$ process. It follows from Proposition C.8 of Lütkepohl [1993] that the first and second moments of the $\mathbf{Y}_t$ process are given by

$$\mathrm{E}[\mathbf{Y}_t] = \lim_{n\to\infty}\sum_{i=0}^{n}\mathbf{A}^i\mathrm{E}[\mathbf{U}_{t-i}] = \lim_{n\to\infty}\sum_{i=0}^{n}\mathbf{A}^i\mathbf{0}_{kp\times1} = \mathbf{0}_{kp\times1}, \quad t = 0, \pm1, \pm2, \ldots,$$

[2]Suppose $\{x_\ell = (x_{1,\ell}, \ldots, x_{k,\ell})'\}$, $\ell = 1, 2, 3, \ldots$, is a sequence of k-dimensional random vectors and $x = (x_1, \ldots, x_k)$ is a k-dimensional random vector.

$x_\ell \stackrel{\text{q.m.}}{\to} x$ if $\lim\ \mathrm{E}\left[(x_\ell - x)'(x_\ell - x)\right] = 0.$

$x_\ell \stackrel{\text{d}}{\to} x$ if $\lim F_\ell(c) = F(c)$ for all continuity points of the cumulative distribution function F.

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{Y}} &= \mathrm{E}\left[\left(\mathbf{Y}_t - \mathrm{E}\left[\mathbf{Y}_t\right]\right)\left(\mathbf{Y}_{t-h} - \mathrm{E}\left[\mathbf{Y}_{t-h}\right]\right)'\right] = \lim_{n\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}\mathbf{A}^i\mathrm{E}\left[\mathbf{U}_{t-i}\mathbf{U}_{t-j}'\right](\mathbf{A}^j)' \\ &= \lim_{n\to\infty}\sum_{i=0}^{n}\mathbf{A}^i\boldsymbol{\Sigma}_{\mathbf{U}}(\mathbf{A}^i)' \\ &= \sum_{i=0}^{\infty}\mathbf{A}^i\boldsymbol{\Sigma}_{\mathbf{U}}(\mathbf{A}^i)', \end{aligned} \tag{5}$$

because $\mathrm{E}\left[\mathbf{U}_t\mathbf{U}_s'\right] = \mathbf{0}_{kp\times 1}$ for $s \neq t$ and $\mathrm{E}\left[\mathbf{U}_t\mathbf{U}_t'\right] = \boldsymbol{\Sigma}_{\mathbf{U}}$ for all t. Now, it follows from standard results for finite sums of independent normal random vectors, e.g., Theorem 3.3.3 of Tong [1990], that

$$\mathbf{W}_{t,n} \equiv \sum_{i=0}^{n}\mathbf{A}^i\mathbf{U}_{t-i} \sim N_{kp}\left[\mathbf{0}_{kp\times 1}, \sum_{i=0}^{n}\mathbf{A}^i\boldsymbol{\Sigma}_{\mathbf{U}}(\mathbf{A}^i)'\right], \quad n = 1, 2, \ldots;$$

see, for example, Corollary 3.3.3 of Tong [1990]. In view of (5), we have that for any $kp \times 1$ real vector $\boldsymbol{\tau}$, the characteristic function of $\mathbf{W}_{t,n}$ satisfies

$$\begin{aligned}\psi_{\mathbf{W}_{t,n}}(\boldsymbol{\tau}) &\equiv \mathrm{E}\left\{\exp\left[\left(\sqrt{-1}\right)\boldsymbol{\tau}'\mathbf{W}_{t,n}\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\boldsymbol{\tau}'\left[\sum_{i=0}^{n}\mathbf{A}^i\boldsymbol{\Sigma}_{\mathbf{U}}(\mathbf{A}^i)'\right]\boldsymbol{\tau}\right\}\end{aligned}$$

and

$$\lim_{n\to\infty}\psi_{\mathbf{W}_{t,n}}(\boldsymbol{\tau}) = \exp\left\{-\frac{1}{2}\boldsymbol{\tau}'\boldsymbol{\Sigma}_{\mathbf{Y}}\boldsymbol{\tau}\right\} \quad \text{for every } \boldsymbol{\tau} \in \Re^{kp},$$

which is the characteristic function for the kp-variate normal distribution with mean $\mathbf{0}_{kp\times 1}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{Y}}$; see Theorem 3.2.3 of Tong [1990]. It follows from the continuity theorem for characteristic functions [Billingsley 1995; Theorem 26.3] together with (5) that as $n \to \infty$

$$\mathbf{W}_{t,n} \xrightarrow{\mathrm{d}} N_{kp}\left[\mathbf{0}_{kp\times 1}, \boldsymbol{\Sigma}_{\mathbf{Y}}\right] \text{ and } \mathbf{W}_{t,n} \xrightarrow{\mathrm{d}} \mathbf{Y}_t \text{ implies } \mathbf{Y}_t \sim N_{kp}\left[\mathbf{0}_{kp\times 1}, \boldsymbol{\Sigma}_{\mathbf{Y}}\right].$$

