

DOCTORAL DISSERTATION
SUBMITTED FOR THE DEGREE OF
PH.D. IN PURE AND APPLIED LOGIC
DEPARTMENT OF PHILOSOPHY
CARNEGIE MELLON UNIVERSITY

Classifying Types

Topics in synthetic homotopy theory

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July 2018

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Introduction

The study of homotopy theoretic phenomena in the language of type theory [33] is sometimes loosely called ‘synthetic homotopy theory’ [8]. Homotopy theory in type theory [4] is only one of the many aspects of homotopy type theory, which also includes the study of the set theoretic semantics (models of homotopy type theory and univalence in a meta-theory of sets or categories [3, 5, 7, 19, 32, 35]), type theoretic semantics (internal models of homotopy type theory), and computational semantics [2], as well as the study of various questions in the internal language of homotopy type theory which are not necessarily motivated by homotopy theory, or questions related to the development of formalized libraries of mathematics based on homotopy type theory. This thesis concerns the development of synthetic homotopy theory.

Homotopy type theory is based on Martin-Löf’s theory of dependent types [25], which was developed during the 1970’s and 1980’s. The novel additions of homotopy type theory are Voevodsky’s univalence axiom [36, 37], and higher inductive types [21, 31, 33]. The univalence axiom characterizes the identity type on the universe, and establishes the universe as an object classifier [29]. Higher inductive types are a generalization of inductive types, in which both point constructors (generators) and path constructors (relations) may be specified. A simple class of higher inductive types, which includes most known higher inductive types, are the homotopy pushouts. When the universe is assumed to be closed under homotopy pushouts, it is also closed under homotopy coequalizers [33], sequential colimits [33], and propositional truncation [15]. For instance, one way of obtaining the n -spheres [22] using homotopy pushouts is by setting $\mathbb{S}^{-1} \equiv \mathbf{0}$ and by inductively defining the $(n + 1)$ -sphere \mathbb{S}^{n+1} to be the pushout of the span $\mathbf{1} \leftarrow \mathbb{S}^n \rightarrow \mathbf{1}$. Then we can attach $(n + 1)$ -cells to a type X . Let $f : (\sum_{(a:A)} B(a)) \rightarrow X$ be a family of attaching maps for some \mathbb{S}^n -bundle $B : A \rightarrow \mathbf{BAut}(\mathbb{S}^n)$, where $\mathbf{BAut}(\mathbb{S}^n)$ is the type $\sum_{(X:\mathcal{U})} \|\mathbb{S}^n \simeq X\|$ of types merely equivalent to the n -sphere. We attach the $(n + 1)$ -disks indexed by A to X by taking the homotopy pushout

$$\begin{array}{ccc} \sum_{(a:A)} B(a) & \xrightarrow{f} & X \\ \pi_1 \downarrow & \lrcorner & \downarrow \text{inr} \\ A & \xrightarrow{\text{inl}} & P. \end{array}$$

In this thesis we study dependent type theory with univalent universes that are closed under homotopy pushouts, further developing the program set out in [33] on synthetic homotopy theory. We will assume in this dissertation that every type family is classified by a univalent universe, and that all universes are closed under homotopy pushouts, as well as under the usual type constructors including identity types, Π - and Σ -types, and a

natural numbers object. The model in cubical sets by Coquand et al. [7] is a constructive model for this setup, although the fact that it is closed under homotopy pushouts is currently unpublished. Furthermore, we will exclusively work with objects that can be described either as (dependent) types, or as their terms. In other words, the objects of our study are all ‘formalizable’ in the sense that they can be encoded in computer implementations of our setup of dependent type theory. Existing implementations of dependent type theory, supporting univalent universes and homotopy pushouts, with libraries developing synthetic homotopy theory, include the proof assistants Agda [26], Coq [13, 6], and Lean [16], and parts of the material in this theses are formalized in each of them.

Overview per chapter

In Chapter 1 we will first establish notation, although we will mostly follow the notation from [33]¹. Furthermore, we recall some of the most basic facts of homotopy type theory, and then we will show that the univalence axiom establishes any universe as an object classifier.

In Chapter 2 we will first recall the most basic properties of homotopy pushouts, and then we will proceed to prove the descent property for pushouts, using the univalence axiom. Note that the coherence problem of type theory plays a role in this chapter: we would much rather have shown that the descent property holds for any homotopy colimit, but this task requires the definition of an internal ∞ -category. Some solace will be offered in Chapter 6, where we will prove a descent property for any modality.

In Chapter 3 we will study reflexive graphs and, most importantly, reflexive coequalizers in type theory. We take interest in reflexive graphs, because the topos of reflexive graphs (over sets) is cohesive over the topos of sets. The left-most adjunction is the reflexive coequalizer which is left adjoint to the discrete functor. We will construct the reflexive coequalizer as a pushout, so it exists under our assumptions, and we know in some cases how to compute them into previously known operations. Here we run again into the limitation of homotopy type theory, because we cannot establish the type of reflexive graphs as the type of objects of an ∞ -category, and neither can we establish the reflexive coequalizer as an ∞ -functor. However, to state and prove that the reflexive coequalizer satisfies the universal property of the left adjoint of the discrete functor, we only need composition of reflexive graph morphisms, and the action on morphisms of the operation equipping a type with the structure of a discrete graph. The universal property is indeed sufficient for many purposes. We then introduce the notion of fibrations for reflexive graphs, and show that the fibrations are right orthogonal to the same maps as the discrete reflexive graphs: morphisms between representables. It follows that a morphism is a fibration if and only if it is cartesian. Then we proceed to prove the descent property of reflexive coequalizers. We also note that diagrams over reflexive graphs are just left fibrations of reflexive graphs, so we also obtain a descent theorem for

¹A difference of practice between [33] and this thesis is that [33] doesn’t use the label ‘Proposition’ for any results at all, whereas we use propositions for statements that might be of independent interest, but are not our main theorem. In this thesis, the label ‘Theorem’ is reserved for the main results that were originally established in my thesis research, and is therefore used much more sparingly.

diagrams over graphs. Of particular interest in the present work is the descent property for sequential colimits. The contents of this chapter are joint work with Bas Spitters.

In Chapter 4 we will show that any function factors as a surjective function followed by an embedding, even though we only assume that the universe is closed under pushouts and the basic type constructors. Of course, it is of essential importance here that the universe contains a natural numbers object. Our construction of the image of a map proceeds by iteratively taking the fiberwise join of a map with itself, so we call it the join construction. It follows from our construction that the image of a map from an essentially small type into a locally small type (notions that are explained in Chapter 1), is again essentially small. The join construction can be used to construct the quotient of a type by a **Prop**-valued equivalence relation (i.e. an equivalence relation in the usual sense), and it can be used to construct the Rezk completion of a pre-category. The construction of set-quotients includes the construction of the set truncation, and the construction of the Rezk completion includes the construction of the 1-truncation, and of Eilenberg-Mac Lane spaces of the form $K(G, 1)$ [23]. Following [23], the Eilenberg Mac-Lane spaces $K(G, n)$ for G abelian and $n \geq 2$ can be constructed once we have constructed the k -truncation for any $k \geq -2$. We note that Eilenberg-Mac Lane spaces are important classifying spaces in higher group theory, and so are the connected components of the universe (which we show to be essentially small).

In Chapter 5 we consider general reflective subuniverses. Examples that we have at our disposal at this point are the (-2) -, (-1) -, 0 -, and 1 -truncations. Most examples of reflective subuniverses are obtained by localization at a family of maps. However, since the only assumed homotopy colimits are pushouts, we will construct localizations only in Chapter 7, and only of families between compact types, because we need more theory in order to establish the necessary basic results. Thus, in Chapter 5 we focus on general reflective subuniverses. We show that for any reflective subuniverse L , the subuniverse L' of L -separated types (i.e. types whose identity types are L -types) is again reflective. It follows at once that the subuniverse of k -truncated types is reflective, for any $k \geq -2$. Furthermore, we will study several classes of maps related to a reflective subuniverse. First of all, we study the L -equivalences, i.e. the maps that become an equivalence by the functorial action of L , and second of all we study the L -connected maps, i.e. the maps with fibers that become trivial after applying L . Clearly, any L -connected map is also an L -equivalence, but the converse is one of the many characterizations of L being lex given in [28]. Furthermore, we study modalities. One of the characterizations of modalities is as reflective subuniverses that are Σ -closed, but we provide three more equivalent definitions of modalities. One particularly important alternative definition is that of a stable orthogonal factorization system, i.e. a pair $(\mathcal{L}, \mathcal{R})$ of two classes of maps such that every map factors as an \mathcal{L} -map followed by a \mathcal{R} -map; the class \mathcal{L} is left orthogonal to the class \mathcal{R} ; and the pullback of an \mathcal{L} -map is again a \mathcal{L} -map. For any modality \circ , the stable orthogonal factorization system associated to it consists of the \circ -connected maps as the \mathcal{L} -maps, and the \circ -modal maps as the \mathcal{R} -maps. A final topic for this chapter is the notion of accessibility for reflective subuniverses and accessible modalities.

The contents from this chapter are selected from [28] and [12]. I began to study reflective subuniverses with Mike Shulman and Bas Spitters, and continued to study them with my MRC teammates Morgan Opie and Luis Scoccola, under the lead of Dan

Christensen.

In Chapter 6 we recall from [38] the notion of \circ -étale map for an arbitrary modality \circ . We prove a modal version of the descent theorem, which asserts that a maps into $\circ X$ from a \circ -modal type are equivalently described as étale maps into X . The \circ -étale maps form the right class of a second orthogonal factorization system associated to any modality: the *reflective* factorization system. The left class of this factorization system is the class of \circ -equivalences. Using this factorization system we obtain that the universal cover of a type X at a point x_0 is the left-right-factorization of the map $\mathbf{1} \rightarrow X$. We then proceed to study the étale maps for the modality of discrete reflexive graphs. Note that the type of all reflexive graphs isn't exactly a universe, so it is not possible to directly apply our previous observations about \circ -étale maps. Nevertheless, most arguments are practically the same. Thus, we treat our section on \circ -étale maps as a blue-print for our section of Δ -étale maps, and do most arguments a second time. Such is the current state of homotopy type theory. What we get out is a generalized flattening lemma, which states that for any morphism $f : \mathcal{B} \rightarrow \mathcal{A}$ factors uniquely as an Δ -equivalence followed by a fibration of graphs. We use this to show that the loop space of the suspension of a pointed type X is the free H-space G with a base-point preserving map $X \rightarrow_* G$. The generalized flattening lemma also applies to diagrams over reflexive graphs, and in particular to sequential colimits. Moreover, the equifibrant replacement can be constructed by a telescope construction. We show in this chapter that sequential colimits commute with Σ - and identity types, and therefore also with pullbacks. In particular, the sequential colimit operation sends sequences of fiber sequences to fiber sequences. Moreover, we show that sequential colimits commute with k -truncation for all $k \geq -2$, from which it follows that sequential colimits commute with π_k for any $k \geq 0$. We expect to be able to use these results also in showing that the spectrification of a pre-spectrum is indeed a spectrum, but we haven't done that yet.

The idea of a modal version of the descent theorem first arose in unpublished work on reflexive graphs with Bas Spitters, in the spring of 2016. However, I only learned about \circ -étale maps much later from Felix Wellen, and many of the results presented in §6.1 came out of a discussion I had with Felix Wellen and Mike Shulman, who also brought my attention to the reflective factorization system of a modality, of which the classical case is due to [11]. The material in §6.2.3 on the equifibrant replacement operation on reflexive graphs is joint work with Bas Spitters. The material in §6.4 on sequential colimits is joint work with Floris van Doorn and Kristina Sojakova [14], and all the results concerning sequential colimits are formalized in the proof assistant Lean.

In Chapter 7 we introduce the notion of (sequentially) compact types, in order to provide an application for the results in Chapter 6. The most important basic result about compact types is that they are closed under pushouts, and in proving this fact we use that sequential colimits commute with pullbacks. Our main purpose here, and the final main result of this dissertation, is to show that for any family f of maps between compact types, the subuniverse of f -local types is reflective, thus providing a fairly large class of reflective subuniverses including types localized away from a prime [12]. It should be noted that all subuniverses of f -local types are reflective if enough higher inductive types are assumed. Furthermore, our result about ω -compact types should in principle hold for κ -compact types for cardinals larger than ω . However, we restrict to the case of ω -compact types since we do not have a good theory of such larger cardinals available in

homotopy type theory, while the natural numbers object is right there (by assumption).

Acknowledgments²

First and foremost, it is my pleasure to thank my advisor, professor Steve Awodey. I could not have done this PhD without his advice, support, and inspiration. I cherish the many pleasant discussions we had: over the blackboard, between us, with our many visitors, while visiting other places, or over a beer somewhere in Pittsburgh. I am very grateful for the guidance you offered in selecting a worthwhile research topic, and I feel honored to have had the privilege to work with you.

Then I would like to express my gratitude to the members of my committee: Jeremy Avigad for transmitting his enthusiasm for interactive theorem proving, and for his wisdom and unparalleled kindness; Ulrik Buchholz for his unabated energy to make large formalization projects possible, introducing me to many concepts of algebraic topology along the way; and Michael Shulman for shining a light on many beautiful subjects related to higher category theory, that would otherwise have remained obscure for me.

I would like to thank my other collaborators, the people I had projects with: Simon Boulier, Dan Christensen, Floris van Doorn, Jonas Frey, Morgan Opie, Luis Scoccola, Kristina Sojakova, Bas Spitters, Nicolas Tabareau, Felix Wellen, and Alexandra Yarosh. I thank you for all your inspiration, insights, energy, and creativity.

I would also like to thank all the people who have generously invited me for a visit. I am deeply indebted to Joachim Kock, who has hosted me for half a year in 2013-2014 at the *Departament de Matemàtiques* of the Universitat Autònoma de Barcelona and helped me finding my PhD position at CMU; I am grateful to Vladimir Voevodsky for inviting me to the Institute for Advanced Study in March 2015; Nicolas Tabareau and his students Kevin Quirin and Simon Boulier for hosting me at INRIA Nantes during the summers of 2015 and 2016; Andrej Bauer for hosting me at the *Fakulteta za Matematiko in Fiziko* of the University of Ljubljana in January 2016; Lars Birkedal and Bas Spitters for hosting me at the Department of Computer Science of Aarhus University in February 2016; Marie-Françoise Roy for inviting me to speak in the Effective Geometry and Algebra seminar at the *Institut de Recherche Mathématiques de Rennes* in June 2016; Dan Christensen and his student Luis Scoccola for hosting me at the University of Western Ontario in London, Ontario in October 2017; Guillaume Brunerie for hosting me at the Institute for Advanced Study in November 2017; Charles Rezk and his student Nima Rasekh for inviting me to speak in the Topology Seminar at the University of Illinois at Urbana-Champaign in December 2017; Tom Hales for inviting me to speak at the Algebra, Combinatorics, and Geometry seminar at the University of Pittsburgh in April 2018; and Pieter Hofstra for inviting me to speak for the Canadian Mathematical Society in Fredericton, New Brunswick, in June 2018. Tak, dankjewel, thank you, merci, danke schön, spasibo, ďankujem, hvala, gracias.

I would like to thank Karin Arnds for lending me her cabin along the Allegheny River, where I wrote and rewrote significant parts of this dissertation.

²I gratefully acknowledge the support of the Air Force Office of Scientific Research through MURI grant FA9550-15-1-0053.

Finally, I thank my dear parents Mieke and Reinier for their unconditional support, and my siblings Jeroen, Sophie, and Fleur, whom I didn't get to see as much as I would have liked because I chose to study overseas. I miss you and I dedicate this thesis to you.

Pittsburgh, June 25, 2019

Chapter 1

The object classifier

In this chapter we establish notation, and we highlight the basic results concerning fiberwise transformations and fiberwise equivalences, which we will use for the descent theorems Theorems 2.2.9 and 3.4.11. Of particular importance are the following theorems:

- (i) The Fundamental Theorem of Identity Types (Theorem 1.2.5), which establishes that a type family B over A with $b : B(a)$ for a given $a : A$ is fiberwise equivalent to the identity type $a = x$ if and only if its total space is contractible. This result appears in [33] as Theorem 5.8.2, which contains other equivalent conditions as well.
- (ii) Proposition 1.4.11, in which we establish that for any fiberwise map

$$g : \prod_{(x:A)} P(x) \rightarrow Q(f(x)),$$

the commuting square

$$\begin{array}{ccc} \sum_{(x:A)} P(x) & \xrightarrow{\text{tot}_f(g)} & \sum_{(y:B)} Q(y) \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ A & \xrightarrow{f} & B, \end{array}$$

where $\text{tot}_f(g)$ is defined as $\lambda(x,p). (f(x), g(x,p))$, is a pullback square if and only if g is a fiberwise equivalence. As a consequence, we obtain that a commuting square

$$\begin{array}{ccc} A & \longrightarrow & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$

is a pullback square if and only if the induced fiberwise transformation

$$\prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_g(h(x))$$

is a fiberwise equivalence. Our main reference [33] does not present many results of homotopy pullbacks, although the material we present about homotopy pullbacks

is surely well-known. The connection between pullbacks and fiberwise equivalences has an important role in the descent theorem in Chapter 2, which is why we devote a section to this result.

- (iii) Theorem 1.6.5, in which we establish that the universe is an object classifier. This result appears in [29] as Theorem 2.31, and in [33] as Theorem 4.8.4

1.1 Notation and preliminary results

We work in Martin-Löf dependent type theory with Π -types, Σ -types and cartesian products, coproducts $A + B$ equipped with $\text{inl} : A \rightarrow A + B$ and $\text{inr} : B \rightarrow A + B$ for any two types A and B , an empty type $\mathbf{0}$, a unit type $\mathbf{1}$ equipped with $\star : \mathbf{1}$, a type $\mathbf{2}$ of booleans equipped with $\mathbf{1}_2, \mathbf{0}_2 : \mathbf{2}$, a type \mathbb{N} of natural numbers equipped with $0 : \mathbb{N}$ and $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$, and identity types.

Remark 1.1.1. As usual, we write $\text{id}_A : A \rightarrow A$ for the **identity function** $\lambda x. x$ on A , and we write $g \circ f : A \rightarrow C$ for the **composite function** $\lambda x. g(f(x))$ of $f : A \rightarrow B$ and $g : B \rightarrow C$. For any two types A and B , and any $b : B$, we write

$$\text{const}_b : A \rightarrow B$$

for the **constant function** $\lambda x. b$. Sometimes we also write $\lambda_. b$ for the constant function.

In the case of Σ -types, the empty type $\mathbf{0}$, and the unit type, we use the following notation to define functions by pattern-matching:

$$\begin{aligned} \lambda(x, y). f(x, y) &: \prod_{(t: \sum_{(x:A)} B(x))} P(t) \\ \lambda \star . y &: \prod_{(t:\mathbf{1})} P(t). \end{aligned}$$

For instance, the first and second projection maps

$$\begin{aligned} \text{pr}_1 &: (\sum_{(x:A)} B(x)) \rightarrow A \\ \text{pr}_2 &: \prod_{(p: \sum_{(x:A)} B(x))} B(\text{pr}_1) \end{aligned}$$

are defined as $\text{pr}_1 := \lambda(x, y). x$ and $\text{pr}_2 := \lambda(x, y). y$. We use similar notation for definitions by iterated pattern-matching. For instance, given a dependent function $f : \prod_{(x:A)} \prod_{(y:B(x))} \prod_{(z:C(x,y))} P((x, y), z)$ we obtain the function

$$\lambda((x, y), z). f(x, y, z) : \prod_{(t: \sum_{(s: \sum_{(x:A)} B(x))} C(s))} P(t).$$

Given a type A in context Γ , the **identity type** of A at $a : A$ is the inductive type family

$$\Gamma, x : A \vdash a =_A x \text{ type}$$

with constructor

$$\Gamma \vdash \text{refl}_a : a =_A a.$$

The induction principle for the identity type of A at a asserts that for any type family

$$\Gamma, x : A, \alpha : a =_A x \vdash P(x, \alpha) \text{ type}$$

there is a term

$$\text{ind}_{a=} : P(a, \text{refl}_a) \rightarrow \prod_{(x:A)} \prod_{(\alpha:a=Ax)} P(x, \alpha)$$

in context Γ , satisfying the computation rule

$$\text{ind}_{a=}(p, a, \text{refl}_a) \equiv p.$$

A term of type $a =_A x$ is also called an **identification** of a with x , or a **path** from a to x . The induction principle for identity types is sometimes called **identification elimination** or **path induction**. Occasionally, we also write Id_A for the identity type on A .

Moreover, we assume that there is a universe \mathcal{U} with a universal family El over \mathcal{U} , that is closed under the type forming operations. For example, there is a map

$$\check{\text{Id}} : \prod_{(A:\mathcal{U})} \text{El}(A) \rightarrow \text{El}(A) \rightarrow \mathcal{U}$$

satisfying

$$\text{El}(\check{\text{Id}}(A, x, y)) \equiv (x =_{\text{El}(A)} y),$$

establishing that the universe is closed under identity types.

Given a type A the **concatenation** operation

$$\text{concat} : \prod_{(x,y,z:A)} (x = y) \rightarrow (y = z) \rightarrow (x = z)$$

is defined by $\text{concat}(\text{refl}_x, q) \equiv q$. We will usually write $p \cdot q$ for $\text{concat}(p, q)$. The concatenation operation satisfies the unit laws

$$\begin{aligned} \text{left_unit}(p) &: \text{refl}_x \cdot p = p \\ \text{right_unit}(p) &: p \cdot \text{refl}_y = p. \end{aligned}$$

The **inverse operation**

$$\text{inv} : \prod_{(x,y:A)} (x = y) \rightarrow (y = x)$$

is defined by $\text{inv}(\text{refl}_x) \equiv \text{refl}_x$. We will usually write p^{-1} for $\text{inv}(p)$. The inverse operation satisfies the inverse laws

$$\begin{aligned} \text{left_inv}(p) &: p^{-1} \cdot p = \text{refl}_y \\ \text{right_inv}(p) &: p \cdot p^{-1} = \text{refl}_x. \end{aligned}$$

The **associativity operation**, which assigns to each $p : x = y$, $q : y = z$, and $r : z = w$ the **associator**

$$\text{assoc}(p, q, r) : (p \cdot q) \cdot r = p \cdot (q \cdot r)$$

is defined by $\text{assoc}(\text{refl}_x, q, r) \equiv \text{refl}_{q \cdot r}$.

Given a map $f : A \rightarrow B$, the **action on paths** of f is an operation

$$\text{ap}_f : \prod_{(x,y:A)} (x = y) \rightarrow (f(x) = f(y))$$

defined by $\mathbf{ap}_f(\mathbf{refl}_x) \equiv \mathbf{refl}_{f(x)}$. Moreover, there are operations

$$\begin{aligned} \mathbf{ap_id}_A &: \prod_{(x,y:A)} \prod_{(p:x=y)} p = \mathbf{ap}_{\mathbf{id}_A}(p) \\ \mathbf{ap_comp}(f, g) &: \prod_{(x,y:A)} \prod_{(p:x=y)} \mathbf{ap}_g(\mathbf{ap}_f(p)) = \mathbf{ap}_{g \circ f}(p) \end{aligned}$$

defined by $\mathbf{ap_id}_A(\mathbf{refl}_x) \equiv \mathbf{refl}_{\mathbf{refl}_x}$ and $\mathbf{ap_comp}(f, g, \mathbf{refl}_x) \equiv \mathbf{refl}_{\mathbf{refl}_{g(f(x))}}$, respectively. It can be shown easily that the action on paths of a map preserves the groupoid operations, and that the groupoid laws are also preserved.

Definition 1.1.2. Let A be a type, and let B be a type family over A . The **transport** operation

$$\mathbf{tr}_B : \prod_{(x,y:A)} (x = y) \rightarrow (B(x) \rightarrow B(y))$$

is defined by $\mathbf{tr}_B(\mathbf{refl}_x) \equiv \mathbf{id}_{B(x)}$.

Definition 1.1.3. Given a dependent function $f : \prod_{(a:A)} B(a)$ and a path $p : x = y$ in A , the **dependent action on paths**

$$\mathbf{apd}_f : \prod_{(x,y:A)} \prod_{(p:x=y)} \mathbf{tr}_B(p, f(x)) = f(y)$$

is defined by $\mathbf{apd}_f(\mathbf{refl}_x) \equiv \mathbf{refl}_{f(x)}$.

Definition 1.1.4. Let $f, g : \prod_{(x:A)} P(x)$ be two dependent functions. The type $f \sim g$ of **homotopies** from f to g is defined as

$$f \sim g \equiv \prod_{(x:A)} f(x) = g(x).$$

Commutativity of diagrams is stated using homotopies. For instance, a triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & X \end{array}$$

is said to commute if it comes equipped with a homotopy $H : f \sim g \circ h$, and a square

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ i \downarrow & & \downarrow f \\ B & \xrightarrow{h} & Y \end{array}$$

is said to commute if it comes equipped with a homotopy $H : h \circ i \sim f \circ g$.

The reflexivity, inverse, and concatenation operations on homotopies are defined pointwise. We will write H^{-1} for $\lambda x. H(x)^{-1}$, and $H \cdot K$ for $\lambda x. H(x) \cdot K(x)$. These operations satisfy the groupoid laws (phrased appropriately as homotopies). Apart from the groupoid operations and their laws, we will occasionally need *whiskering* operations and the naturality of homotopies.

Definition 1.1.5. We define the following **whiskering** operations on homotopies:

(i) Suppose $H : f \sim g$ for two functions $f, g : A \rightarrow B$, and let $h : B \rightarrow C$. We define

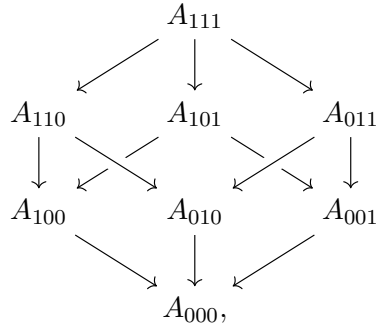
$$h \cdot H := \lambda x. \text{ap}_h(H(x)) : h \circ f \sim h \circ g.$$

(ii) Suppose $f : A \rightarrow B$ and $H : g \sim h$ for two functions $g, h : B \rightarrow C$. We define

$$H \cdot f := \lambda x. H(f(x)) : h \circ f \sim g \circ f.$$

We will frequently make use of commuting cubes. The commutativity of a cube is stated using the whiskering operations on homotopies.

Definition 1.1.6. A commuting cube



consists of

(i) types

$$A_{111}, A_{110}, A_{101}, A_{011}, A_{100}, A_{010}, A_{001}, A_{000},$$

(ii) maps

$$\begin{array}{ll} f_{11\bar{1}} : A_{111} \rightarrow A_{110} & f_{\bar{1}01} : A_{101} \rightarrow A_{001} \\ f_{1\bar{1}1} : A_{111} \rightarrow A_{101} & f_{0\bar{1}1} : A_{011} \rightarrow A_{010} \\ f_{\bar{1}11} : A_{111} \rightarrow A_{011} & f_{0\bar{1}\bar{1}} : A_{011} \rightarrow A_{001} \\ f_{1\bar{1}0} : A_{110} \rightarrow A_{100} & f_{\bar{1}00} : A_{100} \rightarrow A_{000} \\ f_{\bar{1}10} : A_{110} \rightarrow A_{010} & f_{0\bar{1}0} : A_{010} \rightarrow A_{000} \\ f_{10\bar{1}} : A_{101} \rightarrow A_{100} & f_{00\bar{1}} : A_{001} \rightarrow A_{000}, \end{array}$$

(iii) homotopies

$$\begin{array}{ll} H_{1\bar{1}\bar{1}} : f_{1\bar{1}0} \circ f_{11\bar{1}} \sim f_{10\bar{1}} \circ f_{1\bar{1}1} & H_{0\bar{1}\bar{1}} : f_{0\bar{1}0} \circ f_{01\bar{1}} \sim f_{00\bar{1}} \circ f_{0\bar{1}1} \\ H_{\bar{1}1\bar{1}} : f_{\bar{1}10} \circ f_{11\bar{1}} \sim f_{01\bar{1}} \circ f_{\bar{1}11} & H_{\bar{1}0\bar{1}} : f_{\bar{1}00} \circ f_{10\bar{1}} \sim f_{00\bar{1}} \circ f_{\bar{1}01} \\ H_{\bar{1}\bar{1}1} : f_{\bar{1}\bar{1}0} \circ f_{1\bar{1}1} \sim f_{0\bar{1}\bar{1}} \circ f_{\bar{1}11} & H_{\bar{1}\bar{1}0} : f_{\bar{1}\bar{1}0} \circ f_{1\bar{1}0} \sim f_{0\bar{1}\bar{1}} \circ f_{\bar{1}10}, \end{array}$$

(iv) and a homotopy

$$\begin{aligned} C : (f_{\bar{1}00} \cdot H_{1\bar{1}\bar{1}}) \cdot ((H_{\bar{1}0\bar{1}} \cdot f_{\bar{1}11}) \cdot (f_{00\bar{1}} \cdot H_{\bar{1}\bar{1}1})) \\ \sim (H_{\bar{1}\bar{1}0} \cdot f_{\bar{1}11}) \cdot ((f_{0\bar{1}0} \cdot H_{\bar{1}\bar{1}1}) \cdot (H_{0\bar{1}\bar{1}} \cdot f_{\bar{1}11})) \end{aligned}$$

filling the cube.

Definition 1.1.7. Let $f : A \rightarrow B$ be a function. We say that f has a **section** if there is a term of type

$$\text{sec}(f) := \sum_{(g:B \rightarrow A)} f \circ g \sim \text{id}_B.$$

Dually, we say that f has a **retraction** if there is a term of type

$$\text{retr}(f) := \sum_{(h:B \rightarrow A)} h \circ f \sim \text{id}_A.$$

If f has a retraction, we also say that A is a **retract** of B .

Definition 1.1.8. We say that a function $f : A \rightarrow B$ is an **equivalence** if it has both a section and a retraction, i.e. if it comes equipped with a term of type

$$\text{is_equiv}(f) := \text{sec}(f) \times \text{retr}(f).$$

We will write $A \simeq B$ for the type $\sum_{(f:A \rightarrow B)} \text{is_equiv}(f)$.

Clearly, if f is **invertible** in the sense that it comes equipped with a function $g : B \rightarrow A$ such that $f \circ g \sim \text{id}_B$ and $g \circ f \sim \text{id}_A$, then f is an equivalence. We write

$$\text{has_inverse}(f) := \sum_{(g:B \rightarrow A)} (f \circ g \sim \text{id}_B) \times (g \circ f \sim \text{id}_A).$$

The section of an equivalence is also a retraction (and vice versa), so we define the **inverse** of an equivalence to be its section. It follows immediately that the inverse of any equivalence is again an equivalence. The identity function id_A on a type A is an equivalence since it is its own section and its own retraction.

It is straightforward to show that for any two functions $f, g : A \rightarrow B$, we have

$$(f \sim g) \rightarrow (\text{is_equiv}(f) \leftrightarrow \text{is_equiv}(g)).$$

Given a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & X. \end{array}$$

with $H : f \sim g \circ h$, we have:

- (i) If the map h has a section, then f has a section if and only if g has a section.
- (ii) If the map g has a retraction, then f has a retraction if and only if h has a retraction.
- (iii) (The **3-for-2 property** for equivalences.) If any two of the functions

$$f, \quad g, \quad h$$

are equivalences, then so is the third.

In the following theorem we characterize the identity type of a Σ -type as a Σ -type of identity types.

Proposition 1.1.9 (Theorem 2.7.2 of [33]). *Let B be a type family over A , let $s : \sum_{(x:A)} B(x)$, and consider the dependent function*

$$\text{pair_eq}_s : \prod_{(t:\sum_{(x:A)} B(x))} (s = t) \rightarrow \sum_{(\alpha:\text{pr}_1(s)=\text{pr}_1(t))} \text{tr}_B(\alpha, \text{pr}_2(s)) = \text{pr}_2(t)$$

defined by $\text{pair_eq}_s(\text{refl}_s) := (\text{refl}_{\text{pr}_1(s)}, \text{refl}_{\text{pr}_2(s)})$. Then $\text{pair_eq}_{s,t}$ is an equivalence for every $t : \sum_{(x:A)} B(x)$.

We include the proof mainly to introduce some more notation.

Proof. The maps in the converse direction

$$\text{eq_pair}_{s,t} : \left(\sum_{(p:\text{pr}_1(s)=\text{pr}_1(t))} \text{tr}_B(p, \text{pr}_2(s)) = \text{pr}_2(t) \right) \rightarrow (s = t)$$

is defined by

$$\text{eq_pair}_{(x,y),(x',y')}(\text{refl}_x, \text{refl}_y) := \text{refl}_{(x,y)}.$$

The proofs that the function $\text{eq_pair}_{s,t}$ is indeed an inverse of $\text{pair_eq}_{s,t}$ are also by induction. \square

Definition 1.1.10. We say that a type A is **contractible** if there is a term of type

$$\text{is_contr}(A) := \sum_{(c:A)} \prod_{(x:A)} c = x.$$

Given a term $(c, C) : \text{is_contr}(A)$, we call $c : A$ the **center of contraction** of A , and we call $C : \prod_{(x:A)} a = x$ the **contraction** of A .

Suppose A is a contractible type with center of contraction c and contraction C . Then the type of C is (judgmentally) equal to the type

$$\text{const}_c \sim \text{id}_A.$$

In other words, the contraction C is a *homotopy* from the constant function to the identity function.

Definition 1.1.11. Consider a type A with a base point $a : A$. We say that A satisfies **singleton induction** if for every type family B over A , the map

$$\text{ev_pt} : \left(\prod_{(x:A)} B(x) \right) \rightarrow B(a)$$

given by $f \mapsto f(a)$ has a section.

Proposition 1.1.12. *A type A is contractible if and only if it satisfies singleton induction.*

Example 1.1.13. By definition the unit type $\mathbf{1}$ satisfies singleton induction, so it is contractible.

Remark 1.1.14. For any family $P : \left(\sum_{(x:A)} B(x) \right) \rightarrow \mathcal{U}$ there is a map

$$\text{ev_pair} : \left(\prod_{(t:\sum_{(x:A)} B(x))} P(t) \right) \rightarrow \prod_{(x:A)} \prod_{(y:B(x))} P(x, y)$$

that evaluates $f : \prod_{(t:\sum_{(x:A)} B(x))} P(t)$ at pairs (x, y) . In other words, ev_pair is defined by $\lambda f. \lambda x. \lambda y. f(x, y)$. By the induction principle for Σ -types, this map has a section. It is easy to show that ev_pair is in fact an equivalence.

Similarly, there is a map

$$\text{ev_refl} : \left(\prod_{(x:A)} \prod_{(p:a=x)} B(x, p) \right) \rightarrow B(a, \text{refl}_a)$$

given by $\lambda f. f(a, \text{refl}_a)$, for any type family $B : \prod_{(x:A)} (a = x) \rightarrow \mathcal{U}$. By path induction, this map has a section, and again it is easy to show that this map is in fact an equivalence.

Proposition 1.1.15 (Lemma 3.11.8 in [33]). *For any $x : A$, the type*

$$\sum_{(y:A)} x = y$$

is contractible.

