Intensionality, Invariance, and Univalence

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Über die mathematische Logik

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Bekanntlich wurde die Logik als Wissenschaft von Aristoteles gegründet. Alle kennen den Aristotelischen Syllogismus. Die syllogistischen Figuren von Aristoteles bildeten den Hauptinhalt der Logik während des ganzen Mittelalters. Kant soll einmal gesagt haben, dass die Logik die einzige Wissenschaft sei, die seit dem Altertum gar keine Fortschritte gemacht hatte. Dies war vielleicht damals richtig; heute ist es aber nicht mehr so.

In der letzten Zeit ist nämlich der sogenannte Logikkalkul oder die mathematische Logik entwickelt worden, eine Theorie, welche über die Aristotelische Logik weit hinausgekommen ist. Sie ist von Mathematikern entwickelt worden; die Philosophen von Fach haben sich sehr wenig dafür interessiert, vermutlich weil sie diese Theorie zu mathematisch gefunden haben. Anderseits haben auch die meisten Mathematiker sich sehr wenig dafür interessiert, weil sie die Theorie zu philosophisch gefunden haben.

1. Frege's puzzle about equality

What is the meaning of a word or statement?

Frege begins by considering equality:

> *Equality gives rise to challenging questions which are not altogether easy to answer. Is it a relation? a relation between objects? or between names or signs of objects? if we were to regard equality as a relation between that which the names 'a' and 'b' designate, it would seem that $a = b$ could not differ from $a = a$ (provided $a = b$ is true). A relation would thereby be expressed of a thing to itself, and indeed one in which each thing stands to itself but to no other thing. What is intended to be said by $a = b$ seems to be that the signs or names 'a' and 'b' designate the same thing, so that those signs themselves would be under discussion; a relation between them would be asserted.*

Frege, Über Sinn und Bedeuting, 1892

1. Frege's notions of meaning and sense

- He decides that it must be a relation between things, but that every expression (name, predicate, sentence) must have both a meaning (*Bedeutung*) and a sense (*Sinn*).
- The meaning is the thing denoted (*das Bezeichnete*).
- The sense is how the meaning is presented (*Art des Gegebenseins*).
- The meaning of a sentence is its "truth-value" (*Wahrheitswert*).

If our conjecture that the meaning of a sentence is its truth-value is correct, the latter must remain unchanged when part of the sentence is replaced by an expression with the same meaning but different sense. ... What else but the truth-value could be found, that belongs quite generally to every sentence ... and remains unchanged by substitutions of the kind in question?

1. Frege's notions of meaning and sense

> *If now the truth value of a sentence is its meaning, then on the one hand all true sentences have the same meaning and so, on the other hand, do all false sentences.*

This is the conclusion that I would like to avoid: that all true statements mean the same thing, namely True – at least for statements of logic and mathematics.

- For mathematical objects, the role of sense can be played by a presentation of the object.
- The meaning of a theorem should not depend on a choice of a presentation.
- But different theorems may mean something very different, even if both are true.

2. Martin-Löf's intensional type theory

The system of intensional type theory of Per Martin-Löf uses two different kinds of equality.

- $a \equiv b$ *definitional* or *judgmental* equality,
- $a = b$ *propositional* or *typal* equality, also written $\mathrm{Id}(a, b)$.

In this system, $(a \equiv b)$ implies $(a = b)$, but not conversely.
So these can be used to model Frege's distinction:

- $a \equiv b$ says a and b have the same **sense**:
  roughly "same syntactic presentation in the system".
- $a = b$ says a and b have the same **meaning**:
  roughly "same things being reasoned about by the system".

For example, on the type $\mathbb{N}$ of natural numbers we may have

$$\pi_1\langle n, m\rangle \equiv n\,,$$

but only

$$m + n = n + m\,.$$

2. Martin-Löf's intensional type theory

The basic operations of type theory are as follows:

$$0,\ 1,\ A+B,\ A\times B,\ A\to B,\ \Sigma_{x:A}B(x),\ \Pi_{x:A}B(x),$$

These correspond to the logical operations:

$$\bot,\ \top,\ p\vee q,\ p\wedge q,\ p\Rightarrow q,\ \exists x\, p(x),\ \forall x\, p(x).$$

- Unlike in predicate logic, where one is only concerned with entailment $p \vdash q$, in type theory one also has terms $x : A \vdash t : B$ which can be regarded as proofs, computations, witnesses, grounds of truth, etc.
- Provability $\vdash p$ is replaced by the existence of a term $\vdash t : A$.
- Thus the meaning of a type theoretic proposition A is not just true or false, but the collection of its "proofs" $\vdash t : A$.