Since $\mathbf{Y}_t$ is a Gaussian process, and it is equivalently written as

$$\left(\mathbf{Z}_t', \mathbf{Z}_{t-1}', \mathbf{Z}_{t-2}', \ldots, \mathbf{Z}_{t-p+1}'\right)',$$

we can conclude that the process $\mathbf{Z}_t$ is also a Gaussian process (that is, $\mathbf{Z}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_Z(0))$) from Theorem 2.2.C of Rencher [1998]. ∎

Remark 1 *Let $\mathbf{Z}_t$ denote a stable p^{th}-order vector autoregressive process defined as in (1) with Gaussian white noise $\mathbf{u}_t$ and a positive definite autocovariance matrix $\mathbf{\Sigma}_{\mathbf{Z}}$. In Lemma 1, it is shown that the random variable*

$$(\mathbf{Z}'_t, \mathbf{Z}'_{t-1}, \mathbf{Z}'_{t-2}, \ldots, \mathbf{Z}'_{t-p+1})'$$

has a nonsingular kp-dimensional multivariate normal distribution. This result is used in Biller and Nelson [2002] to generate realizations from the corresponding vector autoregressive process.

Theorem 1 *Let $\mathbf{Z}_t$ denote a stable p^{th}-order vector autoregressive process defined as in Lemma 1. The random variable $(Z_{i,t}, Z_{j,t-h})'$, for $i, j = 1, 2, \ldots, k$ and $h = 0, 1, 2, \ldots$ (except $i = j$ when $h = 0$), has a nonsingular bivariate normal distribution with density function*

$$f(\tilde{\mathbf{z}}; \mathbf{\Sigma}_2) = \frac{1}{2\pi |\mathbf{\Sigma}_2|^{\frac{1}{2}}} e^{-\frac{1}{2}\tilde{\mathbf{z}}' \mathbf{\Sigma}_2^{-1} \tilde{\mathbf{z}}}, \ \ \tilde{\mathbf{z}} \in \Re^2, \ \ \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & \rho_{\mathbf{Z}}(i,j,h) \\ \rho_{\mathbf{Z}}(i,j,h) & 1 \end{pmatrix}_{(2\times 2)}.$$

Proof. Define $\mathbf{Y}_t$, $\mathbf{U}_t$, $\mathbf{A}$, and $\mathbf{\Sigma}_{\mathbf{Y}}$ as in Lemma 1. It has been shown that $\mathbf{Y}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{Y}})$ and $\mathbf{Y}_{t-h} \sim N(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{Y}})$. Next, we will find the conditional distribution of $\mathbf{Y}_t$ given $\mathbf{Y}_{t-h}$: If $\mathbf{Y}_{t-h} = \mathbf{y}_{t-h}$, then it follows from (3) of Lemma 1 that $\mathbf{Y}_t = \mathbf{A}^h \mathbf{y}_{t-h} + \sum_{i=0}^{h-1} \mathbf{A}^i \mathbf{U}_{t-i}$. From the definition of multivariate normality given by Rao [1973], it follows that $\mathbf{A}^i \mathbf{U}_{t-i} \sim N\left(\mathbf{0}, \mathbf{A}^i \mathbf{\Sigma}_U (\mathbf{A}^i)'\right)$ and $\sum_{i=0}^{h-1} \mathbf{A}^i \mathbf{U}_{t+h-i} \sim N\left(\mathbf{0}, \sum_{i=0}^{h-1} \mathbf{A}^i \mathbf{\Sigma}_U (\mathbf{A}^i)'\right)$. Then, conditioned on $\mathbf{Y}_{t-h} = \mathbf{y}_{t-h}$, $\mathbf{Y}_t$ has a $N\left(\mathbf{A}^h \mathbf{y}_{t-h}, \varsigma_{\mathbf{Y},h}\right)$ distribution, where $\varsigma_{\mathbf{Y},h} = \sum_{i=0}^{h-1} \mathbf{A}^i \mathbf{\Sigma}_U (\mathbf{A}^i)'$ corresponds to a positive semi-definite matrix. Thus, the probability density function of $\mathbf{Y}_{t-h}$, and the conditional distribution of $\mathbf{Y}_t$, can be written as

$$f(\mathbf{y}_{t-h}) = \frac{1}{(2\pi)^{\frac{kp}{2}} |\mathbf{\Sigma}_{\mathbf{Y}}|^{\frac{1}{2}}} \, e^{-\frac{1}{2}\mathbf{y}'_{t-h} \varsigma_{\mathbf{Y}}^{-1} \mathbf{y}_{t-h}}, \ \ \mathbf{y}_{t-h} \in \Re^{kp}$$