Proof. We have the term $(x, \text{refl}_x) : \sum_{(y:A)} x = y$, and both maps in the composite

$$\prod_{(t:\sum_{(y:A)} x=y)} B(t) \xrightarrow{\text{ev_pair}} \prod_{(y:A)} \prod_{(p:x=y)} B((y, p)) \xrightarrow{\text{ev_refl}} B((x, \text{refl}_x))$$

have sections, so the composite has a section. The composite is ev_pt , so we see that the asserted type satisfies singleton induction. \square

Definition 1.1.16. Let $f : A \rightarrow B$ be a function, and let $b : B$. The **fiber** of f at b is defined to be the type

$$\text{fib}_f(b) := \sum_{(a:A)} f(a) = b.$$

Example 1.1.17 (Lemma 4.8.1 of [33]). Consider a type family B over A . Then the map

$$B(a) \rightarrow \text{fib}_{\text{pr}_1}(a)$$

given by $b \mapsto ((a, b), \text{refl}_a)$ is an equivalence. In other words, the fibers of the projection function $\text{pr}_1 : \left(\sum_{(x:A)} B(x) \right) \rightarrow A$ are just the fibers of the family B .

Definition 1.1.18. We say that a function $f : A \rightarrow B$ is **contractible** if there is a term of type

$$\text{is_contr}(f) := \prod_{(b:B)} \text{is_contr}(\text{fib}_f(b)).$$

We cite Chapter 4 of [33] for the following result, although it is well-known that it can be proven directly and without the use of function extensionality.

Proposition 1.1.19 (Chapter 4 in [33]). *A function is an equivalence if and only if it is contractible.*

1.2 The Fundamental Theorem of Identity Types

Consider a family

$$f : \prod_{(x:A)} B(x) \rightarrow C(x)$$

of maps. Such f is also called a **fiberwise map** or **fiberwise transformation**.

Definition 1.2.1 (Definition 4.7.5 of [33]). We define the map

$$\text{tot}(f) : \sum_{(x:A)} B(x) \rightarrow \sum_{(x:A)} C(x).$$

by $\lambda(x, y). (x, f(x, y))$.

Lemma 1.2.2 (Theorem 4.7.6 of [33]). *For any fiberwise transformation $f : \prod_{(x:A)} B(x) \rightarrow C(x)$, and any $a : A$ and $c : C(a)$, there is an equivalence*

$$\text{fib}_{f(a)}(c) \simeq \text{fib}_{\text{tot}(f)}((a, c)).$$

Example 1.2.3. There are equivalences

$$\text{fib}_{(\text{ap}_f)_{x,y}}(q) \simeq \text{fib}_{\delta_f}((x, y, q))$$

for any $q : f(x) = f(y)$, because the triangle

$$\begin{array}{ccc} A & \xrightarrow{\lambda x. (x, x, \text{refl}_x)} & \sum_{(x,y:A)} x = y \\ & \searrow \delta_f & \swarrow \text{tot}(\text{tot}(\text{ap}_f)) \\ & \sum_{(x,y:A)} f(x) = f(y) & \end{array}$$

commutes, and the top map is an equivalence.

Proposition 1.2.4 (Theorem 4.7.7 of [33]). *Let $f : \prod_{(x:A)} B(x) \rightarrow C(x)$ be a fiberwise transformation. The following are logically equivalent:*

- (i) *For each $x : A$, the map $f_x : B(x) \rightarrow C(x)$ is an equivalence. In this case we say that f is a **fiberwise equivalence**.*
- (ii) *The map $\text{tot}(f) : \sum_{(x:A)} B(x) \rightarrow \sum_{(x:A)} C(x)$ is an equivalence.*

The following theorem is the key to many results about identity types, which we will use instead of the *encode-decode method* of [24]. We refer to it as the **Fundamental Theorem of Identity Types**.

Theorem 1.2.5 (Theorem 5.8.2 of [33]). *Let A be a type with $a : A$, and let B be a type family over A with $b : B(a)$. Then the following are logically equivalent:*

- (i) *The canonical family of maps*

$$\text{ind}_{a=} (b) : \prod_{(x:A)} (a = x) \rightarrow B(x)$$

is a fiberwise equivalence.

(ii) *The total space*

$$\sum_{(x:A)} B(x)$$

is contractible.

Proof. By Proposition 1.2.4 it follows that the fiberwise transformation $\text{ind}_{a=}(b)$ is a fiberwise equivalence if and only if it induces an equivalence

$$\left(\sum_{(x:A)} a = x \right) \simeq \left(\sum_{(x:A)} B(x) \right)$$

on total spaces. We have that $\sum_{(x:A)} a = x$ is contractible. Now it follows by the 3-for-2 property of equivalences, applied in the case

$$\begin{array}{ccc} \sum_{(x:A)} a = x & \xrightarrow{\text{tot}(\text{ind}_{a=}(b))} & \sum_{(x:A)} B(x) \\ & \searrow \simeq & \swarrow \\ & \mathbf{1} & \end{array}$$

that $\text{tot}(\text{ind}_{a=}(b))$ is an equivalence if and only if $\sum_{(x:A)} B(x)$ is contractible. \square

Observe that in the proof of Theorem 1.2.5 we haven't used the actual definition of the fiberwise transformation. Indeed, for any fiberwise transformation

$$f : \prod_{(x:A)} (a = x) \rightarrow B(x)$$

we have that f is a fiberwise equivalence if and only if the total space of B is contractible.

Since retracts of contractible types are again contractible, it follows that the only retract of the identity type is the identity type itself:

Corollary 1.2.6. *Let $a : A$, and let B be a type family over A . If each $B(x)$ is a retract of $a = x$, then $B(x)$ is equivalent to $a = x$ for every $x : A$.*

As a first application of the fundamental theorem we give a quick new proof that equivalences are embeddings. The proof of the corresponding theorem in [33] is more involved.

Definition 1.2.7. An **embedding** is a map $f : A \rightarrow B$ satisfying the property that

$$\text{ap}_f : (x = y) \rightarrow (f(x) = f(y))$$

is an equivalence for every $x, y : A$. We write $\text{is_emb}(f)$ for the type of witnesses that f is an embedding.

Proposition 1.2.8 (Theorem 2.11.1 in [33]). *Any equivalence is an embedding.*

Proof. Let $e : A \simeq B$ be an equivalence, and let $x : A$. By Theorem 1.2.5 it follows that

$$\text{ap}_e : (x = y) \rightarrow (e(x) = e(y))$$

is an equivalence for every $y : A$ if and only if the total space

$$\sum_{(y:A)} e(x) = e(y)$$

is contractible for every $y : A$. Now observe that $\sum_{(y:A)} e(x) = e(y)$ is equivalent to the fiber $\text{fib}_e(e(x))$, which is contractible by Proposition 1.1.19. \square

Definition 1.2.9. A type A is said to be a **proposition** if there is a term of type

$$\text{is_prop}(A) := \prod_{(x,y:A)} \text{is_contr}(x = y).$$

Furthermore, we write $\text{Prop} := \sum_{(X:\mathcal{U})} \text{is_prop}(X)$ for the type of all small propositions.

We will often use either of the following characterizations of propositions.

Lemma 1.2.10 (Lemma 3.11.10 and Exercise 3.5 of [33]). *For any type A the following are equivalent:*

(i) A is a proposition.

(ii) A is **proof irrelevant** in the sense that $\prod_{(x,y:A)} x = y$.

(iii) $A \rightarrow \text{is_contr}(A)$.

Example 1.2.11. Any contractible type is a proposition. The empty type is a proposition by a direct application of the induction principle of the empty type. Furthermore, any retract of a proposition is again a proposition. In particular, propositions are closed under equivalences.

Lemma 1.2.12. *Consider a function $f : A \rightarrow B$, and let $(a, p), (a', p') : \text{fib}_f(b)$ for some $b : B$. Then the canonical map*

$$((a, p) = (a', p')) \rightarrow \text{fib}_{\text{ap}_f}(p \cdot p'^{-1})$$

is an equivalence.

Proof. By Proposition 1.2.4 it suffices to show that the type

$$\sum_{(y:A)} \sum_{(q:f(y)=b)} \sum_{(p:\text{pr}_1(s)=y)} \text{ap}_f(p) = \text{pr}_2(s) \cdot q^{-1}$$

is contractible, which is immediate by two applications of Proposition 1.1.15. \square

Proposition 1.2.13 (Lemma 7.6.2 of [33]). *A map is an embedding if and only if its fibers are propositions.*

Proof. If f is an embedding, then the fibers of ap_f are contractible by Proposition 1.1.19. Thus it follows by Lemma 1.2.12 that the fibers of f are propositions.

Conversely, if the fibers of f are propositions, then we have by Lemma 1.2.12 an equivalence

$$((x, p) = (y, \text{refl}_{f(y)})) \simeq \text{fib}_{\text{ap}_f}(p)$$

for any $p : f(x) = f(y)$, which shows that the fibers of ap_f are contractible. Thus f is an embedding by Proposition 1.1.19. \square

Definition 1.2.14. A type family B over A is said to be a **subtype** of A if for each $x : A$ the type $B(x)$ is a proposition.

Corollary 1.2.15. *A type family B over A is a subtype if and only if the projection map*

$$\text{pr}_1 : \left(\sum_{(x:A)} B(x) \right) \rightarrow A$$

is an embedding.

Proof. Immediate by Example 1.1.17 and Proposition 1.2.13. \square

1.3 Function extensionality

Proposition 1.3.1 (Theorem 4.9.5 of [33]). *The following are equivalent:*

- (i) *The **function extensionality principle**: For every type family B over A , and any two dependent functions $f, g : \prod_{(x:A)} B(x)$, the canonical map*

$$\text{htpy_eq}(f, g) : (f = g) \rightarrow (f \sim g)$$

by path induction (sending refl_f to $\lambda x. \text{refl}_{f(x)}$) is an equivalence. We will write eq_htpy for its inverse.

- (ii) *The **weak function extensionality principle** holds: For every type family B over A one has*

$$\left(\prod_{(x:A)} \text{is_contr}(B(x)) \right) \rightarrow \text{is_contr} \left(\prod_{(x:A)} B(x) \right).$$

From now on we will assume that function extensionality holds.

Corollary 1.3.2 (Theorem 7.1.9 of [33]). *For any type family B over A one has*

$$\left(\prod_{(x:A)} \text{is_prop}(B(x)) \right) \rightarrow \text{is_prop} \left(\prod_{(x:A)} B(x) \right).$$

In particular, if B is a proposition, then $A \rightarrow B$ is a proposition for any type A .

We show in this section that a map $f : A \rightarrow B$ is an equivalence if and only if for any type family P over B , the precomposition map

$$- \circ f : \left(\prod_{(y:B)} P(y) \right) \rightarrow \left(\prod_{(x:A)} P(f(x)) \right)$$

is an equivalence. In the proof we use the notion of *path-split* maps, which was introduced in [28].

Definition 1.3.3. We say that a map $f : A \rightarrow B$ is **path-split** if f has a section, and for each $x, y : A$ the map

$$\text{ap}_f(x, y) : (x = y) \rightarrow (f(x) = f(y))$$

also has a section. We write $\text{path_split}(f)$ for the type

$$\text{sec}(f) \times \prod_{(x,y:A)} \text{sec}(\text{ap}_f(x, y)).$$

We will also use the notion of *half-adjoint equivalences*, which were introduced in [33].

Definition 1.3.4 (Definition 4.2.1 of [33]). We say that a map $f : A \rightarrow B$ is a **half-adjoint equivalence**, in the sense that there are

$$\begin{aligned} g &: B \rightarrow A \\ G &: f \circ g \sim \text{id}_B \\ H &: g \circ f \sim \text{id}_A \\ K &: G \cdot f \sim f \cdot H. \end{aligned}$$

We write $\text{half_adj}(f)$ for the type of such quadruples (g, G, H, K) .

Furthermore, we will need ‘type theoretic choice’.

Proposition 1.3.5 (Theorem 2.15.7 of [33]). *Let $C(x, y)$ be a type in context $\Gamma, x : A, y : B(x)$. Then the map*

$$\varphi : \left(\prod_{(x:A)} \sum_{(y:B(x))} C(x, y) \right) \rightarrow \left(\sum_{(f:\prod_{(x:A)} B(x))} \prod_{(x:A)} C(x, f(x)) \right)$$

given by $\lambda h. (\lambda x. \text{pr}_1(h(x)), \lambda x. \text{pr}_2(h(x)))$ is an equivalence.

Corollary 1.3.6. *For type A and any type family C over B , the map*

$$\left(\sum_{(f:A \rightarrow B)} \prod_{(x:A)} C(f(x)) \right) \rightarrow \left(A \rightarrow \sum_{(y:B)} C(x) \right)$$

given by $\lambda(f, g). \lambda x. (f(x), g(x))$ is an equivalence.

Proposition 1.3.7. *For any map $f : A \rightarrow B$, the following are equivalent:*

- (i) *f is an equivalence.*
- (ii) *f is path-split.*
- (iii) *f is a half-adjoint equivalence.*
- (iv) *For any type family P over B the map*

$$\left(\prod_{(y:B)} P(y) \right) \rightarrow \left(\prod_{(x:A)} P(f(x)) \right)$$

given by $s \mapsto s \circ f$ is an equivalence.

- (v) *For any type X the map*

$$(B \rightarrow X) \rightarrow (A \rightarrow X)$$

given by $g \mapsto g \circ f$ is an equivalence.

Proof. To see that (i) implies (ii) we note that any equivalence has a section, and its action on paths is an equivalence by Proposition 1.2.8 so again it has a section.

To show that (ii) implies (iii), assume that f is path-split. Thus we have $(g, G) : \text{sec}(f)$, and the assumption that $\text{ap}_f : (x = y) \rightarrow (f(x) = f(y))$ has a section for every $x, y : A$ gives us a term of type

$$\prod_{(x:A)} \text{fib}_{\text{ap}_f}(G(f(x))).$$

By Proposition 1.3.5 this type is equivalent to

$$\sum_{(H:\prod_{(x:A)} g(f(x))=x)} \prod_{(x:A)} G(f(x)) = \text{ap}_f(H(x)),$$

so we obtain $H : g \circ f \sim \text{id}_A$ and $K : G \cdot f \sim f \cdot H$, showing that f is a half-adjoint equivalence.

To show that (iii) implies (iv), suppose that f comes equipped with (g, G, H, K) witnessing that f is a half-adjoint equivalence. Then we define the inverse of $- \circ f$ to be the map

$$\varphi : \left(\prod_{(x:A)} P(f(x)) \right) \rightarrow \left(\prod_{(y:B)} P(y) \right)$$

given by $s \mapsto \lambda y. \text{tr}_P(G(y), sg(y))$.

To see that φ is a section of $- \circ f$, let $s : \prod_{(x:A)} P(f(x))$. By function extensionality it suffices to construct a homotopy $\varphi(s) \circ f \sim s$. In other words, we have to show that

$$\text{tr}_P(G(f(x)), s(g(f(x)))) = s(x)$$

for any $x : A$. Now we use the additional homotopy K from our assumption that f is a half-adjoint equivalence. Since we have $K(x) : G(f(x)) = \mathbf{ap}_f(H(x))$ it suffices to show that

$$\text{tr}_P(\mathbf{ap}_f(H(x)), sgf(x)) = s(x).$$

A simple path-induction argument yields that

$$\text{tr}_P(\mathbf{ap}_f(p)) \sim \text{tr}_{P \circ f}(p)$$

for any path $p : x = y$ in A , so it suffices to construct an identification

$$\text{tr}_{P \circ f}(H(x), sgf(x)) = s(x).$$

We have such an identification by $\mathbf{apd}_{H(x)}(s)$.

To see that φ is a retraction of $- \circ f$, let $s : \prod_{(y:B)} P(y)$. By function extensionality it suffices to construct a homotopy $\varphi(s \circ f) \sim s$. In other words, we have to show that

$$\text{tr}_P(G(y), sfg(y)) = s(y)$$

for any $y : B$. We have such an identification by $\mathbf{apd}_{G(y)}(s)$. This completes the proof that (iii) implies (iv).

Note that (v) is an immediate consequence of (iv), since we can just choose P to be the constant family X .

It remains to show that (v) implies (i). Suppose that

$$- \circ f : (B \rightarrow X) \rightarrow (A \rightarrow X)$$

is an equivalence for every type X . Then its fibers are contractible by Proposition 1.1.19. In particular, choosing $X \equiv A$ we see that the fiber

$$\mathbf{fib}_{- \circ f}(\text{id}_A) \equiv \sum_{(h:B \rightarrow A)} h \circ f = \text{id}_A$$

is contractible. Thus we obtain a function $h : B \rightarrow A$ and a homotopy $H : h \circ f \sim \text{id}_A$ showing that h is a retraction of f . We will show that h is also a section of f . To see this, we use that the fiber

$$\mathbf{fib}_{- \circ f}(f) \equiv \sum_{(i:B \rightarrow B)} i \circ f = f$$

is contractible (choosing $X \equiv B$). Of course we have $(\text{id}_B, \text{refl}_f)$ in this fiber. However we claim that there also is an identification $p : (f \circ h) \circ f = f$, showing that $(f \circ h, p)$ is in this fiber, because

$$\begin{aligned} (f \circ h) \circ f &\equiv f \circ (h \circ f) \\ &= f \circ \text{id}_A \\ &\equiv f \end{aligned}$$

Now we conclude by the contractibility of the fiber that there is an identification $(\text{id}_B, \text{refl}_f) = (f \circ h, p)$. In particular we obtain that $\text{id}_B = f \circ h$, showing that h is a section of f . \square

1.4 Homotopy pullbacks

1.4.1 Cartesian squares

Recall that a square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{g} & X \end{array}$$

is said to **commute** if there is a homotopy $H : f \circ p \sim g \circ q$.

Definition 1.4.1. A **cospan** consists of three types A , X , and B , and maps $f : A \rightarrow X$ and $g : B \rightarrow X$. Given a type C , a **cone** on the cospan $A \xrightarrow{f} X \xleftarrow{g} B$ with **vertex** C consists of maps $p : C \rightarrow A$, $q : C \rightarrow B$ and a homotopy $H : f \circ p \sim g \circ q$ witnessing that the square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

commutes. We write

$$\text{cone}(C) := \sum_{(p:C \rightarrow A)} \sum_{(q:C \rightarrow B)} f \circ p \sim g \circ q$$

for the type of cones with vertex C .

Given a cone with vertex C on a span $A \xrightarrow{f} X \xleftarrow{g} B$ and a map $h : C' \rightarrow C$, we construct a new cone with vertex C' in the following definition.

Definition 1.4.2. For any cone (p, q, H) with vertex C and any type C' , we define a map

$$\text{cone_map}(p, q, H) : (C' \rightarrow C) \rightarrow \text{cone}(C')$$

by $h \mapsto (p \circ h, q \circ h, H \cdot h)$.

Definition 1.4.3. We say that a commuting square

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

with $H : f \circ p \sim g \circ q$ is a **pullback square**, or that it is **cartesian**, if it satisfies the **universal property** of pullbacks, which asserts that the map

$$\text{cone_map}(p, q, H) : (C' \rightarrow C) \rightarrow \text{cone}(C')$$

is an equivalence for every type C' .

We often indicate the universal property with a diagram as follows:

$$\begin{array}{ccccc}
 C' & & & & \\
 & \searrow^{q'} & & & \\
 & \text{---} h \text{---} & C & \xrightarrow{q} & B \\
 & & \downarrow p & & \downarrow g \\
 & \searrow^{p'} & A & \xrightarrow{f} & X
 \end{array}$$

since the universal property states that for every cone (p', q', H') with vertex C' , the type of pairs (h, α) consisting of $h : C' \rightarrow C$ equipped with $\alpha : \text{cone_map}((p, q, H), h) = (p', q', H')$ is contractible by Proposition 1.1.19.

Proposition 1.4.4. *Consider a commuting square*

$$\begin{array}{ccc}
 C & \xrightarrow{q} & B \\
 p \downarrow & & \downarrow g \\
 A & \xrightarrow{f} & X
 \end{array}$$

with $H : f \circ p \sim g \circ q$ Then the following are equivalent:

- (i) The square is a pullback square.
- (ii) For every type C' and every cone (p', q', H') with vertex C' , the type of quadruples (h, K, L, M) consisting of

$$\begin{aligned}
 h &: C' \rightarrow C \\
 K &: p \circ h \sim p' \\
 L &: q \circ h \sim q' \\
 M &: (H \cdot h) \cdot (g \cdot L) \sim (f \cdot K) \cdot H'
 \end{aligned}$$

is contractible.

Remark 1.4.5. The homotopy M in Proposition 1.4.4 witnesses that the square

$$\begin{array}{ccc}
 f \circ p \circ h & \xrightarrow{f \cdot K} & f \circ p' \\
 H \cdot h \downarrow & & \downarrow H' \\
 g \circ q \circ h & \xrightarrow{g \cdot L} & g \circ q'
 \end{array}$$

of homotopies commutes.

1.4.2 The unique existence of pullbacks

Definition 1.4.6. Let $f : A \rightarrow X$ and $B \rightarrow X$ be maps. Then we define

$$A \times_X B := \sum_{(x:A)} \sum_{(y:B)} f(x) = g(y)$$

$$\begin{array}{ll}
\pi_1 \equiv \text{pr}_1 & : A \times_X B \rightarrow A \\
\pi_2 \equiv \text{pr}_1 \circ \text{pr}_2 & : A \times_X B \rightarrow B \\
\pi_3 \equiv \text{pr}_2 \circ \text{pr}_2 & : f \circ \pi_1 \sim g \circ \pi_2.
\end{array}$$

The type $A \times_X B$ is called the **canonical pullback** of f and g .

Note that $A \times_X B$ depends on f and g , although this dependency is not visible in the notation.

Proposition 1.4.7 (Exercise 2.11 of [33]). *Given maps $f : A \rightarrow X$ and $g : B \rightarrow X$, the commuting square*

$$\begin{array}{ccc}
A \times_X B & \xrightarrow{\pi_2} & B \\
\pi_1 \downarrow & & \downarrow g \\
A & \xrightarrow{f} & X,
\end{array}$$

is a pullback square.

In the following lemma we establish the uniqueness of pullbacks up to equivalence via a *3-for-2 property* for pullbacks.

Lemma 1.4.8. *Consider the squares*

$$\begin{array}{ccc}
C & \xrightarrow{q} & B \\
p \downarrow & & \downarrow g \\
A & \xrightarrow{f} & X
\end{array}
\quad
\begin{array}{ccc}
C' & \xrightarrow{q'} & B \\
p' \downarrow & & \downarrow g \\
A & \xrightarrow{f} & X
\end{array}$$

with homotopies $H : f \circ p \sim g \circ q$ and $H' : f \circ p' \sim g \circ q'$. Furthermore, suppose we have a map $h : C' \rightarrow C$ equipped with

$$\begin{array}{l}
K : p \circ h \sim p' \\
L : q \circ h \sim q' \\
M : (H \cdot h) \cdot (g \cdot L) \sim (f \cdot K) \cdot H'.
\end{array}$$

If any two of the following three properties hold, so does the third:

- (i) C is a pullback.
- (ii) C' is a pullback.
- (iii) h is an equivalence.

Proof. The type of triples (K, L, M) is equivalent to the type of identifications

$$\text{cone_map}((p, q, H), h) = (p', q', H').$$

Let D be a type, and let $k : D \rightarrow C'$ be a map. We observe that

$$\text{cone_map}((p, q, H), (h \circ k)) \equiv (p \circ (h \circ k), q \circ (h \circ k), H \circ (h \circ k))$$

$$\begin{aligned}
&\equiv ((p \circ h) \circ k, (q \circ h) \circ k, (H \circ h) \circ k) \\
&\equiv \text{cone_map}(\text{cone_map}((p, q, H), h), k) \\
&= \text{cone_map}((p', q', H'), k).
\end{aligned}$$

Thus we see that the triangle

$$\begin{array}{ccc}
(D \rightarrow C') & \xrightarrow{h \circ -} & (D \rightarrow C) \\
& \searrow \text{cone_map}(p', q', H') & \swarrow \text{cone_map}(p, q, H) \\
& & \text{cone}(D)
\end{array}$$

commutes. Therefore it follows from the 3-for-2 property of equivalences that if any two of the following properties hold, then so does the third:

- (i) The map $\text{cone_map}(p, q, H) : (D \rightarrow C) \rightarrow \text{cone}(D)$ is an equivalence,
- (ii) The map $\text{cone_map}(p', q', H') : (D \rightarrow C') \rightarrow \text{cone}(D)$ is an equivalence,
- (iii) The map $h \circ - : (D \rightarrow C') \rightarrow (D \rightarrow C)$ is an equivalence.

Thus the 3-for-2 property for pullbacks follows from the fact that h is an equivalence if and only if $h \circ - : (D \rightarrow C') \rightarrow (D \rightarrow C)$ is an equivalence for any type D . \square

Definition 1.4.9. Given a commuting square

$$\begin{array}{ccc}
C & \xrightarrow{q} & B \\
p \downarrow & & \downarrow g \\
A & \xrightarrow{f} & X
\end{array}$$

with $H : f \circ p \sim g \circ q$, we define the **gap map**

$$\text{gap}(p, q, H) : C \rightarrow A \times_X B$$

by $\lambda z. (p(z), q(z), H(z))$. Furthermore, we will write

$$\text{is_pullback}(f, g, H) \equiv \text{is_equiv}(\text{gap}(p, q, H)).$$

Proposition 1.4.10. Consider a commuting square

$$\begin{array}{ccc}
C & \xrightarrow{q} & B \\
p \downarrow & & \downarrow g \\
A & \xrightarrow{f} & X
\end{array}$$

with $H : f \circ p \sim g \circ q$. The following are equivalent:

- (i) The square is a pullback square
- (ii) There is a term of type

$$\text{is_pullback}(p, q, H) \equiv \text{is_equiv}(\text{gap}(p, q, H)).$$

Proof. Note that there are homotopies

$$\begin{aligned} K &: \pi_1 \circ \mathbf{gap}(p, q, H) \sim p \\ L &: \pi_2 \circ \mathbf{gap}(p, q, H) \sim q \\ M &: (\pi_3 \cdot \mathbf{gap}(p, q, H)) \cdot (g \cdot L) \sim (f \cdot K) \cdot H. \end{aligned}$$

given by

$$\begin{aligned} K &::= \lambda z. \mathbf{refl}_{p(z)} \\ L &::= \lambda z. \mathbf{refl}_{q(z)} \\ M &::= \lambda z. \mathbf{right_unit}(H(z)) \cdot \mathbf{left_unit}(H(z))^{-1}. \end{aligned}$$

Therefore the claim follows by Lemma 1.4.8. \square

1.4.3 Fiberwise equivalences

Proposition 1.4.11. *Let $f : A \rightarrow B$, and let $g : \prod_{(a:A)} P(a) \rightarrow Q(f(a))$ be a fiberwise transformation. The following are equivalent:*

(i) *The commuting square*

$$\begin{array}{ccc} \sum_{(a:A)} P(a) & \xrightarrow{\mathbf{tot}_f(g)} & \sum_{(b:B)} Q(b) \\ \mathbf{pr}_1 \downarrow & & \downarrow \mathbf{pr}_1 \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square.

(ii) *g is a fiberwise equivalence.*

Proof. The gap map factors as follows

$$\begin{array}{ccc} \sum_{(x:A)} P(x) & \xrightarrow{\mathbf{gap}} & A \times_B \left(\sum_{(y:B)} Q(y) \right) \\ \searrow \mathbf{tot}(g) & & \nearrow \mathbf{gap}' := \lambda(x,q). (x, (f(x), q), \mathbf{refl}_{f(x)}) \\ & & \sum_{(x:A)} Q(f(x)) \end{array}$$

Since \mathbf{gap}' is an equivalence, it follows by Proposition 1.2.4 that the gap map is an equivalence if and only if g is a fiberwise equivalence. \square

Lemma 1.4.12. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

with $H : f \circ p \sim g \circ q$, and consider the fiberwise transformation

$$\mathbf{fib}_{(f,q,H)} : \prod_{(a:A)} \mathbf{fib}_p(a) \rightarrow \mathbf{fib}_g(f(a))$$

given by $\lambda a. \lambda(c, u). (q(c), H(c)^{-1} \cdot \mathbf{ap}_f(u))$. Then there is an equivalence

$$\mathbf{fib}_{\mathbf{gap}(p,q,H)}((a, b, \alpha)) \simeq \mathbf{fib}_{\mathbf{fib}_{(f,q,H)}(a)}((b, \alpha^{-1}))$$

Proof. To obtain an equivalence of the desired type we simply concatenate known equivalences:

$$\begin{aligned} \mathbf{fib}_h((a, b, \alpha)) &\equiv \sum_{(z:C)} (p(z), q(z), H(z)) = (a, b, \alpha) \\ &\simeq \sum_{(z:C)} \sum_{(u:p(z)=a)} \sum_{(v:q(z)=b)} H(z) \cdot \mathbf{ap}_g(v) = \mathbf{ap}_f(u) \cdot \alpha \\ &\simeq \sum_{((z,u):\mathbf{fib}_p(a))} \sum_{(v:q(z)=b)} H(z)^{-1} \cdot \mathbf{ap}_f(u) = \mathbf{ap}_g(v) \cdot \alpha^{-1} \\ &\simeq \mathbf{fib}_{\varphi(a)}((b, \alpha^{-1})) \end{aligned} \quad \square$$

Corollary 1.4.13. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

with $H : f \circ p \sim g \circ q$. The following are equivalent:

- (i) The square is a pullback square.
- (ii) The induced map on fibers

$$\mathbf{fib}_{(p,q,H)} : \prod_{(x:A)} \mathbf{fib}_p(x) \rightarrow \mathbf{fib}_g(f(x))$$

is a fiberwise equivalence.

Corollary 1.4.14. *Consider a commuting square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X. \end{array}$$

and suppose that g is an equivalence. Then the following are equivalent:

- (i) The square is a pullback square.
- (ii) The map $p : C \rightarrow A$ is an equivalence.

Proof. If the square is a pullback square, then by Proposition 1.4.11 the fibers of p are equivalent to the fibers of g , which are contractible by Proposition 1.1.19. Thus it follows that p is a contractible map, and hence that p is an equivalence.

If p is an equivalence, then by Proposition 1.1.19 both $\mathbf{fib}_p(x)$ and $\mathbf{fib}_g(f(x))$ are contractible for any $x : X$. It follows that the induced map $\mathbf{fib}_p(x) \rightarrow \mathbf{fib}_g(f(x))$ is an equivalence. Thus we apply Corollary 1.4.13 to conclude that the square is a pullback. \square

1.5 The univalence axiom

The univalence axiom characterizes the identity type of the universe. It is considered to be an *extensionality principle* for types. In the following theorem we introduce the univalence axiom and give two more equivalent ways of stating this.

Proposition 1.5.1. *The following are equivalent:*

(i) The **univalence axiom**: for any $A : \mathcal{U}$ the map

$$\text{equiv_eq} := \text{ind}_{A=(\text{id}_A)} : \prod_{(B:\mathcal{U})} (A = B) \rightarrow (A \simeq B).$$

is a fiberwise equivalence. If this is the case, we write eq_equiv for the inverse of equiv_eq .

(ii) The type

$$\sum_{(B:\mathcal{U})} A \simeq B$$

is contractible for each $A : \mathcal{U}$.

(iii) The principle of **equivalence induction**: for every $A : \mathcal{U}$ and for every type family

$$P : \prod_{(B:\mathcal{U})} (A \simeq B) \rightarrow \text{Type},$$

the map

$$\left(\prod_{(B:\mathcal{U})} \prod_{(e:A \simeq B)} P(B, e) \right) \rightarrow P(A, \text{id}_A)$$

given by $f \mapsto f(A, \text{id}_A)$ has a section.

It is a trivial observation, but nevertheless of fundamental importance, that by the univalence axiom the identity types of \mathcal{U} are equivalent to types in \mathcal{U} , because it provides an equivalence $(A = B) \simeq (A \simeq B)$, and the type $A \simeq B$ is in \mathcal{U} for any $A, B : \mathcal{U}$. Since the identity types of \mathcal{U} are equivalent to types in \mathcal{U} , we also say that the universe is *locally small*.

Definition 1.5.2. (i) A type A is said to be **essentially small** if there is a type $X : \mathcal{U}$ and an equivalence $A \simeq X$. We write

$$\text{ess_small}(A) := \sum_{(X:\mathcal{U})} A \simeq X.$$

(ii) A map $f : A \rightarrow B$ is said to be **essentially small** if for each $b : B$ the fiber $\text{fib}_f(b)$ is essentially small. We write

$$\text{ess_small}(f) := \prod_{(b:B)} \text{ess_small}(\text{fib}_f(b)).$$

(iii) A type A is said to be **locally small** if for every $x, y : A$ the identity type $x = y$ is essentially small. We write

$$\text{loc_small}(A) := \prod_{(x,y:A)} \text{ess_small}(x = y).$$

- (iv) Similarly, a map $f : A \rightarrow X$ is said to be **locally small** if $\delta_f : A \rightarrow A \times_X A$ is essentially small.

Lemma 1.5.3. *The type $\text{ess_small}(X)$ is a proposition for any type X .*

Proof. Let X be a type. Our goal is to show that the type

$$\sum_{(Y:\mathcal{U})} X \simeq Y$$

is a proposition. Suppose there is a type $X' : \mathcal{U}$ and an equivalence $e : X \simeq X'$, then the map

$$(X \simeq Y) \rightarrow (X' \simeq Y)$$

given by precomposing with e^{-1} is an equivalence. This induces an equivalence on total spaces

$$\left(\sum_{(Y:\mathcal{U})} X \simeq Y \right) \simeq \left(\sum_{(Y:\mathcal{U})} X' \simeq Y \right)$$

However, the codomain of this equivalence is contractible by Proposition 1.5.1. Thus it follows that the asserted type is a proposition. \square

Corollary 1.5.4. *For each function $f : A \rightarrow B$, the type $\text{ess_small}(f)$ is a proposition, and for each type X the type $\text{loc_small}(X)$ is a proposition.*

Proof. This follows from the fact that propositions are closed under dependent products, established in Corollary 1.3.2. \square

Remark 1.5.5. The property of essentially smallness is preserved by Π , Σ , and Id . Of course, any contractible type is essentially small, and so is any small type. The property of essentially smallness is preserved by Σ and Id , and the exponent X^A of a locally small type X by an essentially small type A is again locally small. Furthermore, any proposition is locally small, and any universe is locally small with respect to itself.

Definition 1.5.6. Consider two functions $f : A \rightarrow X$ and $g : B \rightarrow X$. We define the type

$$\text{hom}_X(f, g) := \sum_{(h:A \rightarrow B)} f \sim g \circ h.$$

In other words, the type $\text{hom}_X(f, g)$ is the type of functions $h : A \rightarrow B$ equipped with a homotopy witnessing that the triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & X \end{array}$$

Lemma 1.5.7. *Let P and Q be two type families over X , and write pr_1^P and pr_1^Q for their first projections, respectively. Then the map*

$$\text{tot_triangle} : \left(\prod_{(x:X)} P(x) \rightarrow Q(x) \right) \rightarrow \text{hom}_X(\text{pr}_1^P, \text{pr}_1^Q)$$

given by $\text{tot_triangle}(f) := (\text{tot}(f), \lambda(x, y). \text{refl}_x)$, is an equivalence.

Corollary 1.5.8. *For any two maps $f : A \rightarrow X$ and $g : B \rightarrow X$, the map*

$$\text{fib_triangle} : \text{hom}_X(f, g) \rightarrow \prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_g(x)$$

given by $\lambda(h, H). \lambda x. \lambda(a, p). (h(a), H(a)^{-1} \cdot p)$ is an equivalence.