3. Propositions as types

This is also called the Curry-Howard correspondence.

0	1	$A+B$	$A \times B$	$A \to B$	$\Sigma_{x:A} B(x)$	$\Pi_{x:A} B(x)$
$\bot$	T	$p \vee q$	$q \wedge q$	$p \Rightarrow q$	$\exists_x \, p(x)$	$\forall_x \, p(x)$

term : Type	proof : Proposition
$x : A$	assumption of A
$\langle a, b \rangle : A \times B$	$\wedge$-intro
$\lambda x.t(x) : A \to B$	$\Rightarrow$-intro
...	...

3. Propositions as types

There are, at first blush, two kinds of construction involved: constructions of proofs of some proposition and constructions of objects of some type. But I will argue that, from the point of view of foundations of mathematics, there is no difference between the two notions. A proposition may be regarded as a type of object, namely, the type of its proofs. Conversely, a type A may be regarded as a proposition, namely, the proposition whose proofs are the objects of type A. So a proposition A is true just in case there is an object of type A.

W.W. Tait
The law of excluded middle
and the axiom of choice (1994)

4. Equality types

- Under propositions as types, the meaning of a proposition is its not just its truth-value, but the collection of its proofs.
- This is already a better notion of meaning than just the truth-values derived from logical equivalence of propositions.
- But there is an even richer notion of meaning in mathematics, related to isomorphism of algebraic structures, equivalence of categories, etc.
- This can also be captured in type theory, using the **equality type**.

$$\mathsf{Id}_A(a, b)$$

4. Equality types

- For any type A and terms $a, b : A$, there is a type $\mathrm{Id}_A(a, b)$.
- For any A and $a : A$, there is a term $r(a) : \mathrm{Id}_A(a, a)$.
- Two terms $a \equiv b$ are always identified by a term $p : \mathrm{Id}_A(a, b)$,
- The elimination rule from $\mathrm{Id}_A(a, b)$ **does not** imply $a \equiv b$.

Thus different presentations a, b may mean the same thing.

- Moreover, every property $x : A \vdash P(x)$ of A objects respects this notion of meaning, in the sense that given a proof $p : \mathrm{Id}_A(a, b)$, and one $t : P(a)$, there is an associated one $p_* t : P(b)$.
- This is Frege's observation that the truth-value of $P(a)$ does not change when one substitutes a component by another one with the same meaning.
- But here we see that such a substitution also has an effect on the PAT meaning, i.e. the set of proofs.

5. The homotopy interpretation

More is actually true: the Id-type endows each type with a rich structure that is respected by all constructions.

Suppose we have terms of ascending identity types:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathsf{Id}_A(a,b) \\ \alpha,\ \beta &: \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,q) \\ \ldots &: \mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_{\ldots}}}(\ldots) \end{aligned}$$

Consider the interpretation:

$$\begin{array}{rcl} \text{Types} & \rightsquigarrow & \text{Spaces} \\ \text{Terms} & \rightsquigarrow & \text{Maps} \\ a : A & \rightsquigarrow & \text{Points } a : 1 \to A \\ p : \mathsf{Id}_A(a,b) & \rightsquigarrow & \text{Paths } p : a \sim b \\ \alpha : \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p,q) & \rightsquigarrow & \text{Homotopies } \alpha : p \approx q \\ & \vdots & \end{array}$$

6. The fundamental groupoid of a type

In topology, the points and paths in any space bear the structure of a **groupoid**: a category in which every arrow has an inverse.

$$
\begin{tikzcd}
a \arrow[loop left, "1_a"] \arrow[r, "p"'] \arrow[dr, bend right, "q \cdot p"'] & b \arrow[l, bend right, "p^{-1}"'] \arrow[d, "q"] \\
 & c
\end{tikzcd}
$$

In the same way the **terms** $a, b, c : X$ and **identity terms** $p : \mathsf{Id}_X(a, b)$ and $q : \mathsf{Id}_X(b, c)$ of any type X also form a groupoid.