and

$$f(\mathbf{y}_t|\mathbf{y}_{t-h}) = \frac{1}{(2\pi)^{\frac{kp}{2}}|\varsigma_{\mathbf{Y},h}|^{\frac{1}{2}}} \; e^{-\frac{1}{2}(\mathbf{y}_t - \mathbf{A}^h \mathbf{y}_{t-h})' \varsigma^{-1}_{\mathbf{Y},h} (\mathbf{y}_t - \mathbf{A}^h \mathbf{y}_{t-h})}, \; \mathbf{y}_t, \mathbf{y}_{t-h} \in \Re^{kp},$$

respectively. Following Theorem 6 of Rohatgi [1976],

$$\begin{aligned} f(\mathbf{y}_t, \mathbf{y}_{t-h}) &= f(\mathbf{y}_{t-h}) f(\mathbf{y}_t|\mathbf{y}_{t-h}) \\ &= \frac{1}{(2\pi)^{\frac{kp}{2}}|\mathbf{\Sigma_Y}|^{\frac{1}{2}}} \; e^{-\frac{1}{2}\mathbf{y}'_{t-h}\varsigma^{-1}_{\mathbf{Y}}\mathbf{y}_{t-h}} \;\; \frac{1}{(2\pi)^{\frac{kp}{2}}|\varsigma_{\mathbf{Y},h}|^{\frac{1}{2}}} \; e^{-\frac{1}{2}(\mathbf{y}_t - \mathbf{A}^h \mathbf{y}_{t-h})' \varsigma^{-1}_{\mathbf{Y},h} (\mathbf{y}_t - \mathbf{A}^h \mathbf{y}_{t-h})}. \end{aligned} \quad (6)$$

The algebraic simplification of (6) results in the density function given by

$$f(\mathbf{y}_t, \mathbf{y}_{t-h}) = \frac{1}{(2\pi)^{kp}|\mathbf{\Sigma}_{2kp}|^{\frac{1}{2}}} e^{-\frac{1}{2}\tilde{\mathbf{y}}'\mathbf{\Sigma}^{-1}_{2kp}\tilde{\mathbf{y}}}, \;\; \tilde{\mathbf{y}} = (\mathbf{y}'_t, \mathbf{y}'_{t-h})' \in \Re^{2kp}, \mathbf{\Sigma}_{2kp} = \begin{pmatrix} \mathbf{\Sigma}_Y(0) & \mathbf{\Sigma}_Y(h) \\ \mathbf{\Sigma}_Y(h)' & \mathbf{\Sigma}_Y(0) \end{pmatrix}_{2kp \times 2kp}.$$

Hence, the random variable $\left(\mathbf{Y}'_t, \mathbf{Y}'_{t-h}\right)'$ has a $2kp$-dimensional multivariate normal distribution with a positive semi-definite autocovariance matrix $\mathbf{\Sigma}_{2kp}$ [Tong 1990]. Notice that the random variable $(\mathbf{Y}'_t, \mathbf{Y}'_{t-h})'$ can be equivalently written as

$$(Z_{1,t}, \cdots, Z_{k,t}, \cdots, Z_{1,t-p+1}, \cdots, Z_{k,t-p+1}, Z_{1,t-h}, \cdots, Z_{k,t-h}, \cdots, Z_{1,t-h-p+1}, \cdots, Z_{k,t-h-p+1})'.$$

Thus, the random variable $(Z_{i,t}, Z_{j,t-h})'$, for $i, j = 1, 2, \ldots, k$ and $h = 0, 1, \ldots$ has a bivariate normal distribution (Rencher [1998]; Theorem 2.2.C) and it is nonsingular except $i = j$ when $h = 0$ as the autocovariance matrix $\mathbf{\Sigma_Z}$ is assumed to be a positive definite matrix. ∎

Remark 2 *In Lemma 1, it has been shown that the random variable* $(\mathbf{Z}'_t, \mathbf{Z}'_{t-1}, \mathbf{Z}'_{t-2}, \ldots, \mathbf{Z}'_{t-p+1})'$ *has a nonsingular kp-dimensional multivariate normal distribution when* $\mathbf{Z}_t$ *corresponds to a stable* p^{th}*-order vector autoregressive process with a positive definite autocovariance matrix* $\mathbf{\Sigma_Z}$*. Following a discussion similar to the one in Theorem 1, we could conclude that any pair* $(Z_{i,t}, Z_{j,t-h})'$*, for* $i, j = 1, 2, \ldots, k$ *and* $h = 0, 1, \ldots, p-1$ *(except* $i = j$ *when* $h = 0$*) has a nonsingular bivariate normal distribution. However, in order to solve the correlation-matching problem [Biller and Nelson 2002], the statement above should hold also for* $h = p$*, which we demonstrated in Theorem 1.*

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