Theorem 1.5.9. *For any small type $A : \mathcal{U}$ there is an equivalence*

$$(A \rightarrow \mathcal{U}) \simeq \left(\sum_{(X:\mathcal{U})} X \rightarrow A \right).$$

Proof. Note that we have the function

$$\varphi : \lambda B. \left(\sum_{(x:A)} B(x), \text{pr}_1 \right) : (A \rightarrow \mathcal{U}) \rightarrow \left(\sum_{(X:\mathcal{U})} X \rightarrow A \right).$$

The fiber of this map at (X, f) is by univalence and function extensionality equivalent to the type

$$\sum_{(B:A \rightarrow \mathcal{U})} \sum_{(e:(\sum_{(x:A)} B(x)) \simeq X)} \text{pr}_1 \sim f \circ e.$$

By Corollary 1.5.8 this type is equivalent to the type

$$\sum_{(B:A \rightarrow \mathcal{U})} \prod_{(a:A)} B(a) \simeq \text{fib}_f(a),$$

and by ‘type theoretic choice’, which was established in Proposition 1.3.5, this type is equivalent to

$$\prod_{(a:A)} \sum_{(X:\mathcal{U})} X \simeq \text{fib}_f(a).$$

We conclude that the fiber of φ at (X, f) is equivalent to the type $\text{ess_small}(f)$. However, since $f : X \rightarrow A$ is a map between small types it is essentially small. Moreover, since being essentially small is a proposition by Lemma 1.5.3, it follows that $\text{fib}_\varphi((X, f))$ is contractible for every $f : X \rightarrow A$. In other words, φ is a contractible map, and therefore it is an equivalence. \square

Remark 1.5.10. The inverse of the map

$$\varphi : (A \rightarrow \mathcal{U}) \rightarrow \left(\sum_{(X:\mathcal{U})} X \rightarrow A \right).$$

constructed in Theorem 1.5.9 is the map $(X, f) \mapsto \text{fib}_f$.

1.6 The object classifier

Definition 1.6.1. Let $p : E \rightarrow B$ and $p' : E' \rightarrow B'$ be maps. A morphism from p' to p is a triple (f, g, H) consisting of maps $f : B' \rightarrow B$ and $g : E' \rightarrow E$ and a homotopy $H : f \circ p' \sim p \circ g$ witnessing that the square

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array} \quad (1.1)$$

commutes. We write $\text{hom}(p', p)$ for the type of such triples (f, g, H) , and sometimes we write $\text{hom}_f(p', p)$ for the type of pairs (g, H) . A morphism (f, g, H) is said to be **cartesian** if the square in Eq. (1.1) is cartesian. We write $\text{cart}(p', p)$ for the type of cartesian morphisms from p' to p , and we write $\text{cart}_f(p', p)$ for the type of triples (g, H, t) for the type of triples, where $t : \text{is_pullback}(p', g, H)$.

Definition 1.6.2. A morphism $p : E \rightarrow B$ is said to be an **object classifier** if the type $\text{cart}(p', p)$ is a proposition for each $p' : E' \rightarrow B'$. If $p : E \rightarrow B$ is an object classifier, we also write

$$\text{is_classified}(p') := \text{hom}(p', p).$$

Our goal in this section is to show that a univalent universe is an object classifier.

Proposition 1.6.3. *Let $\alpha : I \rightarrow J$ be a map, and let $A : I \rightarrow \mathcal{U}$ and $B : J \rightarrow \mathcal{U}$ be type families. Then the map*

$$\left(\prod_{(i:I)} A_i \rightarrow B_{\alpha(i)} \right) \rightarrow \text{hom}_{\alpha}(\text{pr}_1^A, \text{pr}_1^B)$$

given by $\lambda f. (\text{tot}_{\alpha}(f), \lambda(x, y). \text{refl}_x)$ is an equivalence. Furthermore, the map

$$\left(\prod_{(i:I)} A_i \simeq B_{\alpha(i)} \right) \rightarrow \text{cart}_{\alpha}(\text{pr}_1^A, \text{pr}_1^B)$$

given by $\lambda e. (\text{tot}_{\alpha}(e), \lambda(x, y). \text{refl}_x, t)$ where t is the term constructed in Proposition 1.4.11, is an equivalence.

Proof. We have the equivalences

$$\begin{aligned} \prod_{(i:I)} A_i \rightarrow B_{\alpha(i)} &\simeq \prod_{(i:I)} \prod_{(a:A_i)} \sum_{(j:J)} \sum_{(\gamma:\alpha(i)=j)} B_j \\ &\simeq \prod_{(i:I)} \prod_{(a:A_i)} \sum_{(j:J)} \sum_{(b:B_j)} \alpha(i) = j \\ &\simeq \prod_{((i,a):\sum_{(i:I)} A_i)} \sum_{((j,b):\sum_{(j:J)} B_j)} \alpha(i) = j \\ &\simeq \sum_{(f:(\sum_{(i:I)} A_i) \rightarrow (\sum_{(j:J)} B_j))} \alpha \circ \text{pr}_1^A \sim \text{pr}_1^B \circ f \\ &\equiv \text{hom}_{\alpha}(\text{pr}_1^A, \text{pr}_1^B). \end{aligned}$$

It is easy to check that this composite is the asserted map. The second claim follows from Proposition 1.4.11. \square

Corollary 1.6.4. *Consider a diagram of the form*

$$\begin{array}{ccc} A & & B \\ f \downarrow & & \downarrow g \\ I & \xrightarrow{\alpha} & J. \end{array}$$

Then the map

$$\text{hom}_{\alpha}(f, g) \rightarrow \left(\prod_{(i:I)} \text{fib}_f(i) \rightarrow \text{fib}_g(\alpha(i)) \right)$$

given by $\lambda(h, H). \lambda i. \lambda(a, p). (h(a), H(a)^{-1} \cdot \text{ap}_{\alpha}(p))$ is an equivalence.

Theorem 1.6.5. *Let $f : A \rightarrow B$ be a map, and let \mathcal{U} be a univalent universe with universal family El over \mathcal{U} . Then there is an equivalence*

$$\text{ess_small}(f) \simeq \text{cart}(f, \text{pr}_1^{\text{El}}).$$

In particular, the type $\text{cart}(f, \text{pr}_1^{\text{El}})$ is a proposition for each map f , so the universe is an object classifier in the sense of Definition 1.6.2.

Proof. From Corollary 1.6.4 we obtain that the type of pairs (\tilde{F}, H) is equivalent to the type of fiberwise transformations

$$\prod_{(b:B)} \text{fib}_f(b) \rightarrow F(b).$$

By Corollary 1.4.13 the square is a pullback square if and only if the induced map

$$\prod_{(b:B)} \text{fib}_f(b) \rightarrow F(b)$$

is a fiberwise equivalence. Thus the data (F, \tilde{F}, H, pb) is equivalent to the type of pairs (F, e) where e is a fiberwise equivalence from fib_f to F . By Proposition 1.3.5 the type of pairs (F, e) is equivalent to the type $\text{ess_small}(f)$. \square

Remark 1.6.6. For any type A (not necessarily small), and any $B : A \rightarrow \mathcal{U}$, the square

$$\begin{array}{ccc} \sum_{(x:A)} B(x) & \xrightarrow{\lambda(x,y).(B(x),y)} & \sum_{(X:\mathcal{U})} X \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ A & \xrightarrow{B} & \mathcal{U} \end{array}$$

is a pullback square. Therefore it follows that for any family $B : A \rightarrow \mathcal{U}$ of small types, the projection map $\text{pr}_1 : \sum_{(x:A)} B(x) \rightarrow A$ is an essentially small map. To see that the claim is a direct consequence of Proposition 1.4.11 we write the asserted square in its rudimentary form:

$$\begin{array}{ccc} \sum_{(x:A)} \text{El}(B(x)) & \xrightarrow{\lambda(x,y).(B(x),y)} & \sum_{(X:\mathcal{U})} \text{El}(X) \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ A & \xrightarrow{B} & \mathcal{U}. \end{array}$$

In the following theorem we show that a type is locally small if and only if its diagonal is classified by \mathcal{U} .

Theorem 1.6.7. *Let A be a type. The following are equivalent:*

- (i) *A is locally small.*
- (ii) *The diagonal $\delta_A : A \rightarrow A \times A$ is classified by \mathcal{U} .*

Proof. The identity type $x = y$ is the fiber of δ_A at $(x, y) : A \times A$. Therefore it follows that A is locally small if and only if the diagonal δ_A is essentially small. Now the result follows from Theorem 1.6.5. \square

Chapter 2

Type theoretic descent

In this chapter we study homotopy pushouts, which were established as higher inductive types in homotopy type theory in section 6.8 of [33]. From this chapter on, we will assume that universes are closed under homotopy pushouts. This is the last assumption that we will be making in the present work. In particular, we will not assume the existence of higher inductive types with some self-reference in the constructors (e.g. the propositional truncation).

Our first main result is the descent theorem for homotopy pushouts (Theorem 2.2.9 and Corollary 2.2.10), in which we establish that a cartesian transformation of spans

$$\begin{array}{ccccc}
 A' & \longleftarrow & S' & \longrightarrow & B' \\
 \downarrow & & \lrcorner & \lrcorner & \downarrow \\
 A & \longleftarrow & S & \longrightarrow & B
 \end{array}$$

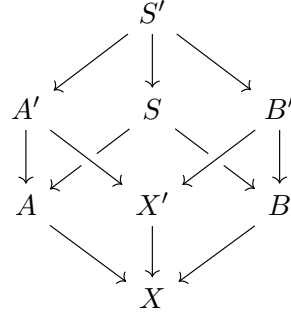
extends uniquely to a cartesian transformation of the pushout squares, i.e. a commuting cube

$$\begin{array}{ccccc}
 & & S' & & \\
 & \swarrow & \downarrow & \searrow & \\
 A' & & S & & B' \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 A & & A' \sqcup^{S'} B' & & B \\
 & \swarrow & \downarrow & \searrow & \\
 & & A \sqcup^S B & &
 \end{array}$$

of which the vertical sides are pullback squares.

The second main theorem of this chapter, Theorem 2.2.12, is an adaption to homotopy type theory of a theorem due to [1]. It is closely related to the descent theorem but can

be stated without a universe: for any commuting cube



of which the two vertical back squares are pullback squares, the two vertical front squares are pullback squares if and only if the square

$$\begin{array}{ccc}
 A' \sqcup^{S'} B' & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 A \sqcup^S B & \longrightarrow & X.
 \end{array}$$

is a pullback square. Even though this statement does not involve a universe, we use the univalence axiom in our proof that this square being pullback implies that the front two vertical squares of the cube are pullback squares. Function extensionality suffices for the converse direction.

2.1 Homotopy pushouts

2.1.1 Pushouts as higher inductive types

Definition 2.1.1. A **span** \mathcal{S} from A to B is a triple (S, f, g) consisting of a type S and maps $f : S \rightarrow A$ and $g : S \rightarrow B$. We write $\text{span}(A, B)$ for the type of small spans from A to B .

Definition 2.1.2. Consider a span $\mathcal{S} \equiv (S, f, g)$ from A to B , and let X be a type. A **cocone** with vertex X on \mathcal{S} is a triple (i, j, H) consisting of maps $i : A \rightarrow X$, $j : B \rightarrow X$, and a homotopy $H : i \circ f \sim j \circ g$ witnessing that the square

$$\begin{array}{ccc}
 S & \xrightarrow{g} & B \\
 f \downarrow & & \downarrow j \\
 A & \xrightarrow{i} & X
 \end{array}$$

commutes. We write $\text{cocone}_{\mathcal{S}}(X)$ for the type of cocones with vertex X on \mathcal{S} .

Definition 2.1.3. Consider a commuting square

$$\begin{array}{ccc}
 S & \xrightarrow{g} & B \\
 f \downarrow & & \downarrow j \\
 A & \xrightarrow{i} & X,
 \end{array}$$

with $H : i \circ f \sim j \circ g$, and let Y be a type. We define the operation

$$\text{cocone_map}((i, j, H), Y) := (X \rightarrow Y) \rightarrow \text{cocone}_{\mathcal{S}}(Y).$$

by $h \mapsto (h \circ i, h \circ j, h \cdot H)$.

Definition 2.1.4. A commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X \end{array}$$

with $H : i \circ f \sim j \circ g$ is said to be a **(homotopy) pushout square** if the cocone (i, j, H) with vertex X on the span $\mathcal{S} \equiv (S, f, g)$ satisfies the **universal property of pushouts**, which asserts that the map

$$\text{cocone_map}(i, j, H) : (X \rightarrow Y) \rightarrow \text{cocone}(Y)$$

is an equivalence for any type Y . Sometimes pushout squares are also called **cocartesian squares**.

Definition 2.1.5. Consider a pushout square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X \end{array}$$

with $H : i \circ f \sim j \circ g$, and consider a cocone (i', j', H') with vertex Y on the same span $\mathcal{S} \equiv (S, f, g)$. Then the unique map $h : X \rightarrow Y$ such that

$$\text{cocone_map}((i, j, H), Y, h) = (i', j', H')$$

is called the **cogap map** of (i', j', H') . We also write $\text{cogap}(i', j', H')$ for the cogap map, and we write

$$\begin{aligned} \text{left_comp}(i', j', H') &: i' \sim \text{cogap}(i', j', H') \circ \text{inl} \\ \text{right_comp}(i', j', H') &: j' \sim \text{cogap}(i', j', H') \circ \text{inr} \\ \text{coh_comp}(i', j', H') &: (\text{left_comp}(i', j', H') \cdot g) \cdot H' \sim (\text{cogap}(i', j', H') \cdot \text{glue}) \cdot (\text{right_comp}(i', j', H') \cdot f). \end{aligned}$$

for the homotopies determining the uniqueness of $\text{cogap}(i', j', H')$.

Proposition 2.1.6. Consider a commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X, \end{array}$$

with $H : i \circ f \sim j \circ g$. The following are equivalent:

(i) The square is a pushout square.

(ii) The square

$$\begin{array}{ccc} Y^X & \xrightarrow{-\circ j} & Y^B \\ -\circ i \downarrow & & \downarrow -\circ g \\ Y^A & \xrightarrow{-\circ f} & Y^S, \end{array}$$

which commutes by the homotopy

$$\lambda h. \text{eq_htpy}(h \cdot H),$$

is a pullback square, for every type Y .

(iii) For every type family P over X , the square

$$\begin{array}{ccc} \prod_{(x:X)} P(x) & \xrightarrow{-\circ j} & \prod_{(b:B)} P(j(b)) \\ -\circ i \downarrow & & \downarrow -\circ g \\ \prod_{(a:A)} P(i(a)) & \xrightarrow{\lambda h. \lambda x. \text{tr}_P(H(x), h(f(x)))} & \prod_{(x:S)} P(j(g(x))), \end{array}$$

which commutes by the homotopy

$$\lambda h. \text{eq_htpy}(\lambda x. \text{apd}_h(H(x)))$$

is a pullback square. This property is also called the **dependent universal property of pushouts**.

(iv) The gap map of the square

$$\begin{array}{ccc} \prod_{(x:X)} P(x) & \longrightarrow & \prod_{(b:B)} P(j(b)) \\ \downarrow & & \downarrow \\ \prod_{(a:A)} P(i(a)) & \longrightarrow & \prod_{(x:S)} P(j(g(x))) \end{array}$$

has a section, for any type family P over X . This property is also called the **induction principle of pushouts**.

Definition 2.1.7. From now on we will assume that any span has a pushout, and moreover that universes are closed under pushouts. We will write $A \sqcup^S B$ for the pushout of the span $\mathcal{S} \equiv (S, f, g)$ from A to B . The type $A \sqcup^S B$ comes equipped with a colimiting cocone $(\text{inl}, \text{inr}, \text{glue})$, as displayed in the pushout square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow \text{inr} \\ A & \xrightarrow{\text{inl}} & A \sqcup^S B. \end{array}$$

Remark 2.1.8. We note that if $\mathcal{S} \equiv (S, f, g)$ is a span of *pointed* types and pointed maps between them, then the pushout $A \sqcup^{\mathcal{S}} B$ of \mathcal{S} is again a pointed type. The cocone $(\text{inl}, \text{inr}, \text{glue})$ consists of two pointed maps and a pointed homotopy filling the square of pointed maps. Moreover, the pushout $A \sqcup^{\mathcal{S}} B$ satisfies a pointed version of the universal property: for any pointed type Y the square

$$\begin{array}{ccc} (A \sqcup^{\mathcal{S}} B \rightarrow_* Y) & \longrightarrow & (B \rightarrow_* Y) \\ \downarrow & & \downarrow \\ (A \rightarrow_* Y) & \longrightarrow & (S \rightarrow_* Y) \end{array}$$

is a pullback square.

2.1.2 Examples of pushouts

Definition 2.1.9. Let X be a type. We define the **suspension** ΣX of X to be the pushout of the span

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \text{inr} \\ \mathbf{1} & \xrightarrow{\text{inl}} & \Sigma X \end{array}$$

We will write $\mathbf{N} := \text{inl}(\star)$ and $\mathbf{S} := \text{inr}(\star)$.

Remark 2.1.10. By the universal property it follows that the map

$$(\Sigma X \rightarrow Y) \rightarrow \sum_{(y, y': Y)} X \rightarrow (y = y')$$

given by $h \mapsto (h(\mathbf{N}), h(\mathbf{S}), h \cdot \text{glue})$ is an equivalence.

Moreover, if X is a pointed type, then the suspension is considered to be a pointed type with base point \mathbf{N} . By the universal property of ΣX it follows that the square

$$\begin{array}{ccc} (\Sigma X \rightarrow_* Y) & \longrightarrow & (\mathbf{1} \rightarrow_* Y) \\ \downarrow & & \downarrow \\ (\mathbf{1} \rightarrow_* Y) & \longrightarrow & (X \rightarrow_* Y) \end{array}$$

is a pullback square. Since $\mathbf{1} \rightarrow_* Y$ is contractible, it follows that

$$(\Sigma X \rightarrow_* Y) \simeq \Omega(X \rightarrow_* Y) \simeq X \rightarrow_* \Omega(Y)$$

Definition 2.1.11. Given a map $f : A \rightarrow B$, we define the **cofiber** cof_f of f as the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \text{inr} \\ \mathbf{1} & \xrightarrow{\text{inl}} & \text{cof}_f. \end{array}$$

The cofiber of a map is sometimes also called the **mapping cone**.

Definition 2.1.12. We define the **join** $X * Y$ of X and Y to be the pushout

$$\begin{array}{ccc} X \times Y & \xrightarrow{\text{pr}_2} & Y \\ \text{pr}_1 \downarrow & & \downarrow \text{inr} \\ X & \xrightarrow{\text{inl}} & X * Y. \end{array}$$

Definition 2.1.13. We define the n -**sphere** \mathbb{S}^n for any $n \geq -1$ by induction on n , by taking

$$\begin{aligned} \mathbb{S}^{-1} &::= \mathbf{0} \\ \mathbb{S}^0 &::= \mathbf{2} \\ \mathbb{S}^{n+1} &::= \mathbf{2} * \mathbb{S}^n. \end{aligned}$$

Definition 2.1.14. Suppose A and B are pointed types, with base points a_0 and b_0 , respectively. The **(binary) wedge** $A \vee B$ of A and B is defined as the pushout

$$\begin{array}{ccc} \mathbf{2} & \longrightarrow & A + B \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & A \vee B. \end{array}$$

Definition 2.1.15. Given a type I , and a family of pointed types A over i , with base points $a_0(i)$. We define the **(indexed) wedge** $\bigvee_{(i:I)} A_i$ as the pushout

$$\begin{array}{ccc} I & \xrightarrow{\lambda i. (i, a_0(i))} & \sum_{(i:I)} A_i \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \bigvee_{(i:I)} A_i. \end{array}$$

Definition 2.1.16. Suppose A and B are pointed types. We define the **wedge inclusion** $\text{wedge_in} : A \vee B \rightarrow A \times B$ to be the unique map obtained via the universal property of pushouts as indicated in the diagram

$$\begin{array}{ccccc} \mathbf{1} & \longrightarrow & B & & \\ \downarrow & & \downarrow & \searrow^{\lambda b. (a_0, b)} & \\ A & \longrightarrow & A \vee B & \xrightarrow{\text{wedge_in}} & A \times B \\ & \searrow^{\lambda a. (a, b_0)} & & \searrow & \end{array}$$

We define the **smash product** $A \wedge B$ of A and B as the cofiber of the wedge inclusion, i.e. as a pushout

$$\begin{array}{ccc} A \vee B & \xrightarrow{\text{wedge_in}} & A \times B \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & A \wedge B. \end{array}$$

2.1.3 Properties of iterated pushouts

The following corollary is also called the **pasteing property** of pullbacks.

edit

Corollary 2.1.17. *Consider a commuting diagram of the form*

$$\begin{array}{ccccc} A & \xrightarrow{k} & B & \xrightarrow{l} & C \\ f \downarrow & & \downarrow g & & \downarrow h \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & Z \end{array}$$

with homotopies $H : i \circ f \sim g \circ k$ and $K : j \circ g \sim h \circ l$, and the homotopy

$$(j \cdot H) \cdot (K \cdot k) : j \circ i \circ f \sim h \circ l \circ k$$

witnessing that the outer rectangle commutes. Furthermore, suppose that the square on the right is a pullback square. Then the following are equivalent:

- (i) The square on the left is a pullback square.
- (ii) The outer rectangle is a pullback square.

Proof. The commutativity of the two squares induces fiberwise transformations

$$\begin{aligned} \prod_{(x:X)} \text{fib}_f(x) &\rightarrow \text{fib}_g(i(x)) \\ \prod_{(y:Y)} \text{fib}_g(y) &\rightarrow \text{fib}_h(j(y)). \end{aligned}$$

By the assumption that the square on the right is a pullback square, it follows from Corollary 1.4.13 that the fiberwise transformation

$$\prod_{(y:Y)} \text{fib}_g(y) \rightarrow \text{fib}_h(j(y))$$

is a fiberwise equivalence. Therefore it follows from 3-for-2 property of equivalences that the fiberwise transformation

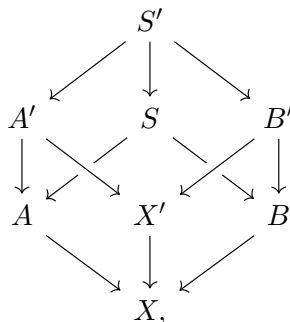
$$\prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_g(i(x))$$

is a fiberwise equivalence if and only if the fiberwise transformation

$$\prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_h(j(i(x)))$$

is a fiberwise equivalence. Now the claim follows from one more application of Corollary 1.4.13. \square

Corollary 2.1.18. *Consider a commuting cube*



of which the two front squares are pullback squares. Then the back left square is a pullback square if and only if the back right square is a pullback square.

Proposition 2.1.19. *Consider the following configuration of commuting squares:*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{k} & C \\ f \downarrow & & g \downarrow & & \downarrow h \\ X & \xrightarrow{j} & Y & \xrightarrow{l} & Z \end{array}$$

with homotopies $H : j \circ f \sim g \circ i$ and $K : l \circ g \sim h \circ k$, and suppose that the square on the left is a pushout square. Then the square on the right is a pushout square if and only if the outer rectangle is a pushout square.

Proof. Let T be a type. Taking the exponent $T^{(-)}$ of the entire diagram of the statement of the theorem, we obtain the following commuting diagram

$$\begin{array}{ccccc} TZ & \xrightarrow{-\circ l} & TY & \xrightarrow{-\circ j} & TX \\ -\circ h \downarrow & & -\circ g \downarrow & & \downarrow -\circ f \\ TC & \xrightarrow{-\circ k} & TB & \xrightarrow{-\circ i} & TA. \end{array}$$

By the assumption that Y is the pushout of $B \leftarrow A \rightarrow X$, it follows that the square on the right is a pullback square. It follows by Corollary 2.1.17 that the rectangle on the left is a pullback if and only if the outer rectangle is a pullback. Thus the statement follows by the second characterization in Proposition 2.1.6. \square

Lemma 2.1.20. *Consider a map $f : A \rightarrow B$. Then the cofiber of the map $\text{inr} : B \rightarrow \text{cof}_f$ is equivalent to the suspension ΣA of A .*

Proposition 2.1.21. *Consider a commuting square*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{j} & Y \end{array}$$

and write $\text{cogap} : X \sqcup^A B \rightarrow Y$ for the cogap map. Then the square

$$\begin{array}{ccc} \text{cof}_f & \longrightarrow & \text{cof}_g \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \text{cof}_{\text{cogap}} \end{array}$$

is a pushout square.

2.2 Descent for pushouts

2.2.1 Type families over pushouts

Definition 2.2.1. Consider a commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X. \end{array}$$

with $H : i \circ f \sim j \circ g$, where all types involved are in \mathcal{U} . The type $\text{Desc}(\mathcal{S})$ of **descent data** for X , is defined to be the type of triples (P_A, P_B, P_S) consisting of

$$\begin{aligned} P_A &: A \rightarrow \mathcal{U} \\ P_B &: B \rightarrow \mathcal{U} \\ P_S &: \prod_{(x:S)} P_A(f(x)) \simeq P_B(g(x)). \end{aligned}$$

Furthermore, we define the map

$$\text{desc_fam}_{\mathcal{S}}(i, j, H) : (X \rightarrow \mathcal{U}) \rightarrow \text{Desc}(\mathcal{S})$$

by $P \mapsto (P \circ i, P \circ j, \lambda x. \text{tr}_P(H(x)))$.

Proposition 2.2.2. Consider a commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X. \end{array}$$

with $H : i \circ f \sim j \circ g$. If the square is a pushout square, then the function

$$\text{desc_fam}_{\mathcal{S}}(i, j, H) : (X \rightarrow \mathcal{U}) \rightarrow \text{Desc}(\mathcal{S})$$

is an equivalence.

Proof. By the 3-for-2 property of equivalences it suffices to construct an equivalence $\varphi : \text{cocone}_{\mathcal{S}}(\mathcal{U}) \rightarrow \text{Desc}(\mathcal{S})$ such that the triangle

$$\begin{array}{ccc} & \mathcal{U}^X & \\ \text{cocone_map}_{\mathcal{S}}(i,j,H) \swarrow & & \searrow \text{desc_fam}_{\mathcal{S}}(i,j,H) \\ \text{cocone}_{\mathcal{S}}(\mathcal{U}) & \xrightarrow[\varphi]{\simeq} & \text{Desc}(\mathcal{S}) \end{array}$$

commutes.

Since we have equivalences

$$\text{equiv_eq} : \left(P_A(f(x)) = P_B(g(x)) \right) \simeq \left(P_A(f(x)) \simeq P_B(g(x)) \right)$$

for all $x : S$, we obtain an equivalence on the dependent products

$$\left(\prod_{(x:S)} P_A(f(x)) = P_B(g(x)) \right) \simeq \left(\prod_{(x:S)} P_A(f(x)) \simeq P_B(g(x)) \right).$$

by post-composing with the equivalences `equiv_eq`. We define φ to be the induced map on total spaces. Explicitly, we have

$$\varphi := \lambda(P_A, P_B, K). (P_A, P_B, \lambda x. \text{equiv_eq}(K(x))).$$

Then φ is an equivalence by Proposition 1.2.4, and the triangle commutes because there is a homotopy

$$\text{equiv_eq}(\text{ap}_P(H(x))) \sim \text{tr}_P(H(x)). \quad \square$$

Corollary 2.2.3. *Consider descent data (P_A, P_B, P_S) for a pushout square as in Proposition 2.2.2. Then the type of quadruples (P, e_A, e_B, e_S) consisting of a family $P : X \rightarrow \mathcal{U}$ equipped with fiberwise equivalences*

$$\begin{aligned} e_A &: \prod_{(a:A)} P_A(a) \simeq P(i(a)) \\ e_B &: \prod_{(b:B)} P_B(a) \simeq P(j(b)) \end{aligned}$$

and a homotopy e_S witnessing that the square

$$\begin{array}{ccc} P_A(f(x)) & \xrightarrow{e_A(f(x))} & P(i(f(x))) \\ P_S(x) \downarrow & & \downarrow \text{tr}_P(H(x)) \\ P_B(g(x)) & \xrightarrow{e_B(g(x))} & P(j(g(x))) \end{array}$$

commutes, is contractible.

Proof. The fiber of `desc_famS(i, j, H)` map at (P_A, P_B, P_S) is equivalent to the type of quadruples (P, e_A, e_B, e_S) as described in the theorem, which are contractible by Proposition 1.1.19. \square

For the remainder of this subsection we consider a pushout square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X. \end{array}$$

with $H : i \circ f \sim j \circ g$, descent data

$$\begin{aligned} P_A &: A \rightarrow \mathcal{U} \\ P_B &: B \rightarrow \mathcal{U} \\ P_S &: \prod_{(x:S)} P_A(f(x)) \simeq P_B(g(x)), \end{aligned}$$

and a family $P : X \rightarrow \mathcal{U}$ equipped with

$$e_A : \prod_{(a:A)} P_A(a) \simeq P(i(a))$$

$$e_B : \prod_{(b:B)} P_B(a) \simeq P(j(b))$$

and a homotopy e_S witnessing that the square

$$\begin{array}{ccc} P_A(f(x)) & \xrightarrow{e_A(f(x))} & P(i(f(x))) \\ P_S(x) \downarrow & & \downarrow \text{tr}_P(H(x)) \\ P_B(g(x)) & \xrightarrow{e_B(g(x))} & P(j(g(x))) \end{array}$$

commutes.

Definition 2.2.4. We define the commuting square

$$\begin{array}{ccc} \sum_{(x:S)} P_A(f(x)) & \xrightarrow{g' := \text{tot}_g(e_S)} & \sum_{(b:B)} P_B(b) \\ f' := \text{tot}_f(\lambda x. \text{id}_{P_A(f(x))}) \downarrow & & \downarrow j' := \text{tot}_j(e_B) \\ \sum_{(a:A)} P_A(a) & \xrightarrow{i' := \text{tot}_i(e_A)} & \sum_{(x:X)} P(x) \end{array}$$

with the homotopy $H' : i' \circ f' \sim j' \circ g'$ defined as

$$\lambda(x, y). \text{eq_pair}(H(x), e_S(x, y)^{-1}).$$

Furthermore, we will write \mathcal{S}' for the span

$$\sum_{(a:A)} P_A(a) \xleftarrow{f'} \sum_{(x:S)} P_A(f(x)) \xrightarrow{g'} \sum_{(b:B)} P_B(b).$$

We now state the flattening lemma for pushouts, which should be compared to the flattening lemma for coequalizers, stated in Lemma 6.12.2 of [33]. We note that, using the dependent universal property of pushouts, our proof is substantially shorter.

Lemma 2.2.5 (The flattening lemma). *The commuting square*

$$\begin{array}{ccc} \sum_{(x:S)} P_A(f(x)) & \xrightarrow{g'} & \sum_{(b:B)} P_B(b) \\ f' \downarrow & & \downarrow j' \\ \sum_{(a:A)} P_A(a) & \xrightarrow{i'} & \sum_{(x:X)} P(x) \end{array}$$

is a pushout square.

Proof. Note that we have a commuting cube

$$\begin{array}{ccccc}
 & & Y^{\sum_{(x:X)} P(x)} & & \\
 & \swarrow & \downarrow \text{ev_pair} & \searrow & \\
 Y^{\sum_{(a:A)} P_A(a)} & & \prod_{(x:X)} Y^{P(x)} & & Y^{\sum_{(b:B)} P_B(b)} \\
 \downarrow \text{ev_pair} & \swarrow & \swarrow & \searrow & \downarrow \text{ev_pair} \\
 \prod_{(a:A)} Y^{P_A(a)} & & Y^{\sum_{(x:S)} P_A(f(x))} & & \prod_{(b:B)} Y^{P_B(b)} \\
 & \swarrow & \downarrow \text{ev_pair} & \searrow & \\
 & & \prod_{(x:S)} Y^{P_A(f(x))} & &
 \end{array}$$

for any type Y . In this cube, the bottom square is a pullback square by property (iii) of Proposition 2.1.6. The vertical maps (of the form ev_pair) are equivalences, so it follows that the top square is a pullback square. We conclude that $\sum_{(x:X)} P(x)$ is a pushout. \square

2.2.2 The descent property for pushouts

Definition 2.2.6. Consider a span \mathcal{S} from A to B , and a span \mathcal{S}' from A' to B' . A **cartesian transformation** of spans from \mathcal{S}' to \mathcal{S} is a tuple

$$(h_A, h_S, h_B, F, G, p_f, p_g)$$

consisting of maps $h_A : A' \rightarrow A$, $h_S : S' \rightarrow S$, and $h_B : B' \rightarrow B$, as indicated in the diagram

$$\begin{array}{ccccc}
 A' & \xleftarrow{f'} & S' & \xrightarrow{g'} & B' \\
 h_A \downarrow & & h_S \downarrow & & \downarrow h_B \\
 A & \xleftarrow{f} & S & \xrightarrow{g} & B,
 \end{array}$$

with homotopies $F : f \circ h_S \sim h_A \circ f'$ and $G : g \circ h_S \sim h_B \circ g'$, satisfying the conditions

$$p_f : \text{is_pullback}(h_S, f', F)$$

$$p_g : \text{is_pullback}(h_S, g', G)$$

that both squares are pullback squares. We write $\text{cart}(\mathcal{S}', \mathcal{S})$ for the type of cartesian transformations from \mathcal{S}' to \mathcal{S} , and we write

$$\text{Cart}(\mathcal{S}) := \sum_{(A', B' : \mathcal{U})} \sum_{(S' : \text{span}(A', B'))} \text{cart}(\mathcal{S}', \mathcal{S}).$$

Given descent data (P_A, P_B, P_S) on a span \mathcal{S} from A to B , we obtain a cartesian transformation

$$\begin{array}{ccccc}
 \sum_{(a:A)} P_A(a) & \xleftarrow{\text{tot}_f(\text{id})} & \sum_{(x:S)} P_A(f(x)) & \xrightarrow{\text{tot}_g(P_S)} & \sum_{(b:B)} P_B(b) \\
 \text{pr}_1 \downarrow & & \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\
 A & \xleftarrow{f} & S & \xrightarrow{g} & B
 \end{array}$$

with the canonical homotopies witnessing that the squares commute. Note that both the left and right commuting squares are pullback squares by Proposition 1.4.11. Thus we obtain an operation

$$\text{cart_desc}_{\mathcal{S}} : \text{Desc}(\mathcal{S}) \rightarrow \text{Cart}(\mathcal{S}).$$

Lemma 2.2.7. *For any span \mathcal{S} , the operation*

$$\text{cart_desc}_{\mathcal{S}} : \text{Desc}(\mathcal{S}) \rightarrow \text{Cart}(\mathcal{S})$$

is an equivalence.