6. The fundamental groupoid of a type

The usual laws of identity provide the **groupoid operations**:

$r : \mathsf{Id}(a, a)$	reflexivity	$a \longrightarrow a$
$s : \mathsf{Id}(a, b) \to \mathsf{Id}(b, a)$	symmetry	$\begin{tikzcd} a \arrow[r] & b \arrow[l, bend right] \end{tikzcd}$
$t : \mathsf{Id}(a, b) \times \mathsf{Id}(b, c) \to \mathsf{Id}(a, c)$	transitivity	$\begin{tikzcd} a \arrow[r] \arrow[dr, bend right] & b \arrow[d] \\ & c \end{tikzcd}$

As in topology, the **groupoid equations**:

$$
\begin{array}{cl}
p \cdot (q \cdot r) = (p \cdot q) \cdot r & \text{associativity} \\
p^{-1} \cdot p = 1 = p \cdot p^{-1} & \text{inverse} \\
1 \cdot p = p = p \cdot 1 & \text{unit}
\end{array}
$$

hold only "**up to homotopy**", i.e. up to higher Id-terms.

6. The fundamental ∞-groupoid of a type

In this way, each type in the system is endowed with the structure of an ∞-**groupoid**, with terms, identities between terms, identities between identities, ...

$$\begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=80, "p"{name=U}] \arrow[r, bend right=80, "q"'{name=D}] & b
\arrow[Rightarrow, from=U, to=D, bend right=40, "\alpha"', ""{name=A}]
\arrow[Rightarrow, from=U, to=D, bend left=40, "\beta", ""'{name=B}]
\arrow[from=A, to=B, phantom, "\overset{\vartheta}{\Rrightarrow}"]
\end{tikzcd}$$

Such structures also occur elsewhere in Mathematics, e.g. in Grothendieck's **homotopy hypothesis**.

7. The hierarchy of n-types

The universe of types is naturally stratified by the level at which the fundamental ∞-groupoid becomes trivial (if it ever does).

$$A \text{ is a } \textbf{proposition}: \quad \prod_{x,y:A} \mathsf{Id}_A(x,y), \ \textit{at most one term } a : A.$$

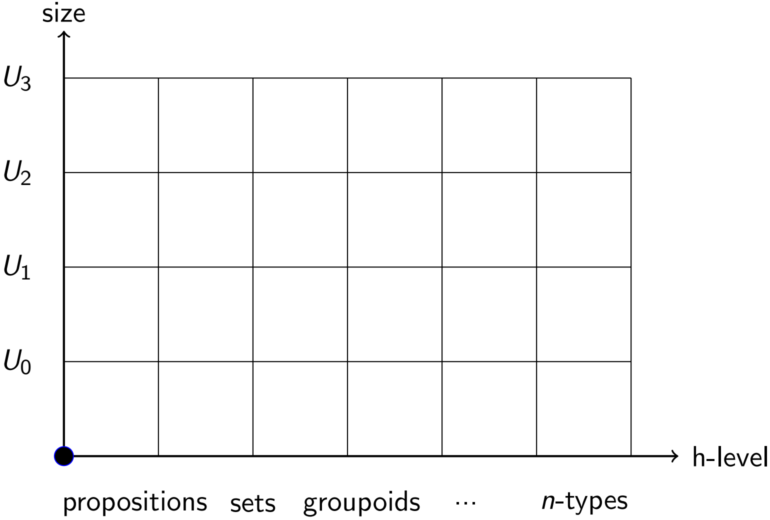
$$A \text{ is a } \textbf{set}: \quad \prod_{x,y:A} \mathsf{Prop}(\mathsf{Id}_A(x,y)), \ \textit{identity is a proposition.}$$

$$A \text{ is a 1-}\textbf{type}: \quad \prod_{x,y:A} \mathsf{Set}(\mathsf{Id}_A(x,y)), \ \textit{identity is a set.}$$

$$A \text{ is an } (n{+}1)\text{-}\textbf{type}: \quad \prod_{x,y:A} \mathsf{nType}(\mathsf{Id}_A(x,y)), \ \textit{identity is an } n\textit{-type.}$$

7. The hierarchy of n-types

This gives a new view of the mathematical universe in which types also have intrinsic higher-dimensional structure.



8. Equivalence of types

- The idea of n-types refines the propositions as types conception: types are now also *higher structures*, rather than mere *propositions*.
- There is a corresponding notion of *equivalence* $A \simeq B$ that is finer than mere *logical equivalence* $A \leftrightarrow B$.
- **Isomorphism** of types is defined as usual:

$$A \cong B \ =_{df} \ \sum_{f:A \to B} \sum_{g:B \to A} \mathsf{Id}(g \circ f, 1_A) \times \mathsf{Id}(f \circ g, 1_B)$$

  This is good for sets, but when A and B are n-types for $n > 0$ the condition "$g \circ f = 1_A$ and $f \circ g = 1_B$" is underspecified.

8. Equivalence of types

Equivalence of types:

$$A \simeq B \ =_{df} \sum_{\substack{f\,:\,A \to B \\ g\,:\,B \to A}} \ \sum_{\substack{\eta\,:\,g \circ f = 1_A \\ \varepsilon\,:\,f \circ g = 1_B}} \mathrm{Coh}(f, g, \eta, \varepsilon)$$

adds a further "coherence" condition $\mathrm{Coh}(f, g, \eta, \varepsilon)$ relating the chosen $\eta : \mathrm{Id}(g \circ f, 1_A)$ and $\varepsilon : \mathrm{Id}(f \circ g, 1_B)$.

Equivalence subsumes:

- *logical equivalence* $A \leftrightarrow B$ for propositions,
- *isomorphism* $A \cong B$ for sets,
- *categorical equivalence* $A \simeq B$ for 1-types (groupoids),

  ...
- *homotopy equivalence* $A \simeq B$ for spaces (∞-groupoids).

8. Equivalence of types

- For every family of types $x : A \vdash B(x)$, we saw that given a term $p : \mathrm{Id}_A(a, b)$, and one $t : B(a)$, there is an associated term $p_* t : B(b)$.
- In fact, the map $p_* : B(a) \to B(b)$ is always an equivalence of types $B(a) \simeq B(b)$.
- Thus equivalence $A \simeq B$ provides a finer notion of meaning than the truth-values derived from logical equivalence $A \leftrightarrow B$.
- This seems like a better answer to Frege's question "what else but the truth-value could be found ...?"

9. A Fregean test

> *The supposition that the truth-value of a sentence is its meaning shall now be put to further test. We have found that the truth-value of a sentence remains unchanged when an expression is replaced by another having the same meaning: but we have not yet considered the case in which the expression to be replaced is itself a sentence. Now if our view is correct, the truth-value of a sentence containing another as part must remain unchanged when the part is replaced by another sentence having the same truth-value.*

Frege, Über Sinn und Bedeuting

Of course, we must replace the truth-value in this test by our new proposal for the meaning of a sentence, namely the $\simeq$-equivalence class, or more briefly the **homotopy type**.

9. A Fregean test

If the type A is, say, a set (a 0-type), and $A \simeq B$, then indeed B is also a set.

More generally, one can prove in the system the statement

$$\mathrm{nType}(A) \times (A \simeq B) \to \mathrm{nType}(B)\,.$$

That is to say, one can construct a term of this type.

This means that the truth-value of the type "A is an n-type" respects equivalence, in the sense that, if "A is an n-type" is true, and "$A \simeq B$" is true, then "B is an n-type" is true.

This is not what is required, however.

9. A Fregean test

Instead, consider the apparently stronger statement:

$$(A \simeq B) \to \big(\mathsf{nType}(A) \simeq \mathsf{nType}(B)\big)$$

This says that the homotopy type of $\mathsf{nType}(A)$ respects the homotopy type of A.