Proof. Note that by Proposition 1.6.3 it follows that the types of triples (f', F, p_f) and (g', G, p_g) are equivalent to the types of fiberwise equivalences

$$\begin{aligned} \prod_{(x:S)} \text{fib}_{h_S}(x) &\simeq \text{fib}_{h_A}(f(x)) \\ \prod_{(x:S)} \text{fib}_{h_S}(x) &\simeq \text{fib}_{h_B}(g(x)) \end{aligned}$$

respectively. Furthermore, by Theorem 1.5.9 the types of pairs (S', h_S) , (A', h_A) , and (B', h_B) are equivalent to the types $S \rightarrow \mathcal{U}$, $A \rightarrow \mathcal{U}$, and $B \rightarrow \mathcal{U}$, respectively. Therefore it follows that the type $\text{Cart}(\mathcal{S})$ is equivalent to the type of tuples $(Q, P_A, \varphi, P_B, P_S)$ consisting of

$$\begin{aligned} Q &: S \rightarrow \mathcal{U} \\ P_A &: A \rightarrow \mathcal{U} \\ P_B &: B \rightarrow \mathcal{U} \\ \varphi &: \prod_{(x:S)} Q(x) \simeq P_A(f(x)) \\ P_S &: \prod_{(x:S)} Q(x) \simeq P_B(g(x)). \end{aligned}$$

However, the type of φ is equivalent to the type $P_A \circ f = Q$. Thus we see that the type of pairs (Q, φ) is contractible, so our claim follows. \square

Definition 2.2.8. Consider a commuting square

$$\begin{array}{ccc} S & \xrightarrow{g} & B \\ f \downarrow & & \downarrow j \\ A & \xrightarrow{i} & X \end{array}$$

with $H : i \circ f \sim j \circ g$. We define an operation

$$\text{cart_map}_{\mathcal{S}} : \left(\sum_{(X':\mathcal{U})} X' \rightarrow X \right) \rightarrow \text{Cart}(\mathcal{S}).$$

Construction. Let $X' : \mathcal{U}$ and $h_X : X' \rightarrow X$. Then we define A' , B' , and S' as the pullbacks

$$\begin{aligned} A' &:\equiv A \times_X X' \\ B' &:\equiv B \times_X X' \\ S' &:\equiv S \times_A A', \end{aligned}$$

resulting in a diagram of the form

$$\begin{array}{ccccc}
 & & S' & & \\
 & \swarrow & \downarrow & \dashrightarrow & \\
 A' & & S & & B' \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 A & & X' & & B \\
 & \swarrow & \downarrow & \swarrow & \\
 & & X & &
 \end{array}$$

By the universal property of B' it follows that there is a unique map $g' : S' \rightarrow B'$ making the cube commute. Moreover, since the two front squares and the back left squares are pullback squares by construction, it follows by Corollary 2.1.17 that also the back right square is a pullback square. Thus we obtain a cartesian transformation of spans. \square

The following theorem is analogous to Proposition 2.2.2.

Theorem 2.2.9 (The descent theorem for pushouts). *Consider a commuting square*

$$\begin{array}{ccc}
 S & \xrightarrow{g} & B \\
 f \downarrow & & \downarrow j \\
 A & \xrightarrow{i} & X
 \end{array}$$

with $H : i \circ f \sim j \circ g$. If this square is a pushout square, then the operation $\text{cart_map}_{\mathcal{S}}$ is an equivalence

$$\left(\sum_{(X':\mathcal{U})} X' \rightarrow X \right) \simeq \text{Cart}(\mathcal{S})$$

Proof. It suffices to show that the square

$$\begin{array}{ccc}
 \mathcal{U}^X & \xrightarrow{\text{desc_fam}_{\mathcal{S}}(i,j,H)} & \text{Desc}(\mathcal{S}) \\
 \text{map_fam}_X \downarrow & & \downarrow \text{cart_desc}_{\mathcal{S}} \\
 \sum_{(X':\mathcal{U})} X^{X'} & \xrightarrow{\text{cart_map}_{\mathcal{S}}} & \text{Cart}(\mathcal{S})
 \end{array}$$

commutes. To see that this suffices, note that the operation map_fam_X is an equivalence by Theorem 1.5.9, the operation $\text{desc_fam}_{\mathcal{S}}(i, j, H)$ is an equivalence by Proposition 2.2.2, and the operation $\text{cart_desc}_{\mathcal{S}}$ is an equivalence by Lemma 2.2.7.

To see that the square commutes, note that the composite

$$\text{cart_map}_{\mathcal{S}} \circ \text{map_fam}_X$$

takes a family $P : X \rightarrow \mathcal{U}$ to the cartesian transformation of spans

$$\begin{array}{ccccc}
 A \times_X \tilde{P} & \longleftarrow & S \times_A (A \times_X \tilde{P}) & \longrightarrow & B \times_X \tilde{P} \\
 \pi_1 \downarrow & & \pi_1 \downarrow & & \downarrow \pi_1 \\
 A & \longleftarrow & S & \longrightarrow & B,
 \end{array}$$

where $\tilde{P} \equiv \sum_{(x:X)} P(x)$.

The composite

$$\text{cart_desc}_S \circ \text{desc_fam}_X$$

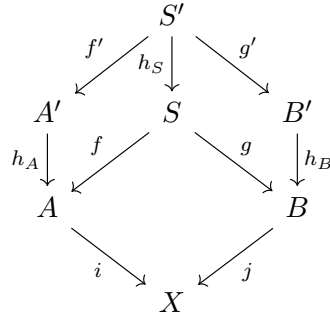
takes a family $P : X \rightarrow \mathcal{U}$ to the cartesian transformation of spans

$$\begin{array}{ccccc} \sum_{(a:A)} P(i(a)) & \longleftarrow & \sum_{(s:S)} P(i(f(s))) & \longrightarrow & \sum_{(b:B)} P(j(b)) \\ \downarrow & & \downarrow & & \downarrow \\ A & \longleftarrow & S & \longrightarrow & B \end{array}$$

These cartesian natural transformations are equal by Proposition 1.4.11. \square

Since cart_map_S is an equivalence it follows that its fibers are contractible.

Corollary 2.2.10. *Consider a diagram of the form*



with homotopies

$$\begin{aligned} F &: f \circ h_S \sim h_A \circ f' \\ G &: g \circ h_S \sim h_B \circ g' \\ H &: i \circ f \sim j \circ g, \end{aligned}$$

and suppose that the bottom square is a pushout square, and the top squares are pullback squares. Then the type of tuples $((X', h_X), (i', I, p), (j', J, q), (H', C))$ consisting of

(i) A type $X' : \mathcal{U}$ together with a morphism

$$h_X : X' \rightarrow X,$$

(ii) A map $i' : A' \rightarrow X'$, a homotopy $I : i \circ h_A \sim h_X \circ i'$, and a term p witnessing that the square

$$\begin{array}{ccc} A' & \xrightarrow{i'} & X' \\ h_A \downarrow & & \downarrow h_X \\ A & \xrightarrow{i} & X \end{array}$$

is a pullback square.

(iii) A map $j' : B' \rightarrow X'$, a homotopy $J : j \circ h_B \sim h_X \circ j'$, and a term q witnessing that the square

$$\begin{array}{ccc} B' & \xrightarrow{j'} & X' \\ h_B \downarrow & & \downarrow h_X \\ B & \xrightarrow{j} & X \end{array}$$

is a pullback square,

(iv) A homotopy $H' : i' \circ f' \sim j' \circ g'$, and a homotopy

$$C : (i \cdot F) \cdot ((I \cdot f') \cdot (h_X \cdot H')) \sim (H \cdot h_S) \cdot ((j \cdot G) \cdot (J \cdot g'))$$

witnessing that the cube

$$\begin{array}{ccccc} & & S' & & \\ & \swarrow & \downarrow & \searrow & \\ A' & & S & & B' \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ A & & X' & & B \\ & \swarrow & \downarrow & \searrow & \\ & & X & & \end{array}$$

commutes,

is contractible.

The following theorem should be compared to the flattening lemma, Lemma 2.2.5.

Theorem 2.2.11. Consider a commuting cube

$$\begin{array}{ccccc} & & S' & & \\ & \swarrow f' & \downarrow h_S & \searrow g' & \\ A' & & S & & B' \\ h_A \downarrow & \swarrow f & \downarrow & \searrow g & \downarrow h_B \\ A & & X' & & B \\ & \swarrow i & \downarrow h_X & \searrow j & \\ & & X & & \end{array}$$

in which the bottom square is a pushout, and the two vertical squares in the back are pullbacks. Then the following are equivalent:

(i) The two vertical squares in the front are pullback squares.

(ii) The top square is a pushout square.

Proof. By Corollary 1.4.13 we have fiberwise equivalences

$$\begin{aligned} F &: \prod_{(x:S)} \mathbf{fib}_{h_S}(x) \simeq \mathbf{fib}_{h_A}(f(x)) \\ G &: \prod_{(x:S)} \mathbf{fib}_{h_S}(x) \simeq \mathbf{fib}_{h_B}(g(x)) \\ I &: \prod_{(a:A)} \mathbf{fib}_{h_A}(a) \simeq \mathbf{fib}_{h_X}(i(a)) \\ J &: \prod_{(b:B)} \mathbf{fib}_{h_B}(b) \simeq \mathbf{fib}_{h_X}(j(b)). \end{aligned}$$

Moreover, since the cube commutes we obtain a fiberwise homotopy

$$K : \prod_{(x:S)} I(f(x)) \circ F(x) \sim J(g(x)) \circ G(x).$$

We define the descent data (P_A, P_B, P_S) consisting of $P_A : A \rightarrow \mathcal{U}$, $P_B : B \rightarrow \mathcal{U}$, and $P_S : \prod_{(x:S)} P_A(f(x)) \simeq P_B(g(x))$ by

$$\begin{aligned} P_A(a) &::= \mathbf{fib}_{h_A}(a) \\ P_B(b) &::= \mathbf{fib}_{h_B}(b) \\ P_S(x) &::= G(x) \circ F(x)^{-1}. \end{aligned}$$

We have

$$\begin{aligned} P &::= \mathbf{fib}_{h_X} \\ e_A &::= I \\ e_B &::= J \\ e_S &::= K. \end{aligned}$$

Now consider the diagram

$$\begin{array}{ccccc} \sum_{(s:S)} \mathbf{fib}_{h_S}(s) & \longrightarrow & \sum_{(s:S)} \mathbf{fib}_{h_A}(f(s)) & \longrightarrow & \sum_{(b:B)} \mathbf{fib}_{h_B}(b) \\ \downarrow & & \downarrow & & \downarrow \\ \sum_{(a:A)} \mathbf{fib}_{h_A}(a) & \longrightarrow & \sum_{(a:A)} \mathbf{fib}_{h_A}(a) & \longrightarrow & \sum_{(x:X)} \mathbf{fib}_{h_X}(x) \end{array}$$

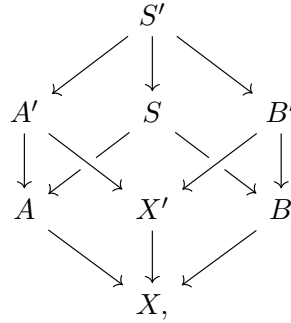
Since the top and bottom map in the left square are equivalences, we obtain that the left square is a pushout square. Moreover, the right square is a pushout by Lemma 2.2.5. Therefore it follows by Proposition 2.1.19 that the outer rectangle is a pushout square.

Now consider the commuting cube

$$\begin{array}{ccccc} & & \sum_{(s:S)} \mathbf{fib}_{h_S}(s) & & \\ & \swarrow & \downarrow & \searrow & \\ \sum_{(a:A)} \mathbf{fib}_{h_A}(a) & & S' & & \sum_{(b:B)} \mathbf{fib}_{h_B}(b) \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ A' & & \sum_{(x:X)} \mathbf{fib}_{h_X}(x) & & B' \\ & \swarrow & \downarrow & \searrow & \\ & & X' & & \end{array}$$

We have seen that the top square is a pushout. The vertical maps are all equivalences, so the vertical squares are all pushout squares. Thus it follows from one more application of Proposition 2.1.19 that the bottom square is a pushout. \square

Theorem 2.2.12. *Consider a commuting cube of types*



and suppose the two vertical squares in the back are pullback squares. Then the following are equivalent:

- (i) The two vertical squares in the front are pullback squares.
- (ii) The commuting square

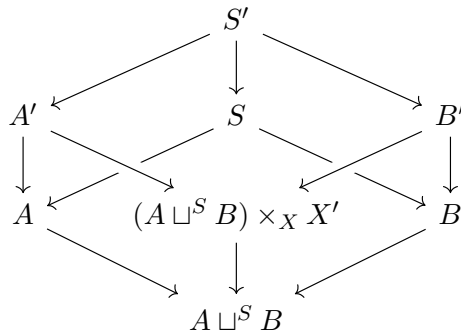
$$\begin{array}{ccc}
 A' \sqcup^{S'} B' & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 A \sqcup^S B & \longrightarrow & X
 \end{array}$$

is a pullback square.

Proof. To see that (i) implies (ii), it suffices to show that the pullback

$$(A \sqcup^S B) \times_X X'$$

has the universal property of the pushout. This follows by the descent theorem, since the vertical squares in the cube



are pullback squares by Corollary 2.1.17.

To prove that (ii) implies (i), we note that in the cube

$$\begin{array}{ccccc}
 & & S' & & \\
 & \swarrow & \downarrow & \searrow & \\
 A' & & S & & B' \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 A & & A' \sqcup^{S'} B' & & B \\
 \swarrow & \downarrow & \searrow & & \\
 & A \sqcup^S B & & &
 \end{array}$$

the two back squares are pullback squares, and the top and bottom squares are pushout squares. Therefore it follows from Theorem 2.2.11 that the two front squares are pullback squares. Now we obtain (i) from the pasting lemma for pushouts. \square

2.3 Applications of the descent theorem for pushouts

2.3.1 Fiber sequences

Definition 2.3.1. A **pointed type** is a pair (X, x) consisting of a type X equipped with a **base point** $x : X$. We will write \mathcal{U}_* for the type $\sum_{(X:\mathcal{U})} X$ of all pointed types.

In the following lemma we characterize the identity type of \mathcal{U}_* .

Lemma 2.3.2. *For any $(A, a), (B, b) : \mathcal{U}_*$ we have an equivalence*

$$\left((A, a) = (B, b) \right) \simeq \left(\sum_{(e:A \simeq B)} e(a) = b \right).$$

Construction. By Proposition 1.1.9 the type on the left hand side is equivalent to the type $\sum_{(p:A=B)} \text{tr}_{\text{El}}(p, a) = b$. By the univalence axiom, the map

$$\text{equiv_eq}_{A,B} : (A = B) \rightarrow (A \simeq B)$$

is an equivalence for each $B : \mathcal{U}$. Therefore, we have an equivalence of type

$$\left(\sum_{(p:A=B)} \text{tr}_{\text{El}}(p, a) = b \right) \simeq \left(\sum_{(e:A \simeq B)} \text{tr}_{\text{El}}(\text{equiv_equiv}(e), a) = b \right)$$

Moreover, by equivalence induction (the analogue of path induction for equivalences), we can compute the transport:

$$\text{tr}_{\text{El}}(\text{equiv_equiv}(e), a) = e(a).$$

It follows that $(\text{tr}_{\text{El}}(\text{equiv_equiv}(e), a) = b) \simeq (e(a) = b)$. \square

Definition 2.3.3. (i) Let $(X, *_X)$ be a pointed type. A **pointed family** over $(X, *_X)$ consists of a type family $P : X \rightarrow \mathcal{U}$ equipped with a base point $*_P : P(*_X)$.

- (ii) Let $(P, *_P)$ be a pointed family over $(X, *_X)$. A **pointed section** of $(P, *_P)$ consists of a dependent function $f : \prod_{(x:X)} P(x)$ and an identification $p : f(*_X) = *_P$. We define the **pointed Π -type** to be the type of pointed sections:

$$\Pi_{(x:X)}^* P(x) := \sum_{(f:\prod_{(x:X)} P(x))} f(*_X) = *_P$$

In the case of two pointed types X and Y , we may also view Y as a pointed family over X . In this case we write $X \rightarrow_* Y$ for the type of pointed functions.

- (iii) Given any two pointed sections f and g of a pointed family P over X , we define the type of pointed homotopies

$$f \sim_* g := \Pi_{(x:X)}^* f(x) = g(x),$$

where the family $x \mapsto f(x) = g(x)$ is equipped with the base point $p \cdot q^{-1}$.

Definition 2.3.4. (i) For any pointed type X , we define the **pointed identity function** $\text{id}_X^* := (\text{id}_X, \text{refl}_*)$.

- (ii) For any two pointed maps $f : X \rightarrow_* Y$ and $g : Y \rightarrow_* Z$, we define the **pointed composite**

$$g \circ_* f := (g \circ f, \text{ap}_g(p_f) \cdot p_g).$$

Definition 2.3.5. Let X be a pointed type with base point x . We define the **loop space** $\Omega(X, x)$ of X at x to be the pointed type $x = x$ with base point refl_x .

Definition 2.3.6. The loop space operation Ω is *functorial* in the sense that

- (i) For every pointed map $f : X \rightarrow_* Y$ there is a pointed map

$$\Omega(f) : \Omega(X) \rightarrow_* \Omega(Y),$$

defined by $\Omega(f)(\omega) := p_f \cdot \text{ap}_f(\omega) \cdot p_f^{-1}$, which is base point preserving by `right_inv(p_f)`.

- (ii) For every pointed type X there is a pointed homotopy

$$\Omega(\text{id}_X^*) \sim_* \text{id}_{\Omega(X)}^*.$$

- (iii) For any two pointed maps $f : X \rightarrow_* Y$ and $g : Y \rightarrow_* Z$, there is a pointed homotopy witnessing that the triangle

$$\begin{array}{ccc} & \Omega(Y) & \\ \Omega(f) \nearrow & & \searrow \Omega(g) \\ \Omega(X) & \xrightarrow{\Omega(g \circ_* f)} & \Omega(Z) \end{array}$$

of pointed types commutes.

Lemma 2.3.7. For any $(e, p), (f, q) : \sum_{(e:A \simeq B)} e(a) = b$, we have an equivalence of type

$$\left((e, p) = (f, q) \right) \simeq \left(\sum_{(h:e \sim f)} p = h(a) \cdot q \right).$$

Construction. The type $(e, p) = (f, q)$ is equivalent to the type $\sum_{(h:e=f)} \mathbf{tr}(h, p) = q$. Note that by the principle of function extensionality, the map $\mathbf{htpy_eq} : (e = f) \rightarrow (e \sim f)$ is an equivalence. Furthermore, it follows by homotopy induction that for any $h : e \sim f$ we have an equivalence of type

$$(\mathbf{tr}(\mathbf{eq_htpy}(h), p) = q) \simeq (p = h(a) \bullet q). \quad \square$$

Definition 2.3.8. A fiber sequence $F \hookrightarrow E \twoheadrightarrow B$ consists of:

- (i) Pointed types $F, E,$ and $B,$ with base points $x_0, y_0,$ and b_0 respectively,
- (ii) Base point preserving maps $i : F \rightarrow_* E$ and $p : E \rightarrow_* B,$ with $\alpha : i(x_0) = y_0$ and $\beta : p(y_0) = b_0,$
- (iii) A pointed homotopy $H : \mathbf{const}_{b_0} \sim_* p \circ_* i$ witnessing that the square

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ \downarrow & & \downarrow p \\ \mathbf{1} & \xrightarrow{\mathbf{const}_{b_0}} & B, \end{array}$$

commutes and is a pullback square.

We will write $\mathbf{Fib_Seq}$ for the type of all fiber sequences in $\mathcal{U}.$

Proposition 2.3.9. *The type of all fiber sequences is equivalent to the type*

$$\sum_{((B,b):\mathcal{U}_*)} \sum_{(P:B \rightarrow \mathcal{U})} P(b).$$

2.3.2 Fiber sequences obtained by the descent property

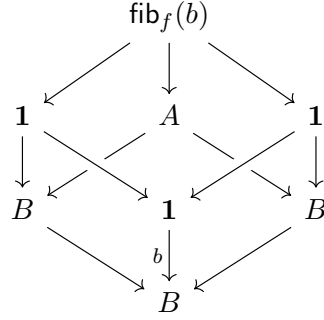
Definition 2.3.10. Let $f : A \rightarrow B$ be a map. The **codiagonal** ∇_f of f is the map obtained from the universal property of the pushout, as indicated in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & \ulcorner & \downarrow \mathbf{inr} \\ B & \xrightarrow{\mathbf{inl}} & B \sqcup^A B \\ & \searrow & \downarrow \nabla_f \\ & & B \end{array}$$

\mathbf{id}_B (curved arrow from B to B)

Proposition 2.3.11. *For any map $f : A \rightarrow B$ and any $y : B,$ there is an equivalence $\mathbf{fib}_{\nabla_f}(y) \simeq \Sigma(\mathbf{fib}_f(y)).$*

Proof. For any $b : B$ we have the commuting cube



of which the vertical sides are pullback squares. Hence we obtain the pullback square

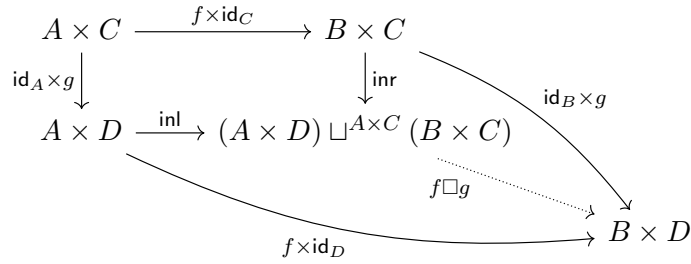
$$\begin{array}{ccc} \Sigma \text{fib}_f(b) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow b \\ B \sqcup^A B & \longrightarrow & B \end{array}$$

from Theorem 2.2.12, from which the claim follows. \square

Definition 2.3.12. Consider two maps $f : A \rightarrow B$ and $g : C \rightarrow D$. The **pushout-product**

$$f \square g : (A \times D) \sqcup^{A \times C} (B \times C) \rightarrow B \times D$$

of f and g is defined by the universal property of the pushout as the unique map rendering the diagram

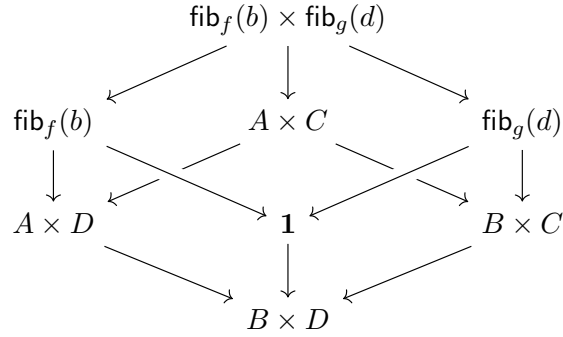


commutative.

Proposition 2.3.13. For any two maps $f : A \rightarrow B$ and $g : C \rightarrow D$, and any $(b, d) : B \times D$, there is an equivalence

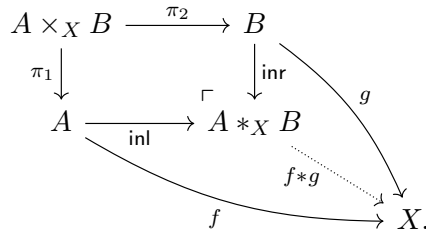
$$\text{fib}_{f \square g}(b, d) \simeq \text{fib}_f(b) * \text{fib}_g(d).$$

Proof. Let $b : B$ and $d : D$. Then we have the commuting cube



of which the vertical sides are pullback squares. Hence the claim follows from Theorem 2.2.12. \square

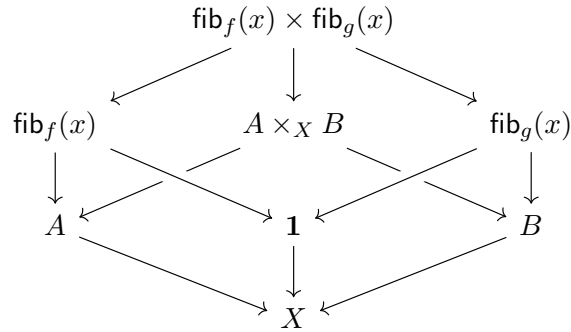
Definition 2.3.14. Let $f : A \rightarrow X$ and $g : B \rightarrow X$ be maps into X . We define the **fiberwise join** $A *_X B$ and the **join**¹ $f * g : A *_X B \rightarrow X$ of f and g , as indicated in the following diagram:



Theorem 2.3.15. Let $f : A \rightarrow X$ and $g : B \rightarrow X$ be maps into X , and let $x : X$. Then there is an equivalence

$$\text{fib}_{f * g}(x) \simeq \text{fib}_f(x) * \text{fib}_g(x).$$

Proof. We have the following commuting cube



in which the vertical squares are pullback squares. Therefore it follows by Theorem 2.2.12 that the square

$$\begin{array}{ccc}
 \text{fib}_f(x) * \text{fib}_g(x) & \longrightarrow & \mathbf{1} \\
 \downarrow & & \downarrow \\
 A *_X B & \longrightarrow & X
 \end{array}$$

¹Warning: By $f * g$ we do not mean the functorial action of the join, applied to (f, g) .

is a pullback square. \square

Remark 2.3.16. The join operation on maps with a common codomain is associative up to homotopy (this was formalized by Brunerie, see Proposition 1.8.6 of [9]), and it is a commutative operation on the generalized elements of a type X . Furthermore, the unique map of type $\mathbf{0} \rightarrow X$ is a unit for the join operation.

Definition 2.3.17. Let A and B be pointed types with base points $a_0 : A$ and $b_0 : B$. The **wedge inclusion** is defined as follows by the universal property of the wedge:

$$\begin{array}{ccc}
 \mathbf{1} & \longrightarrow & B \\
 \downarrow & & \downarrow \text{inr} \\
 A & \xrightarrow{\text{inl}} & A \vee B \\
 & \searrow \text{wedge_in}_{A,B} & \searrow \lambda b. (a_0, b) \\
 & & A \times B \\
 & \nearrow \lambda a. (a, b_0) & \nearrow
 \end{array}$$

Proposition 2.3.18. *There is a fiber sequence*

$$\Omega(A) * \Omega(B) \hookrightarrow A \vee B \twoheadrightarrow A \times B.$$

Proof. We have the commuting cube

$$\begin{array}{ccccc}
 & & \Omega(B) \times \Omega(A) & & \\
 & \swarrow & \downarrow & \searrow & \\
 \Omega(B) & & \mathbf{1} & & \Omega(A) \\
 \text{const}_a \downarrow & \swarrow & \downarrow & \searrow & \downarrow \text{const}_b \\
 A & & \mathbf{1} & & B \\
 & \swarrow & \downarrow & \searrow & \\
 & & A \times B & &
 \end{array}$$

of which the vertical sides are pullback squares. Hence the claim follows from Theorem 2.2.12. \square

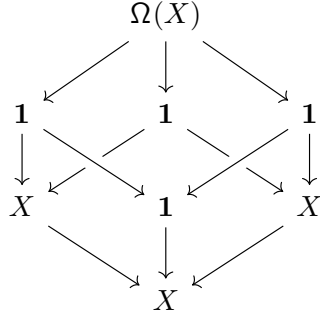
Definition 2.3.19. Consider a pointed type X . We define the map $\text{fold} : X \vee X \rightarrow X$ by the universal property of the wedge as indicated in the diagram

$$\begin{array}{ccc}
 \mathbf{1} & \xrightarrow{x_0} & X \\
 x_0 \downarrow & \lrcorner & \downarrow \text{inr} \\
 X & \xrightarrow{\text{inl}} & X \vee X \\
 & \searrow \text{fold} & \searrow \text{id}_X \\
 & & X \\
 & \nearrow \text{id}_X & \nearrow
 \end{array}$$

Proposition 2.3.20. *There is a fiber sequence*

$$\Sigma\Omega(X) \hookrightarrow X \vee X \twoheadrightarrow X.$$

Proof. We have the commuting cube



of which the vertical sides are pullback squares. Hence the claim follows from Theorem 2.2.12. \square

Remark 2.3.21. As a corollary, there are fiber sequences

$$\begin{aligned}
 \mathbb{S}^1 &\hookrightarrow \mathbb{R}P^\infty \vee \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty \\
 \mathbb{S}^2 &\hookrightarrow \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty.
 \end{aligned}$$

Here we take $\mathbb{R}P^\infty := K(\mathbb{Z}/2, 1)$ and $\mathbb{C}P^\infty := K(\mathbb{Z}, 2)$, where the Eilenberg-Mac Lane space $K(G, n)$ is defined in [23].

Corollary 2.3.22. *There is a fiber sequence*

$$(\Sigma\Omega(X))^{\vee n} \hookrightarrow X^{\vee(n+1)} \rightarrow X.$$

Definition 2.3.23. A **coherent H-space** consists of a type X equipped with a unit $1 : X$, a multiplication operation $\mu : X \rightarrow (X \rightarrow X)$ such that the function $\mu(x, -)$ and $\mu(-, y)$ are equivalences for each $x : X$ and $y : X$, respectively, and *coherent* unit laws

$$\begin{aligned}
 \text{right_unit} &: \prod_{(x:X)} \mu(x, 1) = x \\
 \text{left_unit} &: \prod_{(y:X)} \mu(1, y) = y \\
 \text{coh_unit} &: \text{left_unit}(1) = \text{right_unit}(1).
 \end{aligned}$$

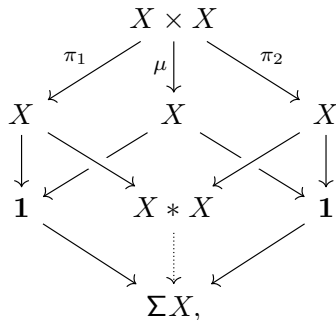
The following theorem is also known as the Hopf-construction.

Theorem 2.3.24. *For any coherent H-space X there is a fiber sequence*

$$X \hookrightarrow X * X \rightarrow \Sigma X.$$

The map $\eta_X : X * X \rightarrow \Sigma X$ is called the **Hopf fibration** for X .

Proof. We have the commuting cube



where the front map is obtained by the universal property of the pushout.

In this cube, the two vertical squares in the back are pullback squares by Proposition 1.4.11, since $\mu(-, y)$ and $\mu(x, -)$ are equivalences for every $x : X$ and $y : X$, respectively. Since the bottom and top squares are pushout squares it follows by Theorem 2.2.12 that the front two squares are pullback squares. \square

Remark 2.3.25. The classical Hopf fibration

$$\mathbb{S}^1 \longleftarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2$$

is now obtained from the fact that $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$, which was established in [9]. Indeed, the circle is a coherent H-space.

Chapter 3

Reflexive coequalizers

The material of this chapter is joint work with Bas Spitters, which we started in the academic year 2012-2013 while I was a research assistant at the Radboud University of Nijmegen.

We begin this chapter with a proof of the type theoretic Yoneda lemma, Lemma 3.1.1. The Yoneda lemma is used to show in Proposition 3.2.5 that the discrete functor $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$ from small types into the type of small reflexive graphs, is an embedding. In Theorem 3.2.6 we show that a reflexive graph \mathcal{A} is discrete if and only if it satisfies any one (and hence all) of the unique extension properties

$$\begin{array}{ccc}
 \mathcal{I} & \longrightarrow & \mathcal{A} \\
 \downarrow & \nearrow & \\
 \mathbf{1} & &
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathcal{A} \\
 0 \downarrow & \nearrow & \\
 \mathcal{I} & &
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathcal{A}, \\
 1 \downarrow & \nearrow & \\
 \mathcal{I} & &
 \end{array}$$

where \mathcal{I} is the walking edge.

The (homotopy) reflexive coequalizer satisfies the universal property of the left adjoint of $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$. We show in Proposition 3.3.7 that the reflexive coequalizer of a reflexive graph \mathcal{A} is just a pushout

$$\begin{array}{ccc}
 \sum_{(x,y:A_0)} A_1(x,y) & \xrightarrow{\pi_2} & A_0 \\
 \pi_1 \downarrow & & \downarrow \\
 A_0 & \longrightarrow & \mathbf{rcoeq}(\mathcal{A}).
 \end{array}$$

In particular it follows that if a universe is closed under pushouts (which is our running assumption), then it is also closed under reflexive coequalizers. More practically, this characterization of reflexive coequalizers as pushouts allows us to compute in Example 3.3.8 many reflexive coequalizers in terms of previously defined operations.

Our next purpose is to study the morphisms $f : \mathcal{B} \rightarrow \mathcal{A}$ of reflexive graphs that are *fibrations* in the sense that they satisfy the right orthogonality conditions

$$\begin{array}{ccc}
 \mathcal{I} & \longrightarrow & \mathcal{B} \\
 \downarrow & \nearrow & \downarrow f \\
 \mathbf{1} & \longrightarrow & \mathcal{A}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathcal{B} \\
 0 \downarrow & \nearrow & \downarrow f \\
 \mathcal{I} & \longrightarrow & \mathcal{A}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathcal{B} \\
 1 \downarrow & \nearrow & \downarrow f \\
 \mathcal{I} & \longrightarrow & \mathcal{A}.
 \end{array}$$

We show in Proposition 3.4.6 that the class of fibrations is precisely the class of *cartesian* morphisms of reflexive graphs, i.e. the morphisms $f : \mathcal{B} \rightarrow \mathcal{A}$ for which the naturality squares

$$\begin{array}{ccc} B_0 & \longleftarrow \sum_{(i,j:B_0)} B_1(i,j) & \longrightarrow B_0 \\ \downarrow & & \downarrow \\ A_0 & \longleftarrow \sum_{(i,j:A_0)} A_1(i,j) & \longrightarrow A_0 \end{array} \quad \begin{array}{ccc} B_0 & \longrightarrow \sum_{(i,j:B_0)} B_1(i,j) & \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow \sum_{(i,j:A_0)} A_1(i,j) & \end{array}$$

are pullback squares. It follows that $f : \mathcal{B} \rightarrow \mathcal{A}$ is a fibration whenever it appears as a pullback

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \Delta Y \\ f \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \Delta X \end{array}$$

of some map between discrete reflexive graphs. Pulling back along $h : \mathcal{A} \rightarrow \Delta X$ for a fixed type X therefore provides an operation

$$h^* : \left(\sum_{(Y:\mathcal{M})} (Y \rightarrow X) \right) \rightarrow \left(\sum_{(\mathcal{B}:\text{rGph})} \text{fib}(\mathcal{B}, \mathcal{A}) \right).$$

One way of stating the descent property for reflexive coequalizers, is that this map is an equivalence whenever $f : \mathcal{A} \rightarrow \Delta X$ is a reflexive coequalizer. As a consequence of the descent property we obtain in Theorem 3.4.11 that for any fibration $f : \mathcal{B} \rightarrow \mathcal{A}$ of reflexive graphs, a commuting square of the form

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \Delta Y \\ f \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \Delta X \end{array}$$

is a pullback square of reflexive graphs if and only if the square

$$\begin{array}{ccc} \text{rcoeq}(\mathcal{B}) & \longrightarrow & Y \\ \text{rcoeq}(f) \downarrow & & \downarrow \\ \text{rcoeq}(\mathcal{A}) & \longrightarrow & X \end{array}$$

is a pullback square. A minor note about the forward direction is that the hypothesis that f is a fibration is implied by the assumption that the square is a pullback square, so this assumption is unrestrictively superfluous. However, this assumption is necessary for the converse direction.

The situation here is that we have an adjunction $F \dashv G$ (in the present case $\text{rcoeq} \dashv \Delta$) satisfying the ‘descent condition’ that if f is a fibration, i.e. the naturality square of the unit $\eta : \text{id} \Rightarrow GF$ at f is a pullback square, then any square of the form

$$\begin{array}{ccc} A & \longrightarrow & GX \\ f \downarrow & & \downarrow Gg \\ B & \longrightarrow & GY \end{array}$$

is a pullback square if and only if the square

$$\begin{array}{ccc} FA & \longrightarrow & X \\ Ff \downarrow & & \downarrow g \\ FB & \longrightarrow & Y \end{array}$$

is a pullback square. We will investigate this situation further in §6.1, where we study the descent condition for general modalities.

In the final section of this chapter we apply the previous results to colimits of diagrams indexed by a (reflexive) graph. Most useful to us are the results on sequential colimits, which we will rely on in later chapters. Here we apply our results on diagrams over (reflexive) graphs to obtain results on sequential colimits. In [14] these results are obtained directly, and moreover they are formalized.

3.1 The Yoneda lemma

The universal property of identity types is sometimes called the type theoretic Yoneda lemma: families of maps out of the identity type are uniquely determined by their action on the reflexivity identification.

Lemma 3.1.1. *Let B be a type family over A , and let $a : A$. Then the map*

$$\text{ev_refl} : \left(\prod_{(x:A)} (a = x) \rightarrow B(x) \right) \rightarrow B(a)$$

given by $\lambda f. f(a, \text{refl}_a)$ is an equivalence.

Proof. The inverse of ev_refl is $\text{ind}_{a=} : B(a) \rightarrow \prod_{(x:A)} (a = x) \rightarrow B(x)$. We have the homotopy $\lambda b. \text{refl}_b : \text{ev_refl} \circ \text{ind}_{a=} \sim \text{id}_{B(a)}$ by the computation rule for identity types, so it is indeed the case that $\text{ind}_{a=}$ is a section of ev_refl .

To see that $\text{ind}_{a=} \circ \text{ev_refl} \sim \text{id}$, let $f : \prod_{(x:A)} (a = x) \rightarrow B(x)$. To show that $\text{ind}_{a=}(f(a, \text{refl}_a)) = f$ we use function extensionality (twice), so it suffices to show that

$$\prod_{(x:A)} \prod_{(p:a=x)} \varphi(f(a, \text{refl}_a), x, p) = f(x, p).$$

This follows by path induction on p , since $\text{ind}_{a=}(f(a, \text{refl}_a), a, \text{refl}_a) \equiv f(a, \text{refl}_a)$. \square

Corollary 3.1.2. *Let B be a type family over A , and let $a : A$. Then there is an equivalence*

$$\left(\text{Id}_A(a) = B \right) \simeq \left(B(a) \times \text{is_contr} \left(\sum_{(x:A)} B(x) \right) \right).$$

Furthermore, for $b : B(a)$ there is an equivalence

$$\left((\text{Id}_A(a), \text{refl}_a) = (B, b) \right) \simeq \text{is_contr} \left(\sum_{(x:A)} B(x) \right).$$

Proof. First we show that there is an equivalence

$$\left(\prod_{(x:A)} (a = x) \simeq B(x) \right) \simeq B(a) \times \text{is_contr} \left(\sum_{(x:A)} B(x) \right).$$

To see this, note that the type of fiberwise equivalences $\prod_{(x:A)}(a = x) \simeq B(x)$ is equivalent to the type

$$\sum_{(f:\prod_{(x:A)}(a=x)\rightarrow B(x))} \prod_{(x:A)} \text{is_equiv}(f)$$

By Theorem 1.2.5, the type $\prod_{(x:A)} \text{is_equiv}(f)$ is equivalent to the contractibility of the total space of B . Now the claim follows from Lemma 3.1.1. The claims in the statement are now easy consequences. \square

Definition 3.1.3. A (small) **reflexive relation** \mathcal{R} on A is a pair (R, ρ) consisting of a binary type-valued relation $R : A \rightarrow (A \rightarrow \mathcal{U})$ equipped with a proof of reflexivity

$$\rho : \prod_{(x:A)} R(x, x).$$

We will write $\text{rRel}(A)$ for the type of all small reflexive relations on A , i.e.

$$\text{rRel}(A) := \sum_{(R:A\rightarrow(A\rightarrow\mathcal{U}))} \prod_{(x:A)} R(x, x).$$

Corollary 3.1.4. Let $\mathcal{R} \equiv (R, \rho)$ be a reflexive relation on A . Then there is an equivalence

$$\left((\text{Id}, \text{refl}) = (R, \rho) \right) \simeq \prod_{(a:A)} \text{is_contr} \left(\sum_{(x:A)} R(a, x) \right).$$

The following theorem, which was proven independently by Escardó [17] around the same time, shows that the canonical map

$$(x = y) \rightarrow (\text{Id}_A(x) = \text{Id}_A(y))$$

is an equivalence, for any $x, y : A$. This will be particularly relevant to us once we study ∞ -equivalence relations.

Proposition 3.1.5. For any type $A : \mathcal{U}$, the map

$$\text{Id}_A : A \rightarrow (A \rightarrow \mathcal{U})$$

is an embedding.

Proof. Let $a : A$. Then we calculate the fiber of Id_A at a type family $B : A \rightarrow \mathcal{U}$ as follows:

$$\begin{aligned} \text{fib}_{\text{Id}_A}(B) &\equiv \sum_{(a:A)} \text{Id}_A(a) = B \\ &\simeq \sum_{(a:A)} \left(B(a) \times \text{is_contr} \left(\sum_{(x:A)} B(x) \right) \right) && \text{(by Corollary 3.1.2)} \\ &\simeq \left(\sum_{(a:A)} B(a) \right) \times \text{is_contr} \left(\sum_{(x:A)} B(x) \right) \\ &\simeq \text{is_contr} \left(\sum_{(x:A)} B(x) \right). \end{aligned}$$

The last equivalence follows since $X \times \text{is_contr}(X) \simeq \text{is_contr}(X)$ for every type X . We conclude that the fibers of Id_A are propositions, so Id_A is an embedding by Proposition 1.2.13. \square

3.2 Discrete reflexive graphs

Definition 3.2.1. A **non-reflexive graph** \mathcal{A} in \mathcal{U} is a pair (A_0, A_1) consisting of

$$\begin{aligned} A_0 &: \mathcal{U} \\ A_1 &: A_0 \rightarrow A_0 \rightarrow \mathcal{U}. \end{aligned}$$

We write \mathbf{Gph} for the type of all graphs in \mathcal{U} . When \mathcal{A} is a graph, we say that A_0 is its type of **vertices**, and that A_1 is its family of **edges**.

A **reflexive graph** \mathcal{A} in \mathcal{U} consists of a graph (A_0, A_1) equipped with a **reflexivity** term

$$\mathbf{rfx}_{\mathcal{A}} : \prod_{(i:A_0)} A_1(i, i).$$

We write

$$\mathbf{rGph} := \sum_{(V:\mathcal{U})} \sum_{(E:V \rightarrow (V \rightarrow \mathcal{U}))} \prod_{(v:V)} E(v, v)$$

for the type of reflexive graphs in \mathcal{U} .

Example 3.2.2. (i) A **reflexive pair** consists of types V and E , and maps s , t , and r between V and E as indicated in the diagram

$$E \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{r} \\ \xrightarrow{t} \end{array} V,$$

equipped with homotopies $H_{sr} : s \circ r \sim \text{id}_V$ and $H_{tr} : t \circ r \sim \text{id}_V$. Given a reflexive pair as above, we obtain a reflexive graph \mathcal{A} by taking

$$\begin{aligned} A_0 &:= V \\ A_1(v, w) &:= \sum_{(e:E)} (s(e) = v) \times (t(e) = w) \\ \mathbf{rfx}_{\mathcal{A}}(v) &:= (r(v), H_{sr}(v), H_{tr}(v)). \end{aligned}$$

By a routine construction it can be shown that the type of small reflexive pairs is equivalent to the type of small reflexive graphs.

(ii) Given a type X , the **discrete graph** $\Delta(X)$ on X is the reflexive graph consisting of

$$\begin{aligned} \Delta(X)_0 &:= X \\ \Delta(X)_1 &:= \text{ld}_X \\ \mathbf{rfx}_{\Delta(X)} &:= \text{refl}. \end{aligned}$$

(iii) Given a map $f : A \rightarrow X$, the **pre-kernel** $k(f)$ of f is the reflexive relation on A given by $x, y \mapsto f(x) = f(y)$.

(iv) The **indiscrete graph** $\nabla(X)$ on X is the reflexive graph consisting of

$$\begin{aligned} \nabla(X)_0 &:= X \\ \nabla(X)_1 &:= \lambda x. \lambda y. \mathbf{1} \end{aligned}$$

$$\text{rfx}_{\nabla(X)} := \star.$$

The reflexive pair corresponding to $\nabla(X)$ is (equivalent to) the reflexive pair

$$X \times X \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{\delta} \\ \xrightarrow{\pi_2} \end{array} X.$$

In particular, we have the **unit** graph $\mathbf{1} := \nabla(\mathbf{1})$, which happens to also be $\Delta(\mathbf{1})$.

- (v) The **walking edge** \mathcal{I} is an example of a reflexive graph. One way of defining it is by taking

$$\begin{aligned} \mathcal{I}_0 &:= \mathbf{2} \\ \mathcal{I}_1(x, y) &:= \tau(x) \rightarrow \tau(y) \\ \text{rfx}_{\mathcal{I}}(x) &:= \text{id}_{\tau(x)} \end{aligned}$$

where $\tau : \mathbf{2} \rightarrow \mathcal{U}$ is the **tautological family** on the type $\mathbf{2}$ of booleans, given by $\tau(0_{\mathbf{2}}) := \mathbf{0}$ and $\tau(1_{\mathbf{2}}) := \mathbf{1}$. We write a for the (unique) edge from $0_{\mathbf{2}}$ to $1_{\mathbf{2}}$. It may be helpful to think of the walking edge \mathcal{I} as an interval, hence the choice of notation.

Definition 3.2.3. A **morphism** of graphs f from \mathcal{A} to \mathcal{B} is a pair

$$\begin{aligned} f_0 &: A_0 \rightarrow B_0 \\ f_1 &: \prod_{(i,j:A_0)} A_1(i, j) \rightarrow B_1(f_0(i), f_0(j)). \end{aligned}$$

We write $\text{Gph}(\mathcal{A}, \mathcal{B})$ for the type of graph morphisms from \mathcal{A} to \mathcal{B} .

A **morphism** of reflexive graphs f from \mathcal{A} to \mathcal{B} is a morphism of graphs equipped with a term

$$\text{rfx}_f : \prod_{(i:A_0)} f_1(\text{rfx}_{\mathcal{A}}(i)) = \text{rfx}_{\mathcal{B}}(f_0(i))$$

witnessing that reflexivity is preserved. We write $\text{rGph}(\mathcal{A}, \mathcal{B})$ for the type of reflexive graph morphisms from \mathcal{A} to \mathcal{B} .

For any reflexive graph there is an identity morphism, and for any composable pair of morphisms there is a composite. Furthermore, these operations are associative, and satisfy the unit laws, both up to homotopy. It should be noted, however, that composition of reflexive graph morphisms is not expected to be associative on the nose, since the reflexivity is only preserved up to higher identification.

Example 3.2.4. (i) Given a function $f : X \rightarrow Y$ we obtain a morphism $\Delta f : \text{rGph}(\Delta X, \Delta Y)$ given by

$$\begin{aligned} \Delta f_0 &:= f \\ \Delta f_1 &:= \text{ap}_f \\ \text{rfx}_{\Delta f} &:= \text{refl}_{\text{refl}_{f(x)}} \end{aligned}$$

The action of Δ on morphisms preserves identity morphisms and compositions, and moreover it preserves the unit laws and associativity.

- (ii) The graph $\nabla(\mathbf{1})$ is the terminal reflexive graph in the sense that for any graph \mathcal{A} , the type $\mathbf{rGph}(\mathcal{A}, \mathbf{1})$ is contractible.
- (iii) For any reflexive graph \mathcal{A} , the map

$$\mathbf{ev_pt} : \mathbf{rGph}(\mathbf{1}, \mathcal{A}) \rightarrow A_0$$

given by $f \mapsto f_0(\star)$, is an equivalence. The analogous statement is false for non-reflexive graphs.

- (iv) The universal property of the walking edge \mathcal{I} is that every edge e in a graph \mathcal{A} , the type of reflexive graph morphisms that map the edge a of \mathcal{I} to e is contractible. In other words, the map

$$\mathbf{ev_edge} : \mathbf{rGph}(\mathcal{I}, \mathcal{A}) \rightarrow \sum_{(i,j:A_0)} A_1(i,j)$$

given by $f \mapsto (f_0(0), f_0(1), f_1(a))$ is an equivalence.

- (v) For the walking edge \mathcal{I} there are reflexive graph morphisms

$$\mathcal{I} \begin{array}{c} \xleftarrow{1} \\ \xrightarrow{0} \end{array} \mathbf{1}$$

This is a *cograph* object of reflexive graphs, since the morphism $\mathcal{I} \rightarrow \mathbf{1}$ is a common *retraction* of the end-point inclusions $0, 1 : \mathbf{1} \rightarrow \mathcal{I}$, whereas in a reflexive pair (a graph object) the morphism r is a common *section* of the source and target maps.

- (vi) Since we do not have the technology available to establish that graphs and reflexive graphs form ∞ -categories, a comparison between the two structures is limited to what we can say directly using the basic categorical operations such as composition. However, this is just enough to establish the universal property of an adjunction. Given a non-reflexive graph $\mathcal{A} \equiv (A_0, A_1)$, we can obtain a reflexive graph $F\mathcal{A}$ by freely adjoining reflexivity:

$$\begin{aligned} F\mathcal{A}_0 &::= A_0 \\ F\mathcal{A}_1(i,j) &::= A_1(i,j) + (i = j) \\ \mathbf{rfx}_{F\mathcal{A}}(i) &::= \mathbf{inr}(\mathbf{refl}_i). \end{aligned}$$

On the other hand, there is the projection $U : \mathbf{rGph} \rightarrow \mathbf{Gph}$ which forgets the reflexivity structure, and for each graph \mathcal{A} there is a graph morphism $\eta : \mathcal{A} \rightarrow UF\mathcal{A}$. Both F and U are functorial in the sense that they act on morphisms, and preserve units and composition in the obvious way, and η is natural in \mathcal{A} .

The universal property of the construction F of freely adjoining reflexivity, is that the map

$$U(-) \circ \eta : \mathbf{rGph}(F\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{Gph}(\mathcal{A}, U\mathcal{B})$$

is an equivalence. Indeed, this is the universal property that establishes F as a left adjoint to U , even though we cannot manifest F and U as functors.

The construction F of freely adjoining reflexivity is not surjective on morphisms. For example, there are no morphisms from $\mathbf{1}$ to the graph \mathcal{A} with a contractible type of vertices but no edges. However $F\mathcal{A}$ is the terminal reflexive graph.

Proposition 3.2.5. *The operation $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$ is an embedding in the following sense:*

- (i) *As a function, $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$ is an embedding.*
- (ii) *For every two types X and Y , the action on morphisms*

$$\Delta : (X \rightarrow Y) \rightarrow \mathbf{rGph}(\Delta X, \Delta Y)$$

is an equivalence.

Proof. Let \mathcal{A} be a reflexive graph. Then the fiber of $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$ is calculated as follows:

$$\begin{aligned} \text{fib}_\Delta(\mathcal{A}) &::= \sum_{(X:\mathcal{U})} \Delta X = \mathcal{A} \\ &\simeq (\text{ld}_{A_0}, \text{refl}) = (A_1, \text{rfx}_{\mathcal{A}}) \\ &\simeq \prod_{(x:A_0)} \text{is_contr} \left(\sum_{(y:A_0)} A_1(x, y) \right), \end{aligned} \quad (\text{by Corollary 3.1.4})$$

which is a proposition.

The map $(X \rightarrow Y) \rightarrow \mathbf{rGph}(\Delta X, \Delta Y)$ is an equivalence, since the type

$$\sum_{(\varphi:\prod_{(x,y:X)}(x=y) \rightarrow (f(x)=f(y)))} \varphi(\text{refl}_x) = \text{refl}_{f(x)}$$

is equivalent to the fiber of the map

$$\text{ev_refl} : \left(\prod_{(x,y:X)}(x=y) \rightarrow (f(x)=f(y)) \right) \rightarrow (f(x)=f(x)),$$

which is contractible by Lemma 3.1.1. □

Theorem 3.2.6. *Let \mathcal{A} be a reflexive graph. The following are equivalent:*

- (i) *The canonical map*

$$\prod_{(i,j:A_0)}(i=j) \rightarrow A_1(i, j)$$

given by $\text{refl}_i \mapsto \text{rfx}_{\mathcal{A}}(i)$ is a fiberwise equivalence. In particular, \mathcal{A} is in the image of $\Delta : \mathcal{U} \rightarrow \mathbf{rGph}$.

- (ii) *The graph \mathcal{A} is \mathcal{I} -null in the sense that the map*

$$\mathbf{rGph}(\mathbf{1}, \mathcal{A}) \rightarrow \mathbf{rGph}(\mathcal{I}, \mathcal{A})$$

given by pre-composition by the unique morphism $\mathbf{rGph}(\mathcal{I}, \mathbf{1})$, is an equivalence.

- (iii) *The map*

$$\mathbf{rGph}(\mathcal{I}, \mathcal{A}) \rightarrow \mathbf{rGph}(\mathbf{1}, \mathcal{A})$$

given by pre-composing with the end-point inclusion $0 : \mathbf{rGph}(\mathbf{1}, \mathcal{I})$, is an equivalence.

- (iv) *The map*

$$\mathbf{rGph}(\mathcal{I}, \mathcal{A}) \rightarrow \mathbf{rGph}(\mathbf{1}, \mathcal{A})$$

given by pre-composing with the end-point inclusion $1 : \mathbf{rGph}(\mathbf{1}, \mathcal{I})$, is an equivalence.

If any of these conditions hold, we say that \mathcal{A} is **discrete**.

Proof. The outline of our argument is as follows:

$$(i) \implies (ii) \begin{array}{l} \implies (iii) \\ \implies (iv) \end{array} \implies (i).$$

Suppose (i) holds. We have a commuting square

$$\begin{array}{ccc} \mathbf{rGph}(\mathbf{1}, \mathcal{A}) & \longrightarrow & \mathbf{rGph}(\mathcal{I}, \mathcal{A}) \\ \text{ev}_{\text{pt}} \downarrow & & \downarrow \text{ev}_{\text{edge}} \\ A_0 & \longrightarrow & \sum_{(i,j:A_0)} A_1(i,j) \end{array}$$

where both vertical maps are equivalences, and the bottom map is an equivalence by assumption. Therefore the top map is an equivalence. This proves that (i) implies (ii).

Since the reflexive graph morphism $\mathbf{rGph}(\mathcal{I}, \mathbf{1})$ is a common retraction of both endpoint inclusions, it follows that the pre-composition map

$$\mathbf{rGph}(\mathbf{1}, \mathcal{A}) \rightarrow \mathbf{rGph}(\mathcal{I}, \mathcal{A})$$

is a common section of both pre-composition maps

$$0^*, 1^* : \mathbf{rGph}(\mathcal{I}, \mathcal{A}) \rightarrow \mathbf{rGph}(\mathbf{1}, \mathcal{A})$$

However, assuming (ii) it follows that both 0^* and 1^* are equivalences, so (ii) implies both (iii) and (iv).

Now suppose that (iii) holds; we will show that (i) follows. We have the commuting square

$$\begin{array}{ccc} \mathbf{rGph}(\mathcal{I}, \mathcal{A}) & \xrightarrow{0^*} & \mathbf{rGph}(\mathbf{1}, \mathcal{A}) \\ \text{ev}_{\text{edge}} \downarrow & & \downarrow \text{ev}_{\text{pt}} \\ \sum_{(i,j:A_0)} A_1(i,j) & \xrightarrow{\pi_1} & A_0 \end{array}$$

in which both vertical maps are equivalences. Therefore the fibers of π_1 are equivalent to the fibers of 0^* . Note that the fibers of π_1 are of the form

$$\sum_{(j:A_0)} A_1(i,j),$$

so it follows from (iii) that these are contractible. Then (i) follows by the fundamental theorem of identity types, Theorem 1.2.5.

The argument that (i) follows from (iv) is similar, using π_2 and 1^* instead of π_1 and 0^* in the square. \square

3.3 Reflexive coequalizers

Definition 3.3.1. Consider a reflexive graph \mathcal{A} and a type X equipped with a morphism $f : \mathbf{rGph}(\mathcal{A}, \Delta(X))$. We say that f is a **reflexive coequalizer** of \mathcal{A} if the map

$$\Delta(-) \circ f : (X \rightarrow Y) \rightarrow \mathbf{rGph}(\mathcal{A}, \Delta(Y))$$

is an equivalence.

Our goal in this section is to show that reflexive coequalizers can be constructed as pushouts. We will use the following lemma, which was discovered jointly with Simon Boulier.

Lemma 3.3.2. Consider a type A with a type family B , and $a : A$. Furthermore, suppose that

$$\alpha : \prod_{(x:A)} B(x) \rightarrow (a = x).$$

Then the **coherence reduction** map

$$\mathbf{coh_red} : \left(\sum_{(y:B(a))} \alpha(y) = \mathbf{refl}_a \right) \rightarrow \left(\sum_{(x:A)} B(x) \right)$$

given by $(y, q) \mapsto (a, y)$ is an equivalence.

Remark 3.3.3. A quick way to see that there is an equivalence

$$\sum_{(x:A)} B(x) \simeq \sum_{(b:B(a))} \alpha_a(b) = \mathbf{refl}_a.$$

is to use the contractibility of the total space of identity types twice:

$$\begin{aligned} \sum_{(x:A)} B(x) &\simeq \sum_{(x:A)} \sum_{(y:B(x))} \sum_{(p:a=x)} \alpha(y) = p \\ &\simeq \sum_{(y:B(a))} \alpha(y) = \mathbf{refl}_a. \end{aligned}$$

However, it is not clear at once that the underlying map of this composite of equivalences is indeed the map $\mathbf{coh_red}$ defined in Lemma 3.3.2.

Proof of Lemma 3.3.2. We show that the fibers are contractible:

$$\begin{aligned} \mathbf{fib}_{\mathbf{coh_red}}((x, y)) &\simeq \sum_{(y':B(a))} \sum_{(q:\alpha(y')=\mathbf{refl}_a)} (a, y') = (x, y) \\ &\simeq \sum_{(y':B(a))} \sum_{(q:\alpha(y')=\mathbf{refl}_a)} \sum_{(p:a=x)} \mathbf{tr}_B(p, y') = y \\ &\simeq \sum_{(y':B(a))} \sum_{(q:\alpha(y')=\mathbf{refl}_a)} \sum_{(p:a=x)} y' = \mathbf{tr}_B(p^{-1}, y) \\ &\simeq \sum_{(p:a=x)} \alpha(\mathbf{tr}_B(p^{-1}, y)) = \mathbf{refl}_a \\ &\simeq \sum_{(p:a=x)} p = \alpha(y). \end{aligned} \quad \square$$

Corollary 3.3.4. Consider a type A with a relation $R : A \rightarrow A \rightarrow \mathcal{U}$ such that

$$\alpha : \prod_{(x,y:A)} R(x, y) \rightarrow (x = y).$$

Then the map

$$\mathbf{coh_red} : \left(\sum_{(x:A)} \sum_{(r:R(x,x))} \alpha(r) = \mathbf{refl}_x \right) \rightarrow \left(\sum_{(x,y:A)} R(x, y) \right).$$

given by $(x, r, c) \mapsto (x, x, r)$ is an equivalence. □

Definition 3.3.5. For any reflexive graph \mathcal{A} we define the span $\tilde{\mathcal{A}}$ of \mathcal{A} to consist of

$$\tilde{A}_0 \xleftarrow{\pi_1} \tilde{A}_1 \xrightarrow{\pi_2} \tilde{A}_0$$

where $\tilde{A}_0 := A_0$ and $\tilde{A}_1 := \sum_{(x,y:A_0)} A_1(x,y)$.

Lemma 3.3.6. For any reflexive graph \mathcal{A} , and any type X , the map

$$f \mapsto (f_0, f_0, \lambda(x,y,e). f_1(e)) : \mathbf{rGph}(\mathcal{A}, \Delta X) \rightarrow \mathbf{cocone}_{\tilde{\mathcal{A}}}(X),$$

is an equivalence, where $\tilde{\mathcal{A}}$ is the span $(\tilde{A}_1, \pi_1, \pi_2)$ from A_0 to A_0 .

Proof. First observe that $\mathbf{rGph}(\mathcal{A}, \Delta X)$ is equivalent to the type

$$\sum_{(f_0:A_0 \rightarrow X)} \sum_{(f_1:\prod_{(x,y:A_0)} A_1(x,y) \rightarrow f(x)=f(y))} \mathbf{eq_htpy}(\lambda x. f_1(\mathbf{rfx}_{\mathcal{A}}(x))) = \mathbf{refl}_{f_0}.$$

We write $\mathbf{rGph}'(\mathcal{A}, \Delta X)$ for the latter type. Furthermore, we observe that the type $\mathbf{cocone}_{\tilde{\mathcal{A}}}(X)$ is equivalent to the type

$$\mathbf{cocone}'_{\tilde{\mathcal{A}}}(X) := \sum_{(f,g:A_0 \rightarrow X)} \prod_{(x,y:A_0)} A_1(x,y) \rightarrow f(x) = f(y).$$

Now we note that we have a commuting square

$$\begin{array}{ccc} \mathbf{rGph}(\mathcal{A}, \Delta X) & \xrightarrow{\varphi} & \mathbf{cocone}_{\tilde{\mathcal{A}}}(X) \\ \downarrow & & \downarrow \\ \mathbf{rGph}'(\mathcal{A}, \Delta X) & \xrightarrow{\mathbf{coh_red}} & \mathbf{cocone}'_{\tilde{\mathcal{A}}}(X). \end{array}$$

where the map φ is the map in the statement, and the coherence reduction map uses the homotopy

$$(f, g, H) \mapsto \mathbf{eq_htpy}(\lambda x. H(\mathbf{rfx}_{\mathcal{A}}(x))).$$

Both vertical maps and the coherence reduction map are equivalences, so it follows that the asserted map is an equivalence. \square

Proposition 3.3.7. Let \mathcal{A} be a reflexive graph, and let X be a type equipped with $f : \mathbf{rGph}(\mathcal{A}, \Delta X)$. Then the following are equivalent:

(i) X is a reflexive coequalizer of \mathcal{A} .

(ii) The square

$$\begin{array}{ccc} \sum_{(x,y:A_0)} A_1(x,y) & \xrightarrow{\pi_2} & A_0 \\ \pi_1 \downarrow & & \downarrow f_0 \\ A_0 & \xrightarrow{f_0} & X \end{array}$$

which commutes by $\lambda(i,j,e). f_1(e)$, is a pushout square.

In particular, there is a reflexive coequalizer for every reflexive graph \mathcal{A} , for which we write

$$\mathbf{constr} : \mathbf{rGph}(\mathcal{A}, \Delta(\mathbf{rcoeq}(\mathcal{A}))).$$

Proof. The triangle

$$\begin{array}{ccc}
 & Y^X & \\
 \Delta(-) \circ f \swarrow & & \searrow \text{cocone_map} \\
 \text{rGph}(\mathcal{A}, \Delta Y) & \xrightarrow{\simeq} & \text{cocone}(Y)
 \end{array}$$

commutes, for any type Y . In this triangle, the bottom map is the map defined in Lemma 3.3.6, which is an equivalence, so the claim follows by the 3-for-2 property of equivalences. \square

Example 3.3.8.

- (i) The reflexive graph quotient of the discrete graph $\Delta(X)$ of a type X is just X itself. It also follows that $\text{constr} : \text{rGph}(\mathcal{A}, \Delta(\text{rcoeq}(\mathcal{A})))$ is an equivalence of reflexive graphs if and only if \mathcal{A} is a discrete graph.
- (ii) The reflexive graph quotient of the indiscrete graph $\nabla(X)$ on a type X is the join square $X * X$.
- (iii) Let \mathcal{A} be a non-reflexive graph, and let $F(\mathcal{A})$ be the reflexive graph obtained by freely adding a proof of reflexivity, as in Item (vi) of Example 3.2.4. Then the non-reflexive graph quotient of \mathcal{A} is the reflexive graph quotient of $F(\mathcal{A})$.
- (iv) Let $f : A \rightarrow B$ be a map. Then the reflexive coequalizer of the reflexive graph $(A, k(f))$ is the fiberwise join $A *_X A$, which was introduced in Definition 2.3.14.
- (v) Let X be a type with base point $x_0 : X$. Define the reflexive graph SX by

$$\begin{aligned}
 SX_0 &::= \mathbf{1} \\
 SX_1 &::= \lambda_. \lambda_. X \\
 \text{rfx}_{SX} &::= \lambda_. x_0.
 \end{aligned}$$

The reflexive graph quotient of SX is the suspension of X .

- (vi) The reflexive coequalizer of the walking edge \mathcal{I} is the interval, which is contractible.

3.4 Descent for reflexive coequalizers

Recall from [1] that morphism $f : X \rightarrow Y$ is right orthogonal to a map $i : A \rightarrow B$ if and only if the square

$$\begin{array}{ccc}
 X^B & \xrightarrow{- \circ i} & X^A \\
 f \circ - \downarrow & & \downarrow f \circ - \\
 Y^B & \xrightarrow{- \circ i} & Y^A
 \end{array}$$

is a pullback square. We use this way of stating the orthogonality condition in our definition of fibrations of reflexive graphs: a morphism $f : \text{rGph}(\mathcal{X}, \mathcal{Y})$ of reflexive graphs

is said to be right orthogonal to a morphism $i : \mathbf{rGph}(\mathcal{A}, \mathcal{B})$ if the square

$$\begin{array}{ccc} \mathbf{rGph}(\mathcal{X}, \mathcal{B}) & \xrightarrow{-oi} & \mathbf{rGph}(\mathcal{X}, \mathcal{A}) \\ f \circ - \downarrow & & \downarrow f \circ - \\ \mathbf{rGph}(\mathcal{Y}, \mathcal{B}) & \xrightarrow{-oi} & \mathbf{rGph}(\mathcal{Y}, \mathcal{A}) \end{array}$$

is a pullback square.¹

Definition 3.4.1. A morphism $f : \mathbf{rGph}(\mathcal{A}, \mathcal{B})$ is said to be a **left fibration** of reflexive graphs if it is right orthogonal to the morphism $0 : \mathbf{rGph}(\mathbf{1}, \mathcal{I})$. Similarly, we say that f is a **right fibration** of reflexive graphs if it is right orthogonal to the morphism $1 : \mathbf{rGph}(\mathbf{1}, \mathcal{I})$, and we say that f is a **fibration** if it is both a left and a right fibration.

Lemma 3.4.2. *Suppose $f : \mathbf{rGph}(\mathcal{B}, \mathcal{A})$ is a left or a right fibration. Then f is right orthogonal to the terminal projection $t : \mathbf{rGph}(\mathcal{I}, \mathbf{1})$.*

Proof. We prove the assertion assuming that f is a left fibration, the case of a right fibration being similar. Consider the diagram

$$\begin{array}{ccccc} \mathbf{rGph}(\mathbf{1}, \mathcal{A}) & \longrightarrow & \mathbf{rGph}(\mathcal{I}, \mathcal{A}) & \longrightarrow & \mathbf{rGph}(\mathbf{1}, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{rGph}(\mathbf{1}, \mathcal{B}) & \longrightarrow & \mathbf{rGph}(\mathcal{I}, \mathcal{B}) & \longrightarrow & \mathbf{rGph}(\mathbf{1}, \mathcal{B}) \end{array}$$

The square on the right is a pullback. Since the composite $\mathbf{1} \rightarrow \mathcal{I} \rightarrow \mathbf{1}$ is the identity morphism of reflexive graphs, the outer rectangle is also a pullback. Therefore the square on the left is a pullback. \square

Definition 3.4.3. A morphism $f : \mathbf{rGph}(\mathcal{A}, \mathcal{B})$ of reflexive graphs is said to be **left cartesian** if the naturality square

$$\begin{array}{ccc} \sum_{(i,j:A_0)} A_1(i,j) & \xrightarrow{\pi_1} & A_0 \\ \downarrow & & \downarrow \\ \sum_{(i,j:B_0)} B_1(i,j) & \xrightarrow{\pi_1} & B_0 \end{array}$$

is a pullback square. Similarly we say that f is **right cartesian** if the naturality square

$$\begin{array}{ccc} \sum_{(i,j:A_0)} A_1(i,j) & \xrightarrow{\pi_2} & A_0 \\ \downarrow & & \downarrow \\ \sum_{(i,j:B_0)} B_1(i,j) & \xrightarrow{\pi_2} & B_0 \end{array}$$

is a pullback square, and we say that f is **cartesian** if it is both left and right cartesian.

¹One may note that the category of reflexive graphs is also locally cartesian closed (has Π -types), so that we could also state the orthogonality condition internally. Although this is straightforward, it is extra work and we will not need it in this thesis.

Example 3.4.4. A common way to obtain a cartesian morphism of reflexive graphs is via equifibered families. An **equifibered family** \mathcal{E} over \mathcal{A} consists of

$$\begin{aligned} E_0 &: A_0 \rightarrow \mathcal{U} \\ E_1 &: \prod_{(i,j:A_0)} A_1(i,j) \rightarrow (E_0(i) \simeq E_0(j)) \\ \text{rfx}_E &: \prod_{(i:A_0)} E_1(\text{rfx}_{\mathcal{A}}(i)) \sim \text{id}_{E_0(i)} \end{aligned}$$

Given an equifibered family \mathcal{E} , we form the reflexive graph $\Sigma(\mathcal{A}, \mathcal{E})$ by

$$\begin{aligned} \Sigma(\mathcal{A}, \mathcal{E})_0 &:\equiv \sum_{(x:A_0)} E_0(x) \\ \Sigma(\mathcal{A}, \mathcal{E})_1((x,u), (y,v)) &:\equiv \sum_{(e:A_1(x,y))} E_1(e, u) = v \\ \text{rfx}_{\Sigma(\mathcal{A}, \mathcal{E})}(x, u) &:\equiv (\text{rfx}_{\mathcal{A}}(x), \text{rfx}_{\mathcal{E}}(x, u)). \end{aligned}$$

There is an obvious projection morphism $\text{pr}_1 : \text{rGph}(\Sigma(\mathcal{A}, \mathcal{E}), \mathcal{A})$, which is cartesian, because the squares in the diagram

$$\begin{array}{ccccc} \Sigma(\mathcal{A}, \mathcal{E})_0 & \longleftarrow & \sum_{(s,t:\Sigma(\mathcal{A}, \mathcal{E}))} \Sigma(\mathcal{A}, \mathcal{E})_1 & \longrightarrow & \Sigma(\mathcal{A}, \mathcal{E})_0 \\ \downarrow \text{pr}_1 & & \text{pr}_1 \downarrow & & \downarrow \\ \mathcal{A}_0 & \longleftarrow & \sum_{(x,y:A_0)} A_1(x,y) & \longrightarrow & \mathcal{A}_0 \end{array}$$

are pullback squares by Proposition 1.4.11.

Lemma 3.4.5. *Consider $f : \text{rGph}(\mathcal{B}, \mathcal{A})$ and $g : \text{rGph}(\mathcal{C}, \mathcal{B})$, and suppose that f is (left/right) cartesian. Then g is (left/right) cartesian if and only if $f \circ g$ is (left/right) cartesian, respectively.*

Proof. Immediate by Corollary 2.1.17. \square

Proposition 3.4.6. *Let $f : \text{rGph}(\mathcal{B}, \mathcal{A})$ be a morphism of reflexive graphs. The following are equivalent:*

- (i) f is a (left/right) fibration.
- (ii) f is (left/right) cartesian.

Proof. We only show that f is a left fibration if and only if f is left cartesian, the right case being similar.

For any morphism $f : \text{rGph}(\mathcal{B}, \mathcal{A})$ we have the commuting cube

$$\begin{array}{ccccc} & & \text{rGph}(\mathcal{I}, \mathcal{B}) & & \\ & \swarrow & \downarrow & \searrow & \\ \text{rGph}(\mathbf{1}, \mathcal{B}) & & \sum_{(i,j:B_0)} B_1(i,j) & & \text{rGph}(\mathcal{I}, \mathcal{A}) \\ \downarrow & \swarrow & \swarrow & \searrow & \downarrow \\ B_0 & & \text{rGph}(\mathbf{1}, \mathcal{A}) & & \sum_{(i,j:A_0)} A_1(i,j) \\ & \swarrow & \downarrow & \searrow & \\ & & A_0 & & \end{array}$$

in which all vertical maps are equivalences. Therefore the top square is a pullback if and only if the bottom square is a pullback, which proves that (ii) holds if and only if (iii) holds. \square

Corollary 3.4.7. *Suppose that $f : \mathbf{rGph}(\mathcal{B}, \mathcal{A})$ is left or right cartesian. Then the square*

$$\begin{array}{ccc} B_0 & \longrightarrow & \sum_{(i,j:B_0)} B_1(i,j) \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & \sum_{(i,j:A_0)} A_1(i,j) \end{array}$$

is a pullback square.

Proposition 3.4.8. *Consider a commuting square*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{X} \end{array}$$

of reflexive graphs. The following are equivalent:

(i) *The square is a pullback square.*

(ii) *The squares*

$$\begin{array}{ccc} \tilde{C}_0 & \longrightarrow & \tilde{B}_0 \\ \downarrow & & \downarrow \\ \tilde{A}_0 & \longrightarrow & \tilde{X}_0 \end{array} \quad \begin{array}{ccc} \tilde{C}_1 & \longrightarrow & \tilde{B}_1 \\ \downarrow & & \downarrow \\ \tilde{A}_1 & \longrightarrow & \tilde{X}_1 \end{array}$$

are pullback squares.

Proof. Straightforward. \square

The following proposition is true more generally:

Proposition 3.4.9. *Consider a pullback square*

$$\begin{array}{ccc} \mathcal{B}' & \longrightarrow & \mathcal{B} \\ f' \downarrow & & \downarrow f \\ \mathcal{A}' & \longrightarrow & \mathcal{A} \end{array}$$

of reflexive graphs, and let $h : \mathbf{rGph}(\mathcal{Y}, \mathcal{X})$ be a morphism of reflexive graphs. If f is right orthogonal to h , then so is f' . In particular, if f is a (left/right) fibration, then so is f' .

Proof. Consider the commuting cube

$$\begin{array}{ccccc}
 & & \text{rGph}(\mathcal{Y}, \mathcal{B}') & & \\
 & \swarrow & \downarrow & \searrow & \\
 \text{rGph}(\mathcal{Y}, \mathcal{A}') & & \text{rGph}(\mathcal{X}, \mathcal{B}') & & \text{rGph}(\mathcal{Y}, \mathcal{B}) \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \text{rGph}(\mathcal{X}, \mathcal{A}') & & \text{rGph}(\mathcal{Y}, \mathcal{A}) & & \text{rGph}(\mathcal{X}, \mathcal{B}) \\
 & \swarrow & \downarrow & \searrow & \\
 & & \text{rGph}(\mathcal{X}, \mathcal{A}) & &
 \end{array}$$

Note $\text{rGph}(\mathcal{X}, -)$ preserves pullbacks for any reflexive graph \mathcal{X} . Therefore it follows that the top and bottom squares are pullback squares. Moreover, the square on the front right is a pullback square by the assumption that f is right orthogonal to $h : \text{rGph}(\mathcal{Y}, \mathcal{X})$. Now it follows by the pasting property of pullbacks that the square on the back left is a pullback square. In other words, f' is right orthogonal to h . \square

Proposition 3.4.10. *For any map $f : X \rightarrow Y$, the morphism $\Delta f : \Delta X \rightarrow \Delta Y$ is a fibration.*

Proof. The commuting square

$$\begin{array}{ccc}
 \text{rGph}(\mathcal{I}, \Delta X) & \xrightarrow{0^*/1^*} & \text{rGph}(\mathbf{1}, \Delta X) \\
 \downarrow & & \downarrow \\
 \text{rGph}(\mathcal{I}, \Delta Y) & \xrightarrow{0^*/1^*} & \text{rGph}(\mathbf{1}, \Delta Y)
 \end{array}$$

is a pullback square, since both horizontal maps are equivalences. \square

Theorem 3.4.11. *Consider a commuting square*

$$\begin{array}{ccc}
 \mathcal{B} & \longrightarrow & \Delta Y \\
 f \downarrow & & \downarrow \\
 \mathcal{A} & \longrightarrow & \Delta X
 \end{array} \tag{3.1}$$

of reflexive graphs, where $f : \mathcal{B} \rightarrow \mathcal{A}$ is a fibration of reflexive graphs. Then the following are equivalent:

(i) *The square is a pullback square of reflexive graphs.*

(ii) *The square*

$$\begin{array}{ccc}
 \text{rcoeq}(\mathcal{B}) & \longrightarrow & Y \\
 \text{rcoeq}(f) \downarrow & & \downarrow \\
 \text{rcoeq}(\mathcal{A}) & \longrightarrow & X
 \end{array}$$

of types is a pullback square.

Proof. The commuting square Eq. (3.1) induces a commuting cube

$$\begin{array}{ccccc}
 & & \tilde{B}_1 & & \\
 & \swarrow & \downarrow & \searrow & \\
 \tilde{B}_0 & & \tilde{A}_1 & & \tilde{B}_0 \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \tilde{A}_0 & & Y & & \tilde{A}_0 \\
 & \swarrow & \downarrow & \searrow & \\
 & & X & &
 \end{array}$$

in which the two vertical squares in the back are pullback squares. Moreover, the square in Eq. (3.1) is a pullback square if and only if the front two squares in the cube are pullback squares. The front two squares are pullback squares if and only if the square

$$\begin{array}{ccc}
 \tilde{B}_0 \sqcup^{\tilde{B}_1} \tilde{B}_0 & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 \tilde{A}_0 \sqcup^{\tilde{A}_1} \tilde{A}_0 & \longrightarrow & Y
 \end{array}$$

is a pullback square. Since the pushouts on the left are reflexive coequalizers, the claim follows. \square

3.5 Colimits of diagrams over graphs

Definition 3.5.1. Let \mathcal{A} be a reflexive graph. A diagram \mathcal{D} over \mathcal{A} is a triple $(D_0, D_1, \text{rfx}_{\mathcal{D}})$ consisting of

$$\begin{aligned}
 D_0 &: A_0 \rightarrow \mathcal{U} \\
 D_1 &: \prod_{(i,j:A_0)} A_1(i,j) \rightarrow (D_0(i) \rightarrow D_0(j)) \\
 \text{rfx}_{\mathcal{D}} &: \prod_{(i:A_0)} D_1(\text{rfx}_{\mathcal{A}}(i)) \sim \text{id}_{D_0(i)}.
 \end{aligned}$$

Definition 3.5.2. Let \mathcal{D} be a diagram over \mathcal{A} . We form the **total graph** $\Sigma(\mathcal{A}, \mathcal{D})$ of \mathcal{D} by

$$\begin{aligned}
 \Sigma(\mathcal{A}, \mathcal{D})_0 &:\equiv \sum_{(i:A_0)} D_0(i) \\
 \Sigma(\mathcal{A}, \mathcal{D})_1((i,x), (j,y)) &:\equiv \sum_{(e:A_1(i,j))} D_1(e, x) = y \\
 \text{rfx}_{\Sigma(\mathcal{A}, \mathcal{D})}((i,x)) &:\equiv (\text{rfx}_{\mathcal{A}}(i), \text{rfx}_{\mathcal{D}}(i, x)).
 \end{aligned}$$

There is an obvious projection $\text{pr}_1 : \text{rGph}(\Sigma(\mathcal{A}, \mathcal{D}), \mathcal{A})$.

Lemma 3.5.3. *For any diagram \mathcal{D} over \mathcal{A} , the projection $\text{pr}_1 : \text{rGph}(\Sigma(\mathcal{A}, \mathcal{D}), \mathcal{A})$ is a left fibration.*

Proof. It suffices to show that

$$\begin{array}{ccc} \sum_{((i,x),(j,y)):\Sigma(\mathcal{A},\mathcal{D})_0} \Sigma(\mathcal{A},\mathcal{D})_1((i,x),(j,y)) & \xrightarrow{\pi_1} & \Sigma(\mathcal{A},\mathcal{D}) \\ \downarrow & & \downarrow \\ \sum_{(i,j:A_0)} A_1(i,j) & \xrightarrow{\pi_1} & A_0 \end{array}$$

is a pullback square. Note that this square is equivalent to the square

$$\begin{array}{ccc} \sum_{(i,j:A_0)} \sum_{(e:A_1(i,j))} D_0(i) & \xrightarrow{\lambda(i,j,e,x)\cdot(i,x)} & \sum_{(i:A_0)} D_0(i) \\ \downarrow & & \downarrow \\ \sum_{(i,j:A_0)} A_1(i,j) & \xrightarrow{\pi_1} & A_0, \end{array}$$

which is clearly a pullback square. \square

Remark 3.5.4. It can be shown that the total graph operation is an equivalence from diagrams over \mathcal{A} to left fibrations over \mathcal{A} .

Definition 3.5.5. Let D be a diagram over \mathcal{A} , and let X be a type. A D -cocone on X is a morphism $f : \mathbf{rGph}(\Sigma(\mathcal{A},\mathcal{D}), \Delta X)$ of reflexive graphs. A D -cocone f on X is colimiting if the map

$$\Delta(-) \circ f : (X \rightarrow Y) \rightarrow \mathbf{rGph}(\Sigma(\mathcal{A},\mathcal{D}), \Delta Y)$$

is an equivalence for every type Y .

Remark 3.5.6. By Proposition 3.3.7 it follows that every diagram D over any reflexive graph \mathcal{A} has a colimit.

Definition 3.5.7. Let \mathcal{D} and \mathcal{D}' be diagrams over \mathcal{A} . A **natural transformation** $\tau : \mathcal{D}' \rightarrow \mathcal{D}$ of diagrams consists of

$$\begin{aligned} \tau_0 &: \prod_{(i:A_0)} D'_0(i) \rightarrow D_0(i) \\ \tau_1 &: \prod_{(i,j:A_0)} \prod_{(e:A_1(i,j))} \prod_{(x:D_0(i))} \tau_0(D'_1(e,x)) = D_1(e, \tau_0(x)) \\ \mathbf{rfx}_\tau &: \prod_{(i:A_0)} \prod_{(x:D_0(i))} \tau_1(\mathbf{rfx}_A(i,x)) = \mathbf{refl}_{\tau_0(x)} \end{aligned}$$

A natural transformation $\tau : \mathcal{D}' \rightarrow \mathcal{D}$ of diagrams over \mathcal{A} is said to be **cartesian** if the commutative squares

$$\begin{array}{ccc} D'_0(i) & \xrightarrow{D'_1(e)} & D'_0(j) \\ \tau_0(i) \downarrow & & \downarrow \tau_0(j) \\ D_0(i) & \xrightarrow{D_1(e)} & D_0(j) \end{array}$$

are all pullback squares.

We will use the following general fact about pullbacks.

Proposition 3.5.8. *Let I be a type, and consider for each $i : I$ a commuting square*

$$\begin{array}{ccc} C_i & \xrightarrow{q_i} & B_i \\ p_i \downarrow & & \downarrow g_i \\ A_i & \xrightarrow{f_i} & X_i. \end{array}$$

with $H_i : f_i \circ p_i \sim g_i \circ q_i$. Then the following are equivalent:

- (i) For each $i : I$, the square is a pullback square.
- (ii) The induced square on total spaces

$$\begin{array}{ccc} \sum_{(i:I)} C_i & \xrightarrow{\text{tot}(q)} & \sum_{(i:I)} B_i \\ \text{tot}(p) \downarrow & & \downarrow \text{tot}(g) \\ \sum_{(i:I)} A_i & \xrightarrow{\text{tot}(f)} & \sum_{(i:I)} X_i. \end{array}$$

which commutes via the homotopy $\text{tot}(H) : \text{tot}(f) \circ \text{tot}(p) \sim \text{tot}(g) \circ \text{tot}(q)$, is a pullback square.

Proof. The gap map of the square in assertion (ii) factors as follows:

$$\begin{array}{ccc} & \sum_{(i:I)} C_i & \\ \text{tot}(\text{gap}(p_i, q_i, H_i)) \swarrow & & \searrow \text{gap}(\text{tot}(p), \text{tot}(q), \text{tot}(H)) \\ \sum_{(i:I)} A_i \times_{X_i} B_i & \xrightarrow{\lambda(i, a, b, p) \cdot ((i, a), (i, b), (\text{refl}_i, p))} & \left(\sum_{(i:I)} A_i \right) \times \left(\sum_{(i:I)} X_i \right) \left(\sum_{(i:I)} B_i \right) \end{array}$$

and the bottom map is an equivalence. Therefore it follows by the 3-for-2 property and an application of Proposition 1.2.4 that $\text{gap}(p_i, q_i, H_i)$ is an equivalence for each $i : I$, if and only if $\text{gap}(\text{tot}(p), \text{tot}(q), \text{tot}(H))$ is an equivalence. \square

Proposition 3.5.9. *Let $\tau : \text{nat}(\mathcal{E}, \mathcal{D})$ be a natural transformation of diagrams over \mathcal{A} . Then τ is cartesian if and only if the induced morphism of reflexive graphs*

$$\text{tot}(\tau) : \text{rGph}(\Sigma(\mathcal{A}, \mathcal{E}), \Sigma(\mathcal{A}, \mathcal{D}))$$

is cartesian.

Proof. Straightforward consequence of Proposition 3.5.8. \square

Corollary 3.5.10. *Let $\tau : \text{cart}(\mathcal{E}, \mathcal{D})$ be a cartesian morphisms of diagrams over \mathcal{A} , and consider a commuting square of reflexive graphs of the following form*

$$\begin{array}{ccc} \Sigma(\mathcal{A}, \mathcal{E}) & \longrightarrow & \Delta Y \\ \text{tot}(\tau) \downarrow & & \downarrow \\ \Sigma(\mathcal{A}, \mathcal{D}) & \longrightarrow & \Delta X. \end{array}$$

Then the following are equivalent:

(i) The square is a pullback square.

(ii) The square

$$\begin{array}{ccc} \text{colim}(\mathcal{E}) & \longrightarrow & Y \\ \text{colim}(\tau) \downarrow & & \downarrow \\ \text{colim}(\mathcal{D}) & \longrightarrow & X \end{array}$$

is a pullback square.

3.6 Sequential colimits

Type sequences are diagrams of the following form.

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots .$$

Their formal specification is as follows.

Definition 3.6.1. An **(increasing) type sequence** \mathcal{A} consists of

$$\begin{aligned} A &: \mathbb{N} \rightarrow \mathcal{U} \\ f &: \prod_{(n:\mathbb{N})} A_n \rightarrow A_{n+1}. \end{aligned}$$

Sequential colimits are characterized by their universal property.

Definition 3.6.2. (i) A **(sequential) cocone** on a type sequence \mathcal{A} with vertex X consists of

$$\begin{aligned} h &: \prod_{(n:\mathbb{N})} A_n \rightarrow X \\ H &: \prod_{(n:\mathbb{N})} f_n \sim f_{n+1} \circ H_n. \end{aligned}$$

We write $\text{cocone}(X)$ for the type of cones with vertex X .

(ii) Given a cone (h, H) with vertex X on a type sequence \mathcal{A} we define the map

$$\text{cocone_map}(h, H) : (X \rightarrow Y) \rightarrow \text{cocone}(X)$$

given by $f \mapsto (f \circ h, \lambda n. \lambda x. \text{ap}_f(H_n(x)))$.

(iii) We say that a cone (h, H) with vertex X is **colimiting** if the map $\text{cocone_map}(h, H)$ is an equivalence for any type Y .

In order to study sequential colimits, we first note that type sequences appear as diagrams over the graph (\mathbb{N}, \prec) with type of vertices \mathbb{N} , and with

$$(n \prec m) := (n + 1 = m).$$

Note that this is a non-reflexive graph. A reflexive graph is obtained by the adjunction explained in Item (vi) of Example 3.2.4, which transforms the relation \prec into the reflexive relation \preceq given by

$$(n \preceq m) := (n + 1 = m) + (n = m).$$

A diagram \mathcal{A} on (\mathbb{N}, \prec) consists of

$$A_0 : \prod_{(n:\mathbb{N})} A_n \rightarrow A_1 : \prod_{(n,m:\mathbb{N})} \prod_{(e:n+1=m)} A_n \rightarrow A_m.$$

Note that the type of A_1 is equivalent to the type $\prod_{(n:\mathbb{N})} A_n \rightarrow A_{n+1}$, so we see indeed that type sequences are equivalently described as diagrams over (\mathbb{N}, \prec) . Similarly, a sequential cocone on \mathcal{A} with vertex X is equivalently described as a cocone on the diagram \mathcal{A} with vertex X . Thus, we also have the following:

Lemma 3.6.3. *Consider a cocone (h, H) with vertex B for a type sequence \mathcal{A} . The following are equivalent:*

- (i) *The cocone (h, H) is colimiting.*
- (ii) *The cocone (h, H) is inductive in the sense that for every type family $P : B \rightarrow \mathcal{U}$, the map*

$$\left(\prod_{(b:B)} P(b) \right) \rightarrow \sum_{(h:\prod_{(n:\mathbb{N})} \prod_{(x:A_n)} P(h_n(x)))} \prod_{(n:\mathbb{N})} \prod_{(x:A_n)} \text{tr}_P(H_n(x), h_n(x)) = h_{n+1}(f_n(x))$$

given by

$$s \mapsto (\lambda n. s \circ h_n, \lambda n. \lambda x. \text{apd}_s(H_n(x)))$$

has a section.

- (iii) *The map in (ii) is an equivalence.*

Definition 3.6.4. The type of **descent data** on a type sequence $\mathcal{A} \equiv (A, f)$ is defined to be

$$\text{Desc}(\mathcal{A}) \equiv \sum_{(B:\prod_{(n:\mathbb{N})} A_n \rightarrow \mathcal{U})} \prod_{(n:\mathbb{N})} \prod_{(x:A_n)} B_n(x) \simeq B_{n+1}(f_n(x)).$$

Equivalently, if \mathcal{A} is seen as a diagram over (\mathbb{N}, \prec) ,

Definition 3.6.5. We define a map

$$\text{desc_fam} : (A_\infty \rightarrow \mathcal{U}) \rightarrow \text{Desc}(\mathcal{A})$$

by $B \mapsto (\lambda n. \lambda x. B(\text{seq_in}(n, x)), \lambda n. \lambda x. \text{tr}_B(\text{seq_glue}(n, x)))$.

Theorem 3.6.6. *The map*

$$\text{desc_fam} : (A_\infty \rightarrow \mathcal{U}) \rightarrow \text{Desc}(\mathcal{A})$$

is an equivalence.

Definition 3.6.7. A **cartesian transformation** of type sequences from \mathcal{A} to \mathcal{B} is a pair (h, H) consisting of

$$\begin{aligned} h &: \prod_{(n:\mathbb{N})} A_n \rightarrow B_n \\ H &: \prod_{(n:\mathbb{N})} g_n \circ h_n \sim h_{n+1} \circ f_n, \end{aligned}$$

such that each of the squares in the diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \\ h_0 \downarrow & & h_1 \downarrow & & h_2 \downarrow & & \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \end{array}$$

is a pullback square. We define

$$\begin{aligned} \text{cart}(\mathcal{A}, \mathcal{B}) &::= \sum_{(h: \prod_{(n:\mathbb{N})} A_n \rightarrow B_n)} \\ &\quad \sum_{(H: \prod_{(n:\mathbb{N})} g_n \circ h_n \sim h_{n+1} \circ f_n)} \prod_{(n:\mathbb{N})} \text{is_pullback}(h_n, f_n, H_n), \end{aligned}$$

and we write

$$\text{Cart}(\mathcal{B}) ::= \sum_{(\mathcal{A}: \text{Seq})} \text{cart}(\mathcal{A}, \mathcal{B}).$$

Definition 3.6.8. We define a map

$$\text{cart_map}(\mathcal{B}) : \left(\sum_{(X': \mathcal{U})} X' \rightarrow X \right) \rightarrow \text{Cart}(\mathcal{B}).$$

which associates to any morphism $h : X' \rightarrow X$ a cartesian transformation of type sequences into \mathcal{B} .

Theorem 3.6.9. *The operation $\text{cart_map}(\mathcal{B})$ is an equivalence.*

The flattening lemma for sequential colimits essentially states that sequential colimits commute with Σ .

Lemma 3.6.10. *Consider*

$$\begin{aligned} B &: \prod_{(n:\mathbb{N})} A_n \rightarrow \mathcal{U} \\ g &: \prod_{(n:\mathbb{N})} \prod_{(x:A_n)} B_n(x) \simeq B_{n+1}(f_n(x)). \end{aligned}$$

and suppose $P : A_\infty \rightarrow \mathcal{U}$ is the unique family equipped with

$$e : \prod_{(n:\mathbb{N})} B_n(x) \simeq P(\text{seq_in}(n, x))$$

and homotopies $H_n(x)$ witnessing that the square

$$\begin{array}{ccc} B_n(x) & \xrightarrow{g_n(x)} & B_{n+1}(f_n(x)) \\ e_n(x) \downarrow & & \downarrow e_{n+1}(f_n(x)) \\ P(\text{seq_in}(n, x)) & \xrightarrow{\text{tr}_P(\text{seq_glue}(n, x))} & P(\text{seq_in}(n+1, f_n(x))) \end{array}$$

commutes. Then $\sum_{(t:A_\infty)} P(t)$ satisfies the universal property of the sequential colimit of the type sequence

$$\sum_{(x:A_0)} B_0(x) \xrightarrow{\text{tot}_{f_0}(g_0)} \sum_{(x:A_1)} B_1(x) \xrightarrow{\text{tot}_{f_1}(g_1)} \sum_{(x:A_2)} B_2(x) \xrightarrow{\text{tot}_{f_2}(g_2)} \cdots$$

In the following theorem we rephrase the flattening lemma in using cartesian transformations of type sequences.

Theorem 3.6.11. *Consider a commuting diagram of the form*

$$\begin{array}{ccccccc}
 A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & \searrow & \swarrow & \downarrow & \\
 & & & & X & & \\
 \downarrow & & \downarrow & \searrow & \swarrow & \downarrow & \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \\
 & & & & Y & &
 \end{array}$$

If each of the vertical squares is a pullback square, and Y is the sequential colimit of the type sequence B_n , then X is the sequential colimit of the type sequence A_n .

Corollary 3.6.12. *Consider a commuting diagram of the form*

$$\begin{array}{ccccccc}
 A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & \searrow & \swarrow & \downarrow & \\
 & & & & X & & \\
 \downarrow & & \downarrow & \searrow & \swarrow & \downarrow & \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \\
 & & & & Y & &
 \end{array}$$

If each of the vertical squares is a pullback square, then the square

$$\begin{array}{ccc}
 A_\infty & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 B_\infty & \longrightarrow & Y
 \end{array}$$

is a pullback square.

Chapter 4

Homotopy images

We have observed in Example 3.3.8 that the reflexive coequalizer of the pre-kernel of a map $f : A \rightarrow X$ is the fiberwise join $A *_X A$, i.e. we have a reflexive coequalizer diagram

$$A \times_X A \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} A \longrightarrow A *_X A.$$

It can also be shown that the colimit of the 2-pre-kernel of a map $f : A \rightarrow X$ is the triple fiberwise join $A *_X A *_X A$, i.e. we have a colimiting

$$A \times_X A \times_X A \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} A \times_X A \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} A \longrightarrow A *_X A *_X A,$$

although one has to take care to use the right amount of coherence data. These results suggest that geometric realization of the Čech nerve of a map $f : A \rightarrow X$, i.e. the homotopy image of f , is the sequential colimit of the type sequence

$$A \longrightarrow A *_X A \longrightarrow A *_X (A *_X A) \longrightarrow \dots$$

and its sequential colimit. We do not have a way of presenting the Čech nerve of a map in type theory, due to the infinite coherence problem of presenting simplicial types in type theory. Nevertheless, we *can* analyze this type sequence and its colimit in homotopy type theory, since it is constructed entirely in terms of known operations. We will show in Theorem 4.2.13 that for any map $f : A \rightarrow X$, the infinite fiberwise join-power $\dots *_X A *_X A *_X A$ is the image of f . We conclude that the image of a map always exists in univalent type theory with homotopy pushouts. We note that an earlier construction of the propositional truncation in a similar setting is due to van Doorn [15], and another one is due to Kraus [20]. The present construction of the image of f is called the join construction implies, as we show in Theorem 4.2.17, that the image of an essentially small type mapping into a locally small type is again essentially small. In particular, the image of a map from a small type into the universe is essentially small. This corollary should be viewed as a type theoretic replacement axiom. This fact has the important consequence that any connected component of the universe is essentially small. We also note that the join construction leads to new constructions of set quotients and of Rezk completions, see [27], where also a construction of the n -truncations is given as an application of the join construction.

4.1 The universal property of the image

Definition 4.1.1. Consider a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{i} & U \\ & \searrow f & \swarrow m \\ & & X \end{array}$$

with $I : f \sim m \circ i$, and where m is an embedding. We say that m has the **universal property of the image of f** if the map

$$(i, I)^* : \text{hom}_X(m, m') \rightarrow \text{hom}_X(f, m')$$

defined by $(i, I)^*(h, H) := (h \circ i, I \cdot (i \cdot H))$, is an equivalence for every embedding $m' : U' \rightarrow X$.

Lemma 4.1.2. *For any $f : A \rightarrow X$ and any embedding $m : U \rightarrow X$, the type $\text{hom}_X(f, m)$ is a proposition.*

Proof. From Corollary 1.5.8 we obtain that the type $\text{hom}_X(f, m)$ is equivalent to the type

$$\prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_m(x),$$

so it suffices to show that this is a proposition. Recall from Proposition 1.2.13 that a map is an embedding if and only if its fibers are propositions. Thus we see that the type $\prod_{(x:X)} \text{fib}_f(x) \rightarrow \text{fib}_m(x)$ is a product of propositions, so it is a proposition by Corollary 1.3.2. \square

Corollary 4.1.3. *Consider a commuting triangle*

$$\begin{array}{ccc} A & \xrightarrow{i} & U \\ & \searrow f & \swarrow m \\ & & X \end{array}$$

with $I : f \sim m \circ i$, and where m is an embedding. Then m satisfies the universal property of the image of f if and only if the implication

$$\text{hom}_X(f, m') \rightarrow \text{hom}_X(m, m')$$

holds for every embedding $m' : U' \rightarrow X$.

Recall that embeddings into the unit type are just propositions. Therefore, the universal property of the image of the map $A \rightarrow \mathbf{1}$ is a proposition P satisfying the universal property of the propositional truncation:

Definition 4.1.4. Let A be a type, and let P be a proposition that comes equipped with a map $f : A \rightarrow P$. We say that $f : A \rightarrow P$ satisfies the **universal property of propositional truncation** if for every proposition Q , the precomposition map

$$- \circ f : (P \rightarrow Q) \rightarrow (A \rightarrow Q)$$

is an equivalence.

4.2 The join construction

4.2.1 Step one: constructing the propositional truncation

Lemma 4.2.1. *Suppose $f : A \rightarrow P$, where A is any type, and P is a proposition. Then the map*

$$(A * B \rightarrow P) \rightarrow (B \rightarrow P)$$

given by $h \mapsto h \circ \text{inr}$ is an equivalence, for any type B .

Proof. Since both types are propositions by Corollary 1.3.2 it suffices to construct a map

$$(B \rightarrow P) \rightarrow (A * B \rightarrow P).$$

Let $g : B \rightarrow P$. Then the square

$$\begin{array}{ccc} A \times B & \xrightarrow{\text{pr}_2} & B \\ \text{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & P \end{array}$$

commutes since P is a proposition. Therefore we obtain a map $A * B \rightarrow P$ by the universal property of the join. \square

The idea of the construction of the propositional truncation is that if we are given a map $f : A \rightarrow P$, where P is a proposition, then it extends uniquely along $\text{inr} : A \rightarrow A * A$ to a map $A * A \rightarrow P$. This extension again extends uniquely along $\text{inr} : A * A \rightarrow A * (A * A)$ to a map $A * (A * A) \rightarrow P$ and so on, resulting in a diagram of the form

$$\begin{array}{ccccccc} A & \xrightarrow{\text{inr}} & A * A & \xrightarrow{\text{inr}} & A * (A * A) & \xrightarrow{\text{inr}} & \dots \\ & \searrow & \vdots & & \swarrow & \searrow & \\ & & P & & & & \end{array}$$

Definition 4.2.2. The **join powers** A^{*n} of a type X are defined by

$$\begin{aligned} A^{*0} &::= \mathbf{0} \\ A^{*1} &::= A \\ A^{*(n+1)} &::= A * A^{*n}. \end{aligned}$$

Furthermore, we define $A^{*\infty}$ to be the sequential colimit of the type sequence

$$A^{*0} \longrightarrow A^{*1} \xrightarrow{\text{inr}} A^{*2} \xrightarrow{\text{inr}} \dots$$

Our goal is now to show that $A^{*\infty}$ is a proposition and satisfies the universal property of the propositional truncation.

Lemma 4.2.3. *Consider a type sequence*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

with sequential colimit A_∞ , and let P be a proposition. Then the map

$$\text{seq_in}^* : (A_\infty \rightarrow P) \rightarrow \left(\prod_{(n:\mathbb{N})} A_n \rightarrow P \right)$$

given by $h \mapsto \lambda n. (h \circ \text{seq_in}_n)$ is an equivalence.

Proof. By the universal property of sequential colimits established in Lemma 3.6.3 we obtain that `cocone_map` is an equivalence. Note that we have a commuting triangle

$$\begin{array}{ccc} & P^{A_\infty} & \\ \text{cocone_map} \swarrow & & \searrow \text{seq_in}^* \\ \text{cocone}(P) & \xrightarrow{\text{pr}_1} & \left(\prod_{(n:\mathbb{N})} A_n \rightarrow P \right) \end{array}$$

Note that for any $g : \prod_{(n:\mathbb{N})} A_n \rightarrow P$ the type

$$\prod_{(n:\mathbb{N})} g_n \sim g_{n+1} \circ f_n$$

is a product of contractible types, since P is a proposition. Therefore it is contractible by Proposition 1.3.1, and it follows that the projection is an equivalence. We conclude by the 3-for-2 property of equivalences that `seq_in*` is an equivalence. \square

Lemma 4.2.4. *Let A be a type, and let P be a proposition. Then the function*

$$- \circ \text{seq_in}_0 : (A^{*\infty} \rightarrow P) \rightarrow (A \rightarrow P)$$

is an equivalence.

Proof. We have the commuting triangle

$$\begin{array}{ccc} & P^{A^{*\infty}} & \\ \text{seq_in}^* \swarrow & & \searrow - \circ \text{seq_in}_0 \\ \left(\prod_{(n:\mathbb{N})} A^{*n} \rightarrow P \right) & \xrightarrow{\lambda h. h_0} & P^A. \end{array}$$

Therefore it suffices to show that the bottom map is an equivalence. Since this is a map between propositions, it suffices to construct a map in the converse direction. Let $f : A \rightarrow P$. We will construct a term of type

$$\prod_{(n:\mathbb{N})} A^{*n} \rightarrow P$$

by induction on $n : \mathbb{N}$. The base case is trivial. Given a map $g : A^{*n} \rightarrow P$, we obtain a map $g : A^{*(n+1)} \rightarrow P$ by Lemma 4.2.1. \square

Lemma 4.2.5. *Consider a type sequence*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

and suppose that each A_n is equipped with a base point $a_n : A_n$, and each f_n is equipped with a homotopy $H_n : \text{const}_{a_{n+1}} \sim f_n$. Then the sequential colimit A_∞ is contractible. \square

Lemma 4.2.6. *The type $A^{*\infty}$ is a proposition for any type A .*

Proof. By Lemma 1.2.10 it suffices to show that $A^{*\infty} \rightarrow \text{is_contr}(A^{*\infty})$, and by Lemma 4.2.4 it suffices to show that

$$A \rightarrow \text{is_contr}(A^{*\infty}),$$

because $\text{is_contr}(A^{*\infty})$ is a proposition.

Let $x : A$. To see that $A^{*\infty}$ is contractible it suffices by Lemma 4.2.5 to show that $\text{inr} : A^{*n} \rightarrow A^{*(n+1)}$ is homotopic to the constant function $\text{const}_{\text{inl}(x)}$. However, we get a homotopy $\text{const}_{\text{inl}(x)} \sim \text{inr}$ immediately from the path constructor `glue`. \square

Theorem 4.2.7. *For any type $A : \mathcal{U}$ there is a proposition $\|A\| : \mathcal{U}$ that comes equipped with a map $\eta : A \rightarrow \|A\|$, and satisfies the universal property of propositional truncation.*

Proof. Let A be a type. Then we define $\|A\| := A^{*\infty}$, and we define $\eta := \text{seq_in}_0 : A \rightarrow A^{*\infty}$. Then $\|A\|$ is a proposition by Lemma 4.2.6, and $\eta : A \rightarrow \|A\|$ satisfies the universal property of propositional truncation by Lemma 4.2.4. \square

4.2.2 Step two: constructing the image of a map

Following Definition 7.6.3 of [33], we recall that the image of a map $f : A \rightarrow X$ can be defined using the propositional truncation:

Definition 4.2.8. For any map $f : A \rightarrow X$ we define the **image** of f to be the type

$$\text{im}(f) := \sum_{(x:X)} \|\text{fib}_f(x)\|$$

and we define the **image inclusion** to be the projection $\text{pr}_1 : \text{im}(f) \rightarrow X$.

However, the construction of the fiberwise join in Definition 2.3.14 suggests that we can also define the image of f as the infinite join power $f^{*\infty}$, where we repeatedly take the fiberwise join of f with itself. Our reason for defining the image in this way is twofold:

- We use this construction to show that the image of a map $f : A \rightarrow B$ from an essentially small type A into a locally small type B is again essentially small.
- Some interesting types, such as the real and complex projective spaces, appear in specific instances of this construction.

Lemma 4.2.9. *Consider a map $f : A \rightarrow X$, an embedding $m : U \rightarrow X$, and $h : \text{hom}_X(f, m)$. Then the map*

$$\text{hom}_X(f * g, m) \rightarrow \text{hom}_X(g, m)$$

is an equivalence for any $g : B \rightarrow X$.

Proof. Note that both types are propositions, so any equivalence can be used to prove the claim. Thus, we simply calculate

$$\begin{aligned} \text{hom}_X(f * g, m) &\simeq \prod_{(x:X)} \text{fib}_{f*g}(x) \rightarrow \text{fib}_m(x) \\ &\simeq \prod_{(x:X)} \text{fib}_f(x) * \text{fib}_g(x) \rightarrow \text{fib}_m(x) \\ &\simeq \prod_{(x:X)} \text{fib}_g(x) \rightarrow \text{fib}_m(x) \\ &\simeq \text{hom}_X(g, m). \end{aligned}$$

The first equivalence holds by Corollary 1.5.8; the second equivalence holds by Theorem 2.3.15; the third equivalence holds by Lemma 4.2.1; the last equivalence again holds by Corollary 1.5.8. \square

For the construction of the image of $f : A \rightarrow X$ we observe that if we are given an embedding $m : U \rightarrow X$ and a map $(i, I) : \text{hom}_X(f, m)$, then (i, I) extends uniquely along $\text{inr} : A \rightarrow A *_X A$ to a map $\text{hom}_X(f * f, m)$. This extension again extends uniquely along $\text{inr} : A *_X A \rightarrow A *_X (A *_X A)$ to a map $\text{hom}_X(f * (f * f), m)$ and so on, resulting in a diagram of the form

$$\begin{array}{ccccccc} A & \xrightarrow{\text{inr}} & A *_X A & \xrightarrow{\text{inr}} & A *_X (A *_X A) & \xrightarrow{\text{inr}} & \dots \\ & \searrow & \downarrow & \nearrow & \nearrow & \nearrow & \\ & & U & & & & \end{array}$$

Definition 4.2.10. Suppose $f : A \rightarrow X$ is a map. Then we define the **fiberwise join powers**

$$f^{*n} := A_X^{*n} \rightarrow X.$$

Construction. Note that the operation $(B, g) \mapsto (A *_X B, f * g)$ defines an endomorphism on the type

$$\sum_{(B:U)} B \rightarrow X.$$

We also have $(\mathbf{0}, \text{ind}_\mathbf{0})$ and (A, f) of this type. For $n \geq 1$ we define

$$\begin{aligned} A_X^{*(n+1)} &:= A *_X A_X^{*n} \\ f^{*(n+1)} &:= f * f^{*n}. \end{aligned} \quad \square$$

Definition 4.2.11. We define $A_X^{*\infty}$ to be the sequential colimit of the type sequence

$$A_X^{*0} \longrightarrow A_X^{*1} \xrightarrow{\text{inr}} A_X^{*2} \xrightarrow{\text{inr}} \dots$$

Since we have a cocone

$$\begin{array}{ccccccc} A_X^{*0} & \longrightarrow & A_X^{*1} & \xrightarrow{\text{inr}} & A_X^{*2} & \xrightarrow{\text{inr}} & \dots \\ & \searrow & \downarrow f^{*1} & \nearrow f^{*2} & \nearrow & \nearrow & \\ & & X & & & & \end{array}$$

we also obtain a map $f^{*\infty} : A_X^{*\infty} \rightarrow X$ by the universal property of $A_X^{*\infty}$.

Lemma 4.2.12. *Let $f : A \rightarrow X$ be a map, and let $m : U \rightarrow X$ be an embedding. Then the function*

$$- \circ \text{seq_in}_0 : \text{hom}_X(f^{*\infty}, m) \rightarrow \text{hom}_X(f, m)$$

is an equivalence.

Theorem 4.2.13. *For any map $f : A \rightarrow X$, the map $f^{*\infty} : A_X^{*\infty} \rightarrow X$ is an embedding that satisfies the universal property of the image inclusion of f .*

4.2.3 Step three: establishing the smallness of the image

Recall from Definition 1.5.2 that a type is said to be locally small if its identity types are equivalent to small types.

Lemma 4.2.14. *Consider a commuting square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

- (i) *If the square is cartesian, B and C are essentially small, and D is locally small, then A is essentially small.*
- (ii) *If the square is cocartesian, and A , B , and C are essentially small, then D is essentially small.*

Corollary 4.2.15. *Suppose $f : A \rightarrow X$ and $g : B \rightarrow X$ are maps from essentially small types A and B , respectively, to a locally small type X . Then $A \times_X B$ is again essentially small.*

Lemma 4.2.16. *Consider a type sequence*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

where each A_n is essentially small. Then its sequential colimit is again essentially small.

Theorem 4.2.17. *For any map $f : A \rightarrow X$ from a small type A into a locally small type X , the image $\text{im}(f)$ is an essentially small type.*

Recall that in set theory, the replacement axiom asserts that for any family of sets $\{X_i\}_{i \in I}$ indexed by a set I , there is a set $X[I]$ consisting of precisely those sets x for which there exists an $i \in I$ such that $x \in X_i$. In other words: the image of a set-indexed family of sets is again a set. Without the replacement axiom, $X[I]$ would be a class. In the following corollary we establish a type-theoretic analogue of the replacement axiom: the image of a family of small types indexed by a small type is again (essentially) small.

Corollary 4.2.18. *For any small type family $B : A \rightarrow \mathcal{U}$, where A is small, the image $\text{im}(B)$ is essentially small. We call $\text{im}(B)$ the **univalent completion** of B .*

Chapter 5

Reflective subuniverses

In this chapter we study reflective subuniverses. Reflective subuniverses were first introduced in section 7.7 of [33], and were studied in much more detail in [28].

In §5.1 we establish the basic closure properties of reflective subuniverses. In particular, we show in Proposition 5.1.19 that pullbacks of L -local types are again L -local. It follows that cartesian products of L -local types and identity types of L -local types are again L -local. In Proposition 5.1.22 we show that any dependent product of L -local types is also L -local, regardless of whether the indexing type is L -local or not. We use this fact in Proposition 5.1.25 to show that the map $\eta \times \eta : X \times Y \rightarrow LX \times LY$ is an L -localization, and that the L -localization of a proposition is again a proposition.

In §5.7 we study accessible reflective subuniverses.

In §5.2 we introduce the notion of L -separated type, for any reflective subuniverse L , and we show that the subuniverse of L -separated types is again a reflective subuniverse. The contents of §5.2 are joint work with Dan Christensen, Morgan Opie, and Luis Scoccola. As a corollary we obtain in Theorem 5.2.17 that the k -truncation can be constructed, for any $k \geq -2$. A more elementary way of obtaining this result appears in [27].

5.1 Localizations

A **subuniverse** is simply a subtype of the universe. Note that we do not require that subuniverses are closed under any type constructors¹ For any subuniverse $P : \mathcal{U} \rightarrow \mathbf{Prop}$, we write $\mathcal{U}_P := \sum_{(X:\mathcal{U})} P(X)$, and we say that $X : \mathcal{U}$ is a P -type if X is in \mathcal{U}_P , i.e. if $P(X)$ holds.

Definition 5.1.1. Let $P : \mathcal{U} \rightarrow \mathbf{Prop}$ be a subuniverse, and let X be a type. A **P -localization** of X is a triple (Y, l, p) consisting of a P -type $Y : \mathcal{U}_P$, a map $l : X \rightarrow Y$, and a term p witnessing that the map

$$l^* : (Y \rightarrow Z) \rightarrow (X \rightarrow Z)$$

is an equivalence, for every $Z : \mathcal{U}_P$. This property is also called the **universal property** of the P -localization of X .

¹In particular, we will see that reflective subuniverses aren't necessarily closed under Σ .

In other words, a P -localization of X is a map $l : X \rightarrow Y$ into a P -type Y , such that every map $f : X \rightarrow Z$ into a P -type Z extends uniquely along l , as indicated in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ l \downarrow & \nearrow & \\ Y & & \end{array}$$

Proposition 5.1.2. *For any subuniverse $P : \mathcal{U} \rightarrow \mathbf{Prop}$ and any type X , the type of P -localizations of X is a proposition.*

Proof. Consider (Y, f, p) and (Y', f', p') of the described type. Since p and p' are terms of a proposition, it suffices to show that $(Y, f) = (Y', f')$. In other words, we have to find an equivalence $g : Y \rightarrow Y'$ such that $g \circ f' = f$.

By $p(Y')$, the type of pairs (g, h) consisting of a function $g : Y \rightarrow Y'$ such that $h : g \circ f = f'$ is contractible. By $p'(Y)$, the type of pairs (g', h') consisting of a function $g' : Y' \rightarrow Y$ such that $h' : g' \circ f' = f$ is contractible.

Now $g' \circ g$ is a function such that $g' \circ g \circ f = g' \circ f' = f$, as is id_Y . By contractibility, it follows that $g' \circ g = \text{id}_Y$. Similarly, $g \circ g' = \text{id}_{Y'}$. \square

Proposition 5.1.3. *Consider a subuniverse $P : \mathcal{U} \rightarrow \mathbf{Prop}$ and a P -localization $l : X \rightarrow Y$. The following are equivalent:*

- (i) X is a P -type (i.e. X is in \mathcal{U}_P).
- (ii) The P -localization $l : X \rightarrow Y$ is an equivalence.
- (iii) The map $l^* : (Y \rightarrow X) \rightarrow (X \rightarrow X)$ is an equivalence.
- (iv) The P -localization $l : X \rightarrow Y$ has a retraction.

Proof. Certainly if l is an equivalence, then X is in P since it is equivalent to the type $Y : \mathcal{U}_P$. Conversely, if X is in P then id_X has the same universal property of l ; so by Proposition 5.1.2 they are equivalent and hence l is an equivalence. This shows that (i) holds if and only if (ii) holds.

It is clear that (ii) implies (iii). Furthermore, (iii) implies (iv) because the fiber of l^* at id_X is contractible. In particular, there is a function $g : Y \rightarrow X$ equipped with a homotopy $g \circ l \sim \text{id}$. In other words, l has a retraction. To see that (iv) implies (ii), suppose g is a retraction of l , i.e. $g \circ l = \text{id}_X$. Then $l \circ g \circ l = l$, so $l \circ g$ is a factorization of l through itself. By uniqueness of such factorizations, $l \circ g = \text{id}_Y$. Thus g is also a section of l , hence l is an equivalence. \square

Corollary 5.1.4. *For any subuniverse P , the unit type $\mathbf{1}$ has a P -localization if and only if $\mathbf{1}$ is already a P -type.*

Proof. Immediate from the fact that $\mathbf{1}$ is a retract of any pointed type. \square

The universal property of P -localization is by itself not sufficient to imply a dependent universal property. However, we have the following ‘constrained’ dependent universal property.

Proposition 5.1.5. *Let P be a subuniverse, and let $l : X \rightarrow Y$ be a P -localization. Furthermore, consider a type family $Z : Y \rightarrow \mathcal{U}$ such that the total space $\sum_{(y:Y)} Z(y)$ is a P -type. Then the precomposition map*

$$l^* : \left(\prod_{(y:Y)} Z(y) \right) \longrightarrow \left(\prod_{(x:X)} Z(l(x)) \right)$$

is an equivalence.

Proof. Since Y is a P -type and $\sum_{(y:Y)} Z(y)$ is a P -type, the precomposition maps l^* in the commuting square

$$\begin{array}{ccc} (Y \rightarrow \sum_{(y:Y)} Z(y)) & \xrightarrow{l^*} & (X \rightarrow \sum_{(y:Y)} Z(y)) \\ \text{pr}_1 \circ - \downarrow & & \downarrow \text{pr}_1 \circ - \\ (Y \rightarrow Y) & \xrightarrow{l^*} & (X \rightarrow Y) \end{array}$$

are equivalences. It follows that they induce an equivalence from the fiber of the left-hand map $\text{pr}_1 \circ -$ at id_Y to the fiber of the right-hand map $\text{pr}_1 \circ -$ at l . In other words, we have an equivalence

$$l^* : \left(\prod_{(y:Y)} Z(y) \right) \longrightarrow \left(\prod_{(x:X)} Z(l(y)) \right). \quad \square$$

Proposition 5.1.6. *Let $P : \mathcal{U} \rightarrow \mathbf{Prop}$ be a subuniverse, and write $\tilde{\mathcal{U}}_P \equiv \sum_{(X:\mathcal{U}_P)} X$. The projection $\text{pr}_1 : \tilde{\mathcal{U}}_P \rightarrow \mathcal{U}_P$ classifies the small maps whose fibers satisfy P .*

Proof. Let $f : Y \rightarrow X$ be any map into X . Then $\text{fib}_f : X \rightarrow \mathcal{U}$ factors through \mathcal{U}_P if and only if all the fibers of f satisfy P . Let us write $P(f)$ for $\prod_{(x:X)} P(\text{fib}_f(x))$. Then we see that the equivalence χ of Theorem 4.8.3 of [34] restricts to an equivalence

$$\chi^P : \left(\sum_{(Y:\mathcal{U})} \sum_{(f:Y \rightarrow X)} P(f) \right) \rightarrow (X \rightarrow \mathcal{U}_P).$$

Now observe that the outer square and the square on the right in the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\lambda_{y. (\text{fib}_f(f(y)), -, (y, \text{refl}_{f(y)}))}} & \tilde{\mathcal{U}}_P & \longrightarrow & \tilde{\mathcal{U}} \\ f \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\text{fib}_f} & \mathcal{U}_P & \longrightarrow & \mathcal{U} \end{array}$$

are pullback squares. Hence the square on the left is a pullback square. \square

Definition 5.1.7. A **reflective subuniverse** L is a subuniverse $\mathcal{U}_L \rightarrow \mathcal{U}$ equipped with an L -localization

$$\eta_X : X \rightarrow LX$$

for every type $X : \mathcal{U}$. The L -localization $\eta_X : X \rightarrow LX$ is sometimes also called the **unit** of the localization. A type in \mathcal{U}_L is also said to be **local**.

Theorem 5.1.8. *The data of any two reflective subuniverses with the same local types are the same.*

Proof. Immediate from the fact that the type of localizations is a proposition. \square

Lemma 5.1.9. *Any reflective subuniverse is a functor up to homotopy: given $f : A \rightarrow B$ we have an induced map $Lf : LA \rightarrow LB$, preserving identities and composition up to homotopy. Moreover, η is a natural transformation up to homotopy, i.e. for any f we have $Lf \circ \eta_A = \eta_B \circ f$.*

Proof. Define Lf to be the unique function such that $Lf \circ \eta_A = \eta_B \circ f$, using the universal property of η_A . The rest is easy to check using further universal properties. \square

Corollary 5.1.10. *The subuniverse of L -local types is closed under retracts.*

Proof. Consider an L -local type Y , and suppose that $i : X \rightarrow Y$ has a retraction $r : Y \rightarrow X$. By the functoriality of L it follows that LX is also a retract of LY , since we have $Lr \circ Li \sim \text{id}$. Therefore we see that the L -localization $\eta : X \rightarrow LX$ is a retract of the L -localization $\eta : Y \rightarrow LY$, as indicated in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ \eta \downarrow & & \eta \downarrow & & \downarrow \eta \\ LX & \xrightarrow{Li} & LY & \xrightarrow{Lr} & LX. \end{array}$$

Since $\eta : Y \rightarrow LY$ is an equivalence, and equivalences are closed under retracts, it follows that $\eta : X \rightarrow LX$ is an equivalence, hence X is L -local. \square

Definition 5.1.11. A map $f : A \rightarrow B$ is said to be an **L -equivalence** if $Lf : LA \rightarrow LB$ is an equivalence.

Proposition 5.1.12. *For a map $f : A \rightarrow B$ the following are equivalent:*

- (i) f is an L -equivalence.
- (ii) For any L -local type X , the precomposition map

$$f^* : (B \rightarrow X) \rightarrow (A \rightarrow X)$$

is an equivalence.

Proof. Suppose first that f is an L -equivalence, and let X be L -local. Then the square

$$\begin{array}{ccc} X^{LB} & \xrightarrow{Lf^*} & X^{LA} \\ \eta^* \downarrow & & \downarrow \eta^* \\ X^B & \xrightarrow{f^*} & X^A \end{array}$$

commutes. In this square the two vertical maps are equivalences by the universal property of localization, and the top map is an equivalence since Lf is an equivalence. Therefore the map $f^* : X^B \rightarrow X^A$ is an equivalence, as desired.

Conversely, assume that $f^* : X^B \rightarrow X^A$ is an equivalence for every L -local type X . By the square above it follows that $Lf^* : X^{LB} \rightarrow X^{LA}$ is an equivalence for every

L -local type X . The fiber of $LA^{LB} \rightarrow LA^{LA}$ at $\text{id} : LA \rightarrow LA$ is contractible, so we obtain a retraction g of Lf . To see that g is also a section observe that the fiber of $LB^{LB} \rightarrow LB^{LA}$ at Lf is contractible. This fiber contains $(\text{id}_{LB}, \text{refl}_{Lf})$. However, we also have an identification $p : Lf^*(Lf \circ g) = Lf$, since

$$Lf^*(Lf \circ g) \equiv (Lf \circ g) \circ Lf \equiv Lf \circ (g \circ Lf) = Lf.$$

Therefore $(Lf \circ g, p)$ is in the fiber of $Lf^* : LB^{LB} \rightarrow LB^{LA}$ at Lf . By the contractibility of the fibers it follows that $(Lf \circ g, p) = (\text{id}_{LB}, \text{refl}_{Lf})$, so it follows that $Lf \circ g = \text{id}_{LB}$. In other words, g is both a retraction and a section of Lf , so Lf is an equivalence. \square

Corollary 5.1.13. *Let $f : X \rightarrow Y$ be a map into an L -local type Y . Then the following are equivalent:*

(i) f is an L -localization.

(ii) f is an L -equivalence.

In particular, the map $\eta_X : X \rightarrow LX$ is an L -equivalence.

Corollary 5.1.14. *Any retract of an L -localization is again an L -localization.*

Proof. Suppose that $f : A \rightarrow B$ is a retract of an L -localization $l : X \rightarrow Y$. Then B is L -local by the previous claim. Moreover, Lf is a retract of Ll , which is an equivalence, so f is an L -equivalence. Therefore f is an L -localization by Corollary 5.1.13. \square

Proposition 5.1.15. *Given a reflective subuniverse L , a type X is L -local if and only if $(-\circ f) : (B \rightarrow X) \rightarrow (A \rightarrow X)$ is an equivalence, for any L -equivalence $f : A \rightarrow B$.*

Proof. If Lf is an equivalence and X is L -local, then by the universal property of η , we have a commutative square

$$\begin{array}{ccc} X^{LB} & \xrightarrow{-\circ Lf} & X^{LA} \\ -\circ \eta_B \downarrow & & \downarrow -\circ \eta_A \\ X^B & \xrightarrow{-\circ f} & X^A \end{array}$$

in which all but the top map are equivalences; thus so is the top map.

Conversely, since $L\eta_X$ is an equivalence, the hypothesis implies that $(-\circ \eta_X) : (LX \rightarrow X) \rightarrow (X \rightarrow X)$ is an equivalence. In particular, its fiber over id_X is inhabited, i.e. η_X has a retraction; hence X is L -local. \square

Proposition 5.1.16. *For any family*

$$f : \prod_{(x:X)} P(x) \rightarrow Q(x)$$

of L -equivalences, the induced map on total spaces

$$\text{tot}(f) : \left(\sum_{(x:X)} P(x) \right) \rightarrow \left(\sum_{(x:X)} Q(x) \right)$$

is an L -equivalence.

Proof. Note that we have a commuting square

$$\begin{array}{ccc} Y^{\sum_{(x:X)} Q(x)} & \xrightarrow{\text{tot}(f)^*} & Y^{\sum_{(x:X)} P(x)} \\ \text{ev_pair} \downarrow & & \downarrow \text{ev_pair} \\ \prod_{(x:X)} Y^{Q(x)} & \xrightarrow{\lambda g. \lambda x. g(x) \circ f(x)} & \prod_{(x:X)} Y^{P(x)} \end{array}$$

in which all but the top map are known to be equivalences. Therefore the top map is an equivalence, so the claim follows by Proposition 5.1.12. \square

Corollary 5.1.17 (Theorem 1.24 of [28]). *For any family $P : X \rightarrow \mathcal{U}$, the map*

$$\lambda(x, y). \eta(x, \eta(y)) : \left(\sum_{(x:X)} P(x) \right) \rightarrow L \left(\sum_{(x:X)} LP(x) \right)$$

is a localization.

Lemma 5.1.18. *For any two maps $f, g : LX \rightarrow Y$ into an L -local type Y , the map*

$$(f \sim g) \rightarrow (f \circ \eta \sim g \circ \eta)$$

given by $H \mapsto H \cdot \eta$, is an equivalence.

Proof. The square

$$\begin{array}{ccc} (f = g) & \xrightarrow{\text{ap}_{\eta^*}} & (f \circ \eta = g \circ \eta) \\ \text{htpy_eq} \downarrow & & \downarrow \text{htpy_eq} \\ (f \sim g) & \xrightarrow{\lambda H. H \cdot \eta} & (f \circ \eta \sim g \circ \eta) \end{array}$$

commutes, and all but one of the maps are known equivalences. Therefore it follows that the bottom map is an equivalence, as claimed. \square

Proposition 5.1.19. *Consider a pullback square*

$$\begin{array}{ccc} C & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

with $H : f \circ p \sim g \circ q$. If A , B , and X are L -local types, then so is C .

Proof. We have the commuting square

$$\begin{array}{ccc} (LC \rightarrow C) & \xrightarrow{\eta^*} & (C \rightarrow C) \\ \text{cone_map} \downarrow & & \downarrow \text{cone_map} \\ \text{cone}(LC) & \xrightarrow{\quad \quad \quad} & \text{cone}(C) \end{array}$$

where the bottom map is the equivalence given by $(\tilde{p}, \tilde{q}, \tilde{H}) \mapsto (\tilde{p} \circ \eta, \tilde{q} \circ \eta, \tilde{H} \cdot \eta)$. This is an equivalence by the assumption that A , B , and X are local, and an application of Lemma 5.1.18. The two vertical maps are equivalences by the assumption that C is a pullback. Therefore it follows that the top map is an equivalence. By Proposition 5.1.3(iii) this suffices to show that C is L -local. \square

Corollary 5.1.20. *Cartesian products of L -local types are L -local.*

Proof. Suppose that X and Y are L -local. Then their cartesian product is a pullback

$$\begin{array}{ccc} X \times Y & \xrightarrow{\text{pr}_2} & Y \\ \text{pr}_1 \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{1}, \end{array}$$

Since the unit type is L -local for any reflective subuniverse by Corollary 5.1.4, the claim follows. \square

Corollary 5.1.21. *If X is L -local, then so is the identity type $x = y$ for any $x, y : X$.*

Proof. This follows at once from the pullback square

$$\begin{array}{ccc} (x = y) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \text{const}_y \\ \mathbf{1} & \xrightarrow{\text{const}_x} & X, \end{array}$$

noting that the unit type is L -local for any reflective subuniverse by Corollary 5.1.4, and X is L -local by assumption. \square

Proposition 5.1.22. *Given a reflective subuniverse, if $P(x)$ is L -local for every $x : X$, then so is $\prod_{(x:X)} P(x)$. In particular, the type Y^X is L -local whenever Y is L -local.*

Proof. Consider the commuting square

$$\begin{array}{ccc} \left(L \left(\prod_{(y:X)} P(y) \right) \rightarrow \prod_{(x:X)} P(x) \right) & \xrightarrow{\eta^*} & \left(\left(\prod_{(y:X)} P(y) \right) \rightarrow \prod_{(x:X)} P(x) \right) \\ \text{swap} \downarrow & & \downarrow \text{swap} \\ \left(\prod_{(x:X)} L \left(\prod_{(y:X)} P(y) \right) \rightarrow P(x) \right) & \xrightarrow{\cong} & \left(\prod_{(x:X)} \left(\prod_{(y:X)} P(y) \right) \rightarrow P(x) \right) \end{array}$$

The vertical maps swap the order of the arguments, and are therefore equivalences. The bottom map is an equivalence by the assumption that each $P(x)$ is L -local. By Proposition 5.1.3(iii) this suffices to show that $\prod_{(x:X)} P(x)$ is L -local. \square

Corollary 5.1.23. *For any two L -local types X and Y , the type of equivalences $X \simeq Y$ is again L -local.*

Proof. The type $X \simeq Y$ is equivalent to the pullback

$$\begin{array}{ccc} (X \simeq Y) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \text{const}_{(\text{id}_X, \text{id}_Y)} \\ Y^X \times X^Y \times X^Y & \xrightarrow{(f,g,h) \mapsto (hf, fg)} & X^X \times Y^Y. \end{array}$$

of L -local types, so it is L -local. \square

Remark 5.1.24. Similarly it follows that $\text{is_trunc}_k(X)$ is L -local for any L -local type X , and $\text{is_trunc}_k(f)$ is L -local for any map $f : X \rightarrow Y$ between L -local types.

Proposition 5.1.25. *For any two types X and Y , the map*

$$\eta \times \eta : X \times Y \rightarrow LX \times LY$$

is an L -localization. Thus L -localization preserves finite products, for any reflective subuniverse L .

Proof. First we note that the product $LX \times LY$ is indeed L -local by Corollary 5.1.20. To see that $\eta \times \eta$ is an L -localization, consider an L -local type Z . Then we have the commuting square

$$\begin{array}{ccc} (LX \times LY \rightarrow Z) & \xrightarrow{(\eta \times \eta)^*} & (X \times Y \rightarrow Z) \\ \text{ev_pair} \downarrow & & \downarrow \text{ev_pair} \\ (LX \rightarrow (LY \rightarrow Z)) & \xrightarrow{\lambda f. \lambda x. \lambda y. f(\eta(x), \eta(y))} & (X \rightarrow (Y \rightarrow Z)). \end{array}$$

The bottom map is an equivalence by the fact that Z and $LY \rightarrow Z$ are both L -local types, and the vertical maps are equivalences too. Therefore $\eta \times \eta$ is an L -localization. \square

Corollary 5.1.26. *Given any reflective subuniverse, the modal operator preserves propositions.*

Proof. A type P is a proposition if and only if the diagonal $P \rightarrow P \times P$ is an equivalence. The result then follows from Proposition 5.1.25. \square

By contrast, localizations, and even modalities, do not generally preserve n -types for any $n \geq 0$. For instance, the “shape” modality of [30] takes the topological circle, which is a 0-type, to the homotopical circle, which is a 1-type, and the topological 2-sphere, which is also a 0-type, to the homotopical 2-sphere, which is (conjecturally) not an n -type for any finite n .

5.2 The reflective subuniverse of separated types

Definition 5.2.1. Consider a subuniverse $P : \mathcal{U} \rightarrow \text{Prop}$. We say that a type X is **P -separated** if the identity types of X are P -types. We write P' for the subuniverse of P -separated types.

Example 5.2.2. We define $\text{is_trunc} : \mathbb{Z}_{\geq -2} \rightarrow \mathcal{U} \rightarrow \mathcal{U}$ by induction on $k : \mathbb{Z}_{\geq -2}$, taking

$$\begin{aligned} \text{is_trunc}_{-2}(A) &::= \text{is_contr}(A) \\ \text{is_trunc}_{k+1}(A) &::= \prod_{(x,y:A)} \text{is_trunc}_k(x = y). \end{aligned}$$

For any type A , we say that A is **k -truncated**, or a **k -type**, if there is a term of type $\text{is_trunc}_k(A)$. We say that a map $f : A \rightarrow B$ is k -truncated if its fibers are k -truncated.

In other words, the subuniverse of $(n + 1)$ -truncated types is precisely the subuniverse of **n -separated types**, i.e. the subuniverse of types whose identity types are n -truncated.

Definition 5.2.3. Let L be a reflective subuniverse and let $X : \mathcal{U}$ be a type. An L' -**localization** of a type X is a localization with respect to the subuniverse of L -separated types.

In other words, a type X is L -separated if its diagonal $\Delta : X \rightarrow X \times X$ is classified by \mathcal{U}_L .

Example 5.2.4. Given $n \geq -2$, the subuniverse of $(n + 1)$ -truncated types is precisely the subuniverse of separated types for the reflective subuniverse of n -truncated types.

Lemma 5.2.5. *Any L -local type is L -separated.*

Proof. Immediate by Corollary 5.1.21. □

Lemma 5.2.6. *Any small subtype of an L -separated type is again L -separated. In particular, any small proposition is L -separated.*

The following lemma can be proven directly. However, it also follows once we have shown that the subuniverse of L -separated types is a reflective subuniverse, so we will omit the proof.

Lemma 5.2.7. *The subuniverse of L -separated types is closed under pullbacks, retracts, and dependent products of families of L -separated types.*

Remark 5.2.8 (Move to modalities). If L is closed under dependent sums, then L' is also closed under dependent sums, by the characterization of identity types of dependent sums [33, Theorem 2.7.2]. So, given that separated types form a reflective subuniverse, it will follow that if L is a modality, then so is L' .

Proposition 5.2.9. *If X is an L -separated type and $P : X \rightarrow \mathcal{U}_L$ is a family of L -local types, then the type $\sum_{(x:X)} P(x)$ is L -separated.*

Proof. For any (x, p) and (y, q) in $\sum_{(x:X)} P(x)$, the type $(x, p) = (y, q)$ is equivalent to the pullback

$$\begin{array}{ccc} (x, p) = (y, q) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow q \\ (x = y) & \xrightarrow{\text{transport}^P(-, p)} & P(y) \end{array}$$

of L -local types, so it is L -local. □

Corollary 5.2.10. *Suppose $l' : X \rightarrow Y'$ is an L' -localization, and let $P : L'X \rightarrow \mathcal{U}_L$ be a family of L -local types. Then the precomposition map*

$$l'^* : \prod_{(y':Y')} P(y') \simeq \prod_{(x:X)} P(l'(x)).$$

is an equivalence.

Proof. This follows immediately from Proposition 5.1.5. □

Proposition 5.2.11. *Any small subtype of the subuniverse \mathcal{U}_L is L -separated.*

Proof. Note that for any two L -local types A and B we have $(A = B) \simeq (A \simeq B)$ by univalence and the fact that being L -local is a mere proposition. Therefore the claim follows from Corollary 5.1.23. \square

The only thing that prevents \mathcal{U}_L from actually being L -separated is the fact that \mathcal{U}_L is not small. In other words, we could say that \mathcal{U}_L is **essentially L -separated**. Using the fact that the image of a small type into \mathcal{U}_L is essentially small, the condition of being essentially L -separated suffices to eliminate from $L'X$ into \mathcal{U}_L .

Lemma 5.2.12. *Consider an L' -localization $l' : X \rightarrow Y'$. Then any type family $P : X \rightarrow \mathcal{U}_L$ of L -local types has a unique extension along l'*

$$\begin{array}{ccc} X & \xrightarrow{P} & \mathcal{U}_L \\ l' \downarrow & \nearrow & \\ Y' & & \end{array}$$

Proof. We prove the first form of the statement. By Proposition 5.2.11, the identity types of \mathcal{U}_L are equivalent to small types, i.e., \mathcal{U}_L is a locally small type. By the join construction [27], the image of P can be taken to be a small type I in \mathcal{U} , so there is a factorization of P into a surjection $\hat{P} : X \rightarrow I$ followed by an embedding $i : I \rightarrow \mathcal{U}_L$:

$$\begin{array}{ccc} X & \xrightarrow{P} & \mathcal{U}_L \\ \eta' \downarrow & \searrow \hat{P} & \uparrow i \\ L'X & & I \end{array}$$

Since the identity types of I are equivalent to identity types of \mathcal{U}_L , and I is small, it follows that the identity types of I are actually L -local. This means that I is an L -separated type, so we can extend \hat{P} to $L'X$ giving us the desired extension of P by composing with i .

Since $X \rightarrow L'X$ is surjective (Theorem 5.2.14), any such extension must factor through the image I . So uniqueness follows from the universal property of L' -localization. \square

Before we show that L' -localizations exist for any type X , we characterize them. To establish our characterization of L' -localizations, we need the following simple lemma, that allows us to construct unique extensions.

Lemma 5.2.13. *Let $g : A \rightarrow B$ and $f : A \rightarrow C$ be maps for which we have a unique extension*

$$\begin{array}{ccc} \text{fib}_g(b) & \xrightarrow{f \circ \text{pr}_1} & C \\ \downarrow & \nearrow & \\ \mathbf{1} & & \end{array}$$

for every $b : B$. Then f extends uniquely along g .

Proof. By assumption we have

$$\prod_{(b:B)} \text{is_contr} \left(\sum_{(c:C)} \prod_{(a:A)} \prod_{(p:g(a)=b)} f(a) = c \right).$$

The center of contraction gives us an extension

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ g \downarrow & \nearrow \tilde{f} & \\ B & & \end{array}$$

and its uniqueness follows from the contraction. \square

Theorem 5.2.14. *Consider a map $l' : X \rightarrow Y'$, where Y' is assumed to be L -separated. Then the following are equivalent:*

- (i) *The map $l' : X \rightarrow Y'$ is an L' -localization.*
- (ii) *The map $l' : X \rightarrow Y'$ is surjective, and for each $x, y : X$, the map*

$$\text{ap}_{l'} : (x = y) \rightarrow (l'(x) = l'(y))$$

is an L -localization.

Proof. First, suppose that $l' : X \rightarrow Y'$ is an L' -localization. To see that l' is surjective, we note that $\text{im}(l')$ is L -separated since it is a subtype of Y' , so the surjective map $q : X \rightarrow \text{im}(l')$ extends uniquely along l' .

$$\begin{array}{ccc} X & \xrightarrow{q} & \text{im}(l') \\ l' \downarrow & \nearrow h & \\ Y' & & \end{array}$$

By the universal property of L' -localization it follows that h is a section of the image inclusion $\text{im}(l') \rightarrow Y'$. In particular, the image inclusion is both surjective and an embedding, so it must be an equivalence. It follows that l' is surjective.

Next, we need to show that for each $x, y : X$, the map

$$\text{ap}_{l'} : (x = y) \rightarrow (l'(x) = l'(y))$$

is an L -localization. Fix $x : X$. Since l' is an L' -localization, there is a unique extension

$$\begin{array}{ccc} X & \xrightarrow{y \mapsto L(x=y)} & \mathcal{U}_L \\ l' \downarrow & \nearrow P & \\ Y' & & \end{array}$$

The family P comes equipped with a point $p_0 : P(l'(x))$ that is induced by $\eta(\text{refl}_x) : L(x = x)$. Moreover, by the fact that P extends $y \mapsto L(x = y)$ we have a pullback square

$$\begin{array}{ccc} \sum_{(y:X)} L(x = y) & \longrightarrow & \sum_{(y':Y')} P(y') \\ \downarrow & & \downarrow \\ X & \xrightarrow{l'} & Y' \end{array}$$

Thus we see that our claim follows by Theorem 1.2.5, once we show that the total space of P is contractible.

For the center of contraction of $\sum_{(y:L'X)} P(y)$ we take $(l'(x), \eta(\text{refl}_x))$. It remains to construct a contraction

$$\prod_{(y':Y')} \prod_{(p:P(y'))} (l'(x), \eta(\text{refl}_x)) = (y', p).$$

Since the fibers of P are L -local, it follows by Proposition 5.2.9 that the total space of P is L -separated. Therefore we obtain by Proposition 5.1.22 that the type

$$\prod_{(p:P(y'))} (l'(x), \eta(\text{refl}_x)) = (y', p)$$

is L -local for every $y' : Y'$. Thus Corollary 5.2.10 reduces the problem to constructing a term of type

$$\prod_{(y:X)} \prod_{(p:L(x=y))} (l'(x), \eta(\text{refl}_x)) = (l'(y), p).$$

Furthermore, for $y : X$ we have equivalences

$$\begin{aligned} \left(\sum_{(p:L(x=y))} (l'(x), \eta(\text{refl}_x)) = (l'(y), p) \right) &\simeq \left(\sum_{(p:L(x=y))} \sum_{(\alpha:l'(x)=l'(y))} \alpha_* (\eta(\text{refl}_x)) = p \right) \\ &\simeq \sum_{(\alpha:l'(x)=l'(y))} \mathbf{1} \\ &\simeq l'(x) = l'(y), \end{aligned}$$

where the last type is clearly L -local. So we can apply Proposition 5.1.5 to reduce the problem to the problem of constructing a term of type

$$\prod_{(y:X)} \prod_{(p:x=y)} (l'(x), \eta(\text{refl}_x)) = (l'(y), \eta(p)).$$

This can be done by a simple application of path induction. This completes the proof that (i) implies (ii).

To show that (ii) implies (i), assume that l' is surjective, and that for every $x, y : X$ the map

$$\text{ap}_{l'} : (x = y) \rightarrow (l'(x) = l'(y))$$

is an L -localization. Our goal is to show that l' satisfies the universal property of L' -localization, so assume $f : X \rightarrow Z$ is a map into an L -separated type Z .

By Lemma 5.2.13, it is enough to show that f restricts to a unique constant map on the fibers of l' . This means that we must show that

$$\sum_{(z:Z)} \prod_{(x:X)} (l'(x) = y') \longrightarrow (f(x) = z)$$

is contractible for every $y' : Y'$. Since this is a mere proposition, and l' is surjective, we can assume that $y' = l'(y)$. In other words, it suffices to show that

$$\sum_{(z:Z)} \prod_{(x:X)} (l'(x) = l'(y)) \longrightarrow (f(x) = z)$$

is contractible for every $y : X$.

Since Z is assumed to be L -separated and $\text{ap}_{l'}$ is assumed to be an L -localization, this type is equivalent to

$$\sum_{(z:Z)} \prod_{(x:X)} (x = y) \longrightarrow (f(x) = z)$$

and it is easy to see that this is a contractible type by applying the contractibility of the total space of the path fibration twice. \square

Our final goal for this section is to show that L' is a reflective subuniverse, i.e. that there is an L' -localization for every type $X : \mathcal{U}$. We will use a ‘local version’ of the type theoretic Yoneda Lemma.

Lemma 5.2.15. *For each $y : X$ and each $P : X \rightarrow \mathcal{U}_L$, the map*

$$\text{ev_locrefl} : \left(\prod_{(z:X)} L(y = z) \rightarrow P(z) \right) \longrightarrow P(y)$$

given by $f \mapsto f(y, \eta(\text{refl}_y))$ is an equivalence.

Proof. By the universal property of $L(y = z)$ and identity elimination, the map in the statement can be factored as follows:

$$\begin{aligned} \left(\prod_{(z:X)} L(y = z) \rightarrow P(z) \right) &\simeq \left(\prod_{(z:X)} (y = z) \rightarrow P(z) \right) \\ &\simeq P(y). \end{aligned} \quad \square$$

Theorem 5.2.16. *For any reflective subuniverse L , the subuniverse of L -separated types is again reflective. We will write*

$$\eta' : X \rightarrow L'X$$

for the L' -localization of a type X .

Proof. Fix a type $X : \mathcal{U}$. Let $\mathcal{Y}_L : X \rightarrow (X \rightarrow \mathcal{U})$ be given by

$$\mathcal{Y}_L(x, y) := L(x = y).$$

We would like to define $L'X$ to be $\text{im}(\mathcal{Y}_L)$, but this is a subtype of $X \rightarrow \mathcal{U}$, so it is not small (i.e., it does not live in \mathcal{U}). However, since \mathcal{U} is locally small, so is $X \rightarrow \mathcal{U}$. Thus the join construction [27] implies that the image is equivalent to a small type which we denote $L'X$. This comes equipped with a surjective map

$$\eta' : X \longrightarrow L'X,$$

which we take to be the unit of the reflective subuniverse.

To show that η' is a localization, we apply Theorem 5.2.14. First we show that $L'X$ is L -separated. Since η' is surjective and being L -local is a proposition, it is enough to show that $\eta'(x) = \eta'(y)$ is L -local for x and y in X . Since $L'X$ embeds in $X \rightarrow \mathcal{U}$, we have an equivalence between $\eta'(x) = \eta'(y)$ and $(\lambda z. L(x = z)) = (\lambda z. L(y = z))$. The latter is equivalent to $\prod_{(z:X)} L(x = z) = L(y = z)$, which is L -local by Proposition 5.2.11.

It remains to show that the canonical map $L(x = y) \rightarrow (\eta'(x) = \eta'(y))$ is an equivalence. By the above argument, combined with univalence, the problem reduces to showing that the canonical map

$$L(x = y) \longrightarrow \left(\prod_{(z:X)} L(x = z) \simeq L(y = z) \right)$$

is an equivalence. Using symmetry of equivalences, it suffices to show that the map

$$L(x = y) \longrightarrow \left(\prod_{(z:X)} L(y = z) \simeq L(x = z) \right)$$

is an equivalence. Moreover, since the forgetful map from equivalences to maps is an embedding, it is enough to show that the composite map

$$\alpha_{x,y} : L(x = y) \longrightarrow \left(\prod_{(z:X)} L(y = z) \rightarrow L(x = z) \right)$$

is an equivalence. Indeed, if i is an embedding and $i \circ g$ is an equivalence, then i is surjective. Therefore i is an equivalence and hence so is g .

By the local Yoneda Lemma 5.2.15, with $P(z) := L(x = z)$, there is an equivalence

$$\beta_{x,y} : L(x = y) \longrightarrow \left(\prod_{(z:X)} L(y = z) \rightarrow L(x = z) \right)$$

which sends $p : L(x = y)$ to the unique function f such that $f(x, \eta(\text{refl}_x)) = p$. So it suffices to show that $\alpha_{x,y} = \beta_{x,y}$. By the universal property of $L(x = y)$, it is enough to show that $\alpha_{x,y} \circ \eta = \beta_{x,y} \circ \eta$ as maps $(x = y) \rightarrow \left(\prod_{(z:X)} L(y = z) \rightarrow L(x = z) \right)$. Letting x and y vary and using path induction, we reduce the problem to showing that $\alpha_{x,x}(\eta(\text{refl}_x)) = \beta_{x,x}(\eta(\text{refl}_x))$.

Since $\alpha_{x,y}$ is defined by path induction, it is easy to see that $\alpha_{x,x}(\eta(\text{refl}_x))$ is equal to $\lambda z. \text{id}_{L(x=z)}$. On the other hand, $\beta_{x,x}(\eta(\text{refl}_x))$ is the unique function f such that $f(x, \eta(\text{refl}_x)) = \eta(\text{refl}_x)$. Therefore, this f must also equal $\lambda z. \text{id}_{L(x=z)}$, showing that $\alpha_{x,x}(\eta(\text{refl}_x)) = \beta_{x,x}(\eta(\text{refl}_x))$. \square

Theorem 5.2.17. *For each $k \geq -2$, the subuniverse of k -truncated types is reflective.*

Proof. The subuniverse of contractible types is obviously reflective, it's localization is just the constant function $\lambda X. \mathbf{1}$ mapping every type to the unit type. Since the subuniverse of $(k + 1)$ -truncated types is precisely the subuniverse of k -separated types, the claim follows inductively by Theorem 5.2.16. \square

5.3 L -local maps

Definition 5.3.1. A map $f : A \rightarrow B$ is said to be **L -local** if its fibers are L -local.

Theorem 5.3.2. *Let B be a type family over A . Then the following are equivalent:*

(i) *For each $x : A$ the type $B(x)$ is L -local.*

(ii) *The projection map*

$$\text{pr}_1 : \left(\sum_{(x:A)} B(x) \right) \rightarrow A$$

is an L -local map.

Proof. Immediate from the equivalences $B(x) \simeq \text{fib}_{\text{pr}_1}(x)$. \square

Theorem 5.3.3. *Let $f : A \rightarrow B$ be a map. The following are equivalent:*

(i) *The map f is L' -local.*

(ii) *For each $x, y : A$, the map*

$$\text{ap}_f : (x = y) \rightarrow (f(x) = f(y))$$

is L -local.

(iii) The diagonal $\delta_f : A \rightarrow A \times_B A$ of f is L -local.

Proof. First we show that for any $s, t : \text{fib}_f(b)$ there is an equivalence

$$(s = t) \simeq \text{fib}_{\text{ap}_f}(\text{pr}_2(s) \cdot \text{pr}_2(t)^{-1})$$

We do this by Σ -induction on s and t , and then we calculate using basic manipulations of identifications that

$$\begin{aligned} ((x, p) = (y, q)) &\simeq \sum_{(r:x=y)} \text{tr}_{f(-)=b}(r, p) = q \\ &\simeq \sum_{(r:x=y)} \text{ap}_f(r)^{-1} \cdot p = q \\ &\simeq \sum_{(r:x=y)} \text{ap}_f(r) = p \cdot q^{-1} \\ &\equiv \text{fib}_{\text{ap}_f}(p \cdot q^{-1}). \end{aligned}$$

By these equivalences, it follows that if ap_f is L -local, then for each $s, t : \text{fib}_f(b)$ the identity type $s = t$ is an L -local type.

For the converse, note that we have equivalences

$$\text{fib}_{\text{ap}_f}(p) \simeq ((x, p) = (y, \text{refl}_{f(y)})).$$

Therefore it follows that if f is L' -local, then the identity type $(x, p) = (y, \text{refl}_{f(y)})$ in $\text{fib}_f(f(y))$ is L -local for any $p : f(x) = f(y)$, and therefore $\text{fib}_{\text{ap}_f}(p)$ is L -local. \square

Theorem 5.3.4. *Let $f : \prod_{(x:A)} B(x) \rightarrow C(x)$ be a fiberwise transformation. Then the following are equivalent:*

(i) For each $x : A$ the map $f(x)$ is L -local.

(ii) The induced map

$$\text{tot}(f) : \left(\sum_{(x:A)} B(x) \right) \rightarrow \left(\sum_{(x:A)} C(x) \right)$$

is L -local.

Proof. This follows directly from Lemma 1.2.2. \square

5.4 Quasi-left-exactness of L' -localization

We now explain how L and L' together behave similarly to a **lex modality**, i.e., a modality that preserves pullbacks. Theorem 3.1 of [28] gives 13 equivalent characterizations of a lex modality, and it turns out that these hold for any reflective subuniverse if the modal operator is replaced by L and L' in the appropriate way. The propositions in this section show this for parts (ix), (x), (xii) and (xi) of Theorem 3.1, respectively. The proofs use the dependent elimination of L' in a crucial way, but do not use the specific construction of L' -localization, just the existence.

Before proving the next result, we need a lemma, which follows directly from the dependent elimination of L' .

Lemma 5.4.1. *Let $P : L'X \rightarrow \mathcal{U}$ be a type family over $L'X$. Then the map*

$$f : \left(\sum_{(x:X)} P(\eta'(x)) \right) \rightarrow \left(\sum_{(y:L'X)} P(y) \right)$$

given by $(x, p) \mapsto (\eta'(x), p)$ is an L -equivalence.

Proof. For any L -local type Z we have the commuting square

$$\begin{array}{ccc} \left(\left(\sum_{(y:L'X)} P(y) \right) \rightarrow Z \right) & \xrightarrow{\lambda f. \lambda(x,p). f(\eta'(x), p)} & \left(\sum_{(x:X)} P(\eta'x) \right) \rightarrow Z \\ \text{ev_pair} \downarrow & & \downarrow \text{ev_pair} \\ \prod_{(y:L'X)} (P(y) \rightarrow Z) & \xrightarrow{\lambda f. \lambda x. \lambda y. f(\eta'(x), y)} & \prod_{(x:X)} (P(\eta'x) \rightarrow Z) \end{array}$$

The bottom map is an equivalence by Corollary 5.2.10, using that Z and $P(y) \rightarrow Z$ are L -local. The two vertical maps are also equivalences, so it follows that the top map is an equivalence. The claim now follows from Proposition 5.1.12. \square

Proposition 5.4.2. *Consider a commuting cube of the form*

$$\begin{array}{ccccc} & & A \times_X B & & \\ & \swarrow & \vdots & \searrow & \\ A & & L'A \times_{L'X} L'B & & B \\ \eta' \downarrow & \swarrow & \downarrow & \searrow & \downarrow \eta' \\ L'A & & X & & L'B \\ & \swarrow & \eta' \downarrow & \searrow & \\ & & L'X & & \end{array}$$

Then the map $A \times_X B \rightarrow L'A \times_{L'X} L'B$ is an L -equivalence.

Proof. Consider the following commuting square

$$\begin{array}{ccc} \sum_{(x:A)} \sum_{(y:B)} f(x) = g(y) & \dashrightarrow & \sum_{(x':L'A)} \sum_{(y':L'B)} L'f(x') = L'g(y') \\ \downarrow & & \uparrow \\ \sum_{(x:A)} \sum_{(y:B)} \eta'(f(x)) = \eta'(g(y)) & \longrightarrow & \sum_{(x:A)} \sum_{(y:B)} L'f(\eta'(x)) = L'g(\eta'(y)) \end{array}$$

In this square, the downwards morphism on the left is the induced map on total spaces of the map $\text{ap}_{\eta'} : (f(x) = g(y)) \rightarrow (\eta'(f(x)) = \eta'(g(y)))$, which is an L -equivalence by Theorem 5.2.14 and Proposition 5.1.16. The bottom map is an equivalence, obtained from the naturality squares $\eta' \circ f \sim L'f \circ \eta'$ and $\eta' \circ g \sim L'g \circ \eta'$. In particular, it is an L -equivalence. The upwards map on the right is an L -equivalence by Lemma 5.4.1. Therefore, the asserted map is a composite of L -equivalences, so it is also an L -equivalence. \square

As a consequence we get a result about the preservation of certain fiber sequences.

Corollary 5.4.3. *Given a fiber sequence $F \hookrightarrow E \twoheadrightarrow B$, there is a map of fiber sequences*

$$\begin{array}{ccccc} F & \hookrightarrow & E & \xrightarrow{p} \twoheadrightarrow & B \\ \downarrow & & \eta' \downarrow & & \downarrow \eta' \\ F' & \hookrightarrow & L'E & \xrightarrow{L'p} \twoheadrightarrow & L'B \end{array}$$

in which the left vertical map is an L -equivalence. □

5.5 Connected maps

Definition 5.5.1. A map $f : A \rightarrow B$ is said to be **L -connected** if $L(\text{fib}_f(b))$ is contractible for every $b : B$. In particular, a type A is L -connected if and only if LA is contractible.

In the following proposition we characterize L' -connected types. We will establish a similar claim for maps in Corollary 5.5.12

Proposition 5.5.2. *A type A is L' -connected if and only if A is merely inhabited (i.e. $\|A\|$ holds), and the identity types of A are L -connected.*

Proof. A is L' -connected if and only if $A \rightarrow \mathbf{1}$ is an L' -localization. By Theorem 5.2.14 this holds if and only if $A \rightarrow \mathbf{1}$ is surjective, and the maps $(x = y) \rightarrow \Omega(\mathbf{1})$ are L -localizations — in other words: if and only if A is merely inhabited and the identity types of A are L -connected. □

In the following proposition we provide two equivalent conditions to a map being L -connected.

Proposition 5.5.3. *Consider a map $f : A \rightarrow B$. The following are equivalent:*

(i) f is L -connected.

(ii) For every family $P : B \rightarrow \mathcal{U}_L$ of L -local types, the map

$$\lambda s. s \circ f : \left(\prod_{(b:B)} P(b) \right) \rightarrow \left(\prod_{(a:A)} P(f(a)) \right).$$

is an equivalence.

(iii) For every family $P : B \rightarrow \mathcal{U}_L$ of L -local types, the map

$$\lambda s. s \circ f : \left(\prod_{(b:B)} P(b) \right) \rightarrow \left(\prod_{(a:A)} P(f(a)) \right).$$

has a section.

Proof. First suppose f is L -connected and let $P : B \rightarrow \mathcal{U}_L$. Then the map

$$\lambda p. \text{const}_p : P(b) \rightarrow (\text{fib}_f(b) \rightarrow P(b))$$

is an equivalence for every $b : B$, since $L(\text{fib}_f(b))$ is assumed to be contractible. Therefore we obtain a commuting square

$$\begin{array}{ccc} \prod_{(b:B)} P(b) & \xrightarrow{\quad\quad\quad} & \prod_{(a:A)} P(f(a)) \\ \downarrow & & \uparrow \text{ev_refl} \\ \prod_{(b:B)} (\text{fib}_f(b) \rightarrow P(b)) & \xrightarrow{\text{evpair}} & \prod_{(b:B)} \prod_{(a:A)} (f(a) = b) \rightarrow P(b) \end{array}$$

in which three out of four maps are known equivalences. The remaining map must therefore also be an equivalence.

Thus, (i) \Rightarrow (ii), and clearly (ii) \Rightarrow (iii). To show (iii) \Rightarrow (i), let $P(b) := L(\text{fib}_f(b))$. Then (iii) yields a map $c : \prod_{(b:B)} L(\text{fib}_f(b))$ with $c(f(a)) = \eta(a, \text{refl}_{f(a)})$. To show that each $L(\text{fib}_f(b))$ is contractible, we will show that $c(b) = w$ for any $b : B$ and $w : L(\text{fib}_f(b))$. In other words, we must show that the identity function $L(\text{fib}_f(b)) \rightarrow L(\text{fib}_f(b))$ is equal to the constant function at $c(b)$. By the universal property of $L(\text{fib}_f(b))$, it suffices to show that they become equal when precomposed with $\eta_{\text{fib}_f(b)}$, i.e. we may assume that $w = \eta(a, p)$ for some $a : A$ and $p : f(a) = b$. But now path induction on p reduces our goal to the given $c(f(a)) = \eta(a, \text{refl}_{f(a)})$. \square

Corollary 5.5.4. *A type A is L -connected if and only if the “constant functions” map $B \rightarrow (A \rightarrow B)$ is an equivalence for every modal type B .* \square

Dually, we will prove in Corollary 5.6.29 that when L is a modality, if this holds for all L -connected A then B is L -local.

Corollary 5.5.5. *If $f : A \rightarrow B$ is an L -connected map into an L -local type B , then f is an L -localization. (The converse only holds when L is a modality.)*

Proposition 5.5.6. *Consider a map $f : A \rightarrow B$.*

- (i) *If f is an L' -equivalence, then f is L -connected.*
- (ii) *If f is L -connected, then f is an L -equivalence.*

Proof. The first statement is a direct consequence of Corollary 5.4.3, since we get for each $b : B$ a map of fiber sequences

$$\begin{array}{ccccc} \text{fib}_f(b) & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow & & \eta' \downarrow & & \downarrow \eta' \\ \text{fib}_{L'f}(\eta'b) & \longrightarrow & L'A & \xrightarrow{L'f} & L'B \end{array}$$

in which the induced map $\text{fib}_f(b) \rightarrow \text{fib}_{L'f}(\eta'b)$ is an L -equivalence. Since $L'f$ is an equivalence, it follows that $\text{fib}_{L'f}(\eta'b)$ is contractible, so we conclude that $\text{fib}_f(b)$ is L -connected. It follows that f is L -connected.

For the second statement, we observe that if f is L -connected, then by Proposition 5.5.3 it induces an equivalence

$$f^* : (B \rightarrow X) \rightarrow (A \rightarrow X)$$

for every L -local type X . Therefore we conclude by Proposition 5.1.12 that f is an L -equivalence. \square

Proposition 5.5.7. *Consider a commuting triangle*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & X \end{array}$$

(i) *If h is L -connected, then the following are equivalent:*

- a) *f is L -connected.*
- b) *g is L -connected.*

(ii) *If f is L' -connected, then the following are equivalent:*

- a) *g is L' -connected.*
- b) *h is L -connected.*

In particular, g is $(n+1)$ -connected if and only if h is n -connected, assuming that f is $(n+1)$ -connected.

Proof. For the first statement suppose that h is L -connected. For any $z : C$ we have

$$\begin{aligned} L(\mathbf{fib}_{g \circ h}(z)) &\simeq L(\sum_{(p:\mathbf{fib}_g(z))} \mathbf{fib}_h(\mathbf{pr}_1(p))) \\ &\simeq L(\sum_{(p:\mathbf{fib}_g(z))} L(\mathbf{fib}_h(\mathbf{pr}_1(p)))) \\ &\simeq L(\sum_{(p:\mathbf{fib}_g(z))} \mathbf{1}) \\ &\simeq L(\mathbf{fib}_g(z)). \end{aligned}$$

using the fact that h is L -connected. Thus, one is contractible if and only if the other is. We conclude that f is L -connected if and only if g is.

For the second statement, suppose that f is L' -connected. If g is L' -connected, then both $g \circ h$ and g are L' -equivalences, and thus h is an L' -equivalence. Then Proposition 5.5.6 implies that h is L -connected.

For the converse, notice that taking fibers over each $x : X$ reduces the problem to showing that given an L -connected map $h : A \rightarrow B$ such that A is L' -connected, it follows that B is L' -connected.

Since $L'A$ is contractible, it follows that $L'B$ is contractible if and only if $L'h : L'A \rightarrow L'B$ is an equivalence. Therefore it suffices to show that the fibers of $L'h$ are contractible. Moreover, since $\eta' : B \rightarrow L'B$ is a surjective map by Theorem 5.2.14, it is enough to show that $\mathbf{fib}_{L'h}(\eta'(b))$ is contractible for every $b : B$. First we observe that, $\mathbf{fib}_{L'h}(\eta'(b)) \simeq \Omega(L'B, \eta'(b))$ since $L'A$ is contractible. Now we observe using Corollary 5.4.3 that we have for every $b : B$ a morphism of fiber sequences

$$\begin{array}{ccccc} \mathbf{fib}_h(b) & \hookrightarrow & A & \xrightarrow{h} & B \\ \downarrow & & \eta' \downarrow & & \downarrow \eta' \\ \Omega(L'B, \eta'(b)) & \hookrightarrow & L'A & \xrightarrow{L'h} & L'B \end{array}$$

in which the map on the left is an L -equivalence. However, the type $\Omega(L'B, \eta'(b))$ is L -local, so it follows that the map

$$\text{fib}_h(b) \rightarrow \Omega(L'B, \eta(b))$$

is an L -localization. Now it follows by our assumption that h is L -connected that $\Omega(L'B, \eta(b))$ is contractible. We conclude that $\text{fib}_{L'h}(\eta'(b))$ is contractible, and therefore that $L'B$ is contractible. \square

Remark 5.5.8. In general it is not true that if g and $g \circ h$ are \circ -connected then h is; this is one of the equivalent characterizations of lex modalities (Theorem 3.1 of [28]).

The above proposition almost gives us a 3-for-2 property that combines L and L' . However the map $\mathbf{0} \rightarrow \mathbf{1}$ is (-2) -connected, and $\mathbf{1}$ is (-1) -connected, whereas $\mathbf{0}$ is not (-1) -connected. So the remaining implication of the 3-for-2 property does not hold. One can show the weaker result that the composite of an L -connected map followed by an L' -connected map is L -connected.

Remark 5.5.9. Theorem 5.2.14 allows us to give a concrete description of the extension defined in Lemma 5.2.12. Using an argument similar to the one used in the proof of Proposition 5.4.2, one can show that given an L' -localization $\eta' : X \rightarrow L'X$ and a map $f : Y \rightarrow X$ with L -local fibers, f is the pullback of the fiberwise L -localization of $\eta' \circ f$.

Definition 5.5.10. A commuting square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{i} & Y \end{array}$$

is said to be **L -cartesian** if its gap map is L -connected.

Proposition 5.5.11. *Consider a commuting square*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{i} & Y. \end{array}$$

(i) *Suppose that g is surjective, and that the square is L -cartesian. Then the following are equivalent:*

- a) *The map $h : A \rightarrow B$ is L -connected.*
- b) *The map $i : X \rightarrow Y$ is L -connected.*

(ii) *Suppose that $i : X \rightarrow Y$ is L' -connected. Then the following are equivalent:*

- a) *The map $h : A \rightarrow B$ is L' -connected.*
- b) *The square is L -cartesian.*

Proof. For the first statement we observe that by Proposition 5.5.7 h is an L -connected if and only if g^*i is an L -connected. Since g is assumed to be surjective, it follows that g^*i is L -connected if and only if i is L -connected.

For the second statement we observe that, since $i : X \rightarrow Y$ is assumed to be L' -connected, the map $g^*i : X \times_Y B \rightarrow B$. Now it follows by Proposition 5.5.7 that the gap map is L -connected if and only if h is L' -connected. \square

Corollary 5.5.12. *A map $f : A \rightarrow B$ is a L' -connected if and only if f is surjective and $\delta_f : A \rightarrow A \times_B A$ is a L -connected.*

Proof. Since every proposition is L' -local, it follows that $\|X\| = \|L'X\|$ for any type X . In particular, if X is L' -connected, then $\|X\| = \mathbf{1}$. From this observation it follows that if f is L' -connected, then f is surjective. Furthermore, since the identity function $\text{id}_A : A \rightarrow A$ is obviously L' -connected, it follows that the square

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is L -cartesian. In other words: $\delta_f : A \rightarrow A \times_B A$ is L -connected.

Now suppose that f is surjective and that δ_f is L -connected. Then the fibers of δ_f are L -connected. Since we have equivalences

$$((x, p) = (y, q)) \simeq \text{fib}_{\delta_f}(x, y, p \cdot q^{-1})$$

for any $(x, p), (y, q) : \text{fib}_f(b)$ it follows by Proposition 5.5.2 that $\text{fib}_f(b)$ is L' -connected for every $b : B$. In other words: f is L' -connected. \square

A commuting square is said to be a **quasi-pullback** if its gap map is surjective. In other words, quasi-pullback squares are the same as (-1) -cartesian squares.

Corollary 5.5.13. *A map $f : A \rightarrow B$ is n -connected if and only if the gap map of the commuting square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & & \downarrow \\ A^{\mathbb{S}^k} & \xrightarrow{f^{\mathbb{S}^k}} & B^{\mathbb{S}^k} \end{array}$$

is a quasi-pullback square, for each $-1 \leq k \leq n$. A map is therefore ∞ -connected if this square is a quasi-pullback for all k

Corollary 5.5.14. *A pointed connected type A is L' -connected if and only if the map $\mathbf{1} \rightarrow A$ is L -connected.*

Proof. If A is L' -connected, then its loop space is L -connected. It follows that the square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & A \end{array}$$

is L -cartesian. Since the map $\mathbf{1} \rightarrow \mathbf{1}$ is obviously L -connected and the map $\mathbf{1} \rightarrow A$ is surjective by the assumption that A is connected, it follows that the map $\mathbf{1} \rightarrow A$ is L -connected.

For the converse, suppose that the map $\mathbf{1} \rightarrow A$ is L -connected. Then the identity types of A are L -connected, since all identity types of A are merely equal to $\Omega(A)$. Therefore it follows by Proposition 5.5.2 that A is L' -connected. \square

Corollary 5.5.15. *Let $P : A \rightarrow \mathcal{U}$ and $Q : B \rightarrow \mathcal{U}$ be type families, and let*

$$g : \prod_{(x:A)} P(x) \rightarrow Q(f(x))$$

be a fiberwise transformation over $f : A \rightarrow B$.

(i) *Suppose that each $g_x : P(x) \rightarrow Q(f(x))$ is L -connected. Then we have the following:*

- a) *If f is L -connected, then $\text{tot}_f(g)$ is L -connected.*
- b) *If each $Q(x)$ is merely inhabited and $\text{tot}_f(g)$ is L -connected, then f is L -connected.*

(ii) *Suppose that f is L' -connected. Then the following are equivalent:*

- a) *Each $g_x : P(x) \rightarrow Q(f(x))$ is L -connected.*
- b) *The map $\text{tot}_f(g)$ is L' -connected.*

Proposition 5.5.16. *Let $P, Q : A \rightarrow \text{Type}$ be type families and $f : \prod_{(a:A)} (P(a) \rightarrow Q(a))$. Then the following are equivalent*

- (i) *Each $f(a) : P(a) \rightarrow Q(a)$ is L -connected.*
- (ii) *The map $\text{tot}(f) : \sum_{(a:A)} P(a) \rightarrow \sum_{(a:A)} Q(a)$ is an L -connected.*

Proof. By Lemma 1.2.2 we have $\text{fib}_{\text{tot}(f)}((x, v)) \simeq \text{fib}_{f(x)}(v)$ for each $x : A$ and $v : Q(x)$. Hence $L(\text{fib}_{\text{tot}(f)}((x, v)))$ is contractible if and only if $L(\text{fib}_{f(x)}(v))$ is contractible. \square

Of course, the “if” direction of Proposition 5.5.16 is a special case of Corollary 5.5.15. This suggests a similar generalization of the “only if” direction of Proposition 5.5.16, which would be a version of Corollary 5.5.15 asserting that if f and φ are L -connected then so is each g_a .

5.6 Modalities

In this section we will introduce the following four notions of modality and prove that they are all equivalent:

- (i) Higher modalities
- (ii) Uniquely eliminating modalities
- (iii) Σ -closed reflective subuniverses
- (iv) Stable orthogonal factorization systems

After their equivalence has been established, we will call all of them simply *modalities*.

The first three definitions have the following data in common: by a **modal operator** we mean a function $\circ : \mathcal{U} \rightarrow \mathcal{U}$, and by a **modal unit** we mean a family of functions $\eta^\circ : \prod_{(A:\mathcal{U})} A \rightarrow \circ A$. Given these data, we say a type X is **modal** if $\eta_X : X \rightarrow \circ X$ is an equivalence, and we write $\mathcal{U}_\circ := \sum_{(X:\mathcal{U})} \text{is_modal}(X)$ for the **subuniverse of modal types**.

Definition 5.6.1. A **higher modality** consists of a modal operator and modal unit together with

- (i) for every $A : \mathcal{U}$ and every dependent type $P : \circ A \rightarrow \mathcal{U}$, a function

$$\text{ind}_{\circ A} : \left(\prod_{(a:A)} \circ(P(\eta(a))) \right) \rightarrow \prod_{(z:\circ A)} \circ(P(z)).$$

- (ii) An identification

$$\text{comp}_{\circ A}(f, x) : \text{ind}_{\circ A}(f)(\eta(x)) = f(x)$$

for each $f : \prod_{(x:A)} \circ(P(\eta(x)))$.

- (iii) For any $x, y : \circ A$ the modal unit $\eta_{(x=y)} : x = y \rightarrow \circ(x = y)$ is an equivalence.

Definition 5.6.2. A **uniquely eliminating modality** consists of a modal operator and modal unit such that the function

$$\lambda f. f \circ \eta_A : \left(\prod_{(x:\circ A)} \circ(P(x)) \right) \rightarrow \left(\prod_{(a:A)} \circ(P(\eta_A(a))) \right)$$

is an equivalence for any A and any $P : \circ A \rightarrow \mathcal{U}$.

Definition 5.6.3. A reflective subuniverse L is said to be **Σ -closed** if $\sum_{(x:X)} P(x)$ is L -local for every family $P : X \rightarrow \mathcal{U}_L$ of L -local types over an L -local type X .

Note that unlike Definitions 5.6.1 and 5.6.2, in Definition 5.6.3 the notion of “modal type” is part of the data. However, we will show in Proposition 5.1.3 that $\text{is_modal}(A)$ is and only if η_A is an equivalence.

Definition 5.6.4. An **orthogonal factorization system** consists of predicates $\mathcal{L}, \mathcal{R} : \prod_{(A,B:\mathcal{U})} (A \rightarrow B) \rightarrow \text{Prop}$ such that

- (i) \mathcal{L} and \mathcal{R} are closed under composition and contain all identities (i.e. they are subcategories of the category of types that contain all the objects), and

(ii) the type $\mathbf{fact}_{\mathcal{L},\mathcal{R}}(f)$ of factorizations

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow f_{\mathcal{L}} & \nearrow f_{\mathcal{R}} \\ & \mathbf{im}_{\mathcal{L},\mathcal{R}}(f) & \end{array}$$

of f , with $f_{\mathcal{L}}$ in \mathcal{L} and $f_{\mathcal{R}}$ in \mathcal{R} , is contractible.

More precisely, the type $\mathbf{fact}_{\mathcal{L},\mathcal{R}}(f)$ is defined to be the type of tuples

$$(\mathbf{im}_{\mathcal{L},\mathcal{R}}(f), (f_{\mathcal{L}}, p), (f_{\mathcal{R}}, q), h)$$

consisting of a type $\mathbf{im}_{\mathcal{L},\mathcal{R}}(f)$, a function $f_{\mathcal{L}} : A \rightarrow \mathbf{im}_{\mathcal{L},\mathcal{R}}(f)$ with $p : \mathcal{L}(f_{\mathcal{L}})$, a function $f_{\mathcal{R}} : \mathbf{im}_{\mathcal{L},\mathcal{R}}(f) \rightarrow B$ with $q : \mathcal{R}(f_{\mathcal{R}})$, and an identification $h : f = f_{\mathcal{R}} \circ f_{\mathcal{L}}$. The type $\mathbf{im}_{\mathcal{L},\mathcal{R}}(f)$ is called the $(\mathcal{L}, \mathcal{R})$ -**image** of f .

A type X is said to be $(\mathcal{L}, \mathcal{R})$ -**modal** if the map $! : X \rightarrow \mathbf{1}$ is in \mathcal{R} (and hence $!_{\mathcal{L}}$ is an equivalence).

An orthogonal factorization system is said to be **stable** if the class \mathcal{L} is stable under pullbacks (By Lemma 5.6.26, \mathcal{R} is always stable under pullbacks).

Remark 5.6.5. By univalence, the fact that \mathcal{L} and \mathcal{R} contain all identities implies that they each contain all equivalences. Conversely, if $f \in \mathcal{L} \cap \mathcal{R}$, then (\mathbf{id}, f) and (f, \mathbf{id}) are both $(\mathcal{L}, \mathcal{R})$ -factorizations of f , and hence equal; which implies that f is an equivalence. Thus, $\mathcal{L} \cap \mathcal{R}$ consists exactly of the equivalences.

We now consider a few examples. Since we will eventually prove all the definitions to be equivalent, we can use any one of them to describe any particular example.

Example 5.6.6. The prime example is the n -**truncation modality** $\|-\|_n$ as studied in [33, Chapter 7]. This can be given as a higher modality, using its induction principle and the fact that $\|A\|_n$ is an n -type and the identity types of an n -type are again n -types (indeed, $(n-1)$ -types). The corresponding stable orthogonal factorization system, consisting of n -connected and n -truncated maps, is also constructed in [33, Chapter 7]; our construction in Theorem 5.6.20 will be a generalization of this.

Example 5.6.7. Let Q be a mere proposition. The **open modality** determined by Q is defined by $\mathbf{Op}_Q A = (Q \rightarrow A)$, with unit $\eta_A(x) = \lambda_.x : A \rightarrow (Q \rightarrow A)$. To show that this is a higher modality, suppose we have $P : (Q \rightarrow A) \rightarrow \mathcal{U}$ and $f : \prod_{(a:A)} Q \rightarrow P(\lambda_.a)$. Then for any $z : Q \rightarrow A$ and $q : Q$ we have $f(z(q), q) : P(\lambda_.z(q))$. And since Q is a mere proposition, we have $z(q) = z(q')$ for any $q' : Q$, hence $e(z, q) : (\lambda_.z(q)) = z$ by function extensionality. This gives

$$\lambda z. \lambda q. e(z, q) * ((f(z(q), q))) : \prod_{(z:Q \rightarrow A)} Q \rightarrow P(z)$$

For the computation rule, we have

$$\begin{aligned} (\lambda z. \lambda q. e(z, q) * ((f(z(q), q))))(\lambda_