More generally, let $\Phi[X]$ be any type expression containing a type X, and consider whether the homotopy type of $\Phi[X]$ respects the homotopy type of X in the foregoing sense:

$$(A \simeq B) \to \big(\Phi[A] \simeq \Phi[B]\big)$$

This can be shown by induction on the type constructors in $\Phi[X]$.

This is an **invariance principle** for the type theoretic language with respect to type equivalence.

10. Tarski's invariance proposal

Now suppose we continue this idea, and consider still wider classes of transformations. In the extreme case, we would consider the class of all one-one transformations of the space, or universe of discourse, or 'world', onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have very few notions, all of a very general character. I suggest that they are the logical notions, that we call a notion 'logical' if it is invariant under all possible one-one transformations of the world onto itself.

Tarski, What are logical notions? (1966)

10. Tarski's invariance proposal

Tarski observes that all concepts Φ that are definable in Russell's theory of types (higher-order logic) are invariant under *isomorphism*, and proposes this as a definition of the notion of a "logical concept".

We have just seen that all concepts in Martin-Löf type theory are invariant under a wider class than the one considered by Tarski, namely *homotopy equivalence*.

(Cardinality is an example of a concept definable in HOL that is not invariant under homotopy equivalence.)

11. Internalizing invariance

We can state the invariance principle *in the system of type theory* by adding a universe of types U, so that we have variables $X : U$.

We then replace the type $\Phi[X]$ by an arbitrary family of types

$$X : U \vdash P(X)\,.$$

Finally, we consider the type:

$$(A \simeq B) \to \big(P(A) \simeq P(B)\big)$$

This formulates the invariance principle internally.

11. Internalizing invariance

Given the invariance principle

$$(A \simeq B) \to \big(P(A) \simeq P(B)\big)\,,$$

we could take $P(X)$ to be $\mathsf{Id}_U(A, X)$ to get

$$(A \simeq B) \to \big(\mathsf{Id}_U(A, A) \simeq \mathsf{Id}_U(A, B)\big).$$

From this, we easily get

$$(A \simeq B) \to \mathsf{Id}_U(A, B).$$

This says that *equivalent types are equal*.

The celebrated **Univalence Axiom** of Voevodsky says something even stronger, namely that equivalence is *equivalent* to equality:

$$(A \simeq B) \simeq \mathsf{Id}_U(A, B)\,.$$

12. Invariance and Univalence

Univalence

$$(A \simeq B) \simeq \mathsf{Id}_U(A, B)$$

easily implies invariance:

- Given an equivalence $e : A \simeq B$, by univalence we get an equality $\overline{e} : \mathsf{Id}_U(A, B)$.
- But then the equality $\overline{e}$ acts on any type family $X : U \vdash P(X)$ to give an equivalence $\overline{e}_* : P(A) \simeq P(B)$.
- So univalence implies the invariance principle

$$(A \simeq B) \to \big(P(A) \simeq P(B)\big)\,.$$

12. Invariance and Univalence

Univalence

$$(A \simeq B) \simeq \mathsf{Id}_U(A, B)$$

is an *internalization* of the principle of invariance: it asserts that all concepts in the system are invariant under equivalence.

Applied to itself in a higher universe U', the statement of univalence becomes

$$\mathsf{Id}_{U'}\big((A \simeq B), \mathsf{Id}_U(A, B)\big)\,.$$

Switching from Id to $=$ this becomes the more readable

$$(A \simeq B) \,=\, (A = B)\,.$$

13. Univalence and Intensionality

Now consider this last version of univalence

$$(A \simeq B) = (A = B)$$

under our interpretation of equality as *sameness of meaning*, not only for terms $a, b : A$, but now also for type expressions A, B, i.e. mathematical statements formulated in type theory.

Such type expressions $A, B, \ldots$ are regarded as *presentations* of mathematical propositions and structures, and they present *the same* mathematical object if $A = B$.

Under this interpretation, the univalence principle says that two such presentations A, B have the same homotopy type just in case they mean the same thing – indeed it says something stronger: to say that they have the same homotopy type means the same thing as to say that they have the same meaning.

The meaning of a mathematical statement is its homotopy type.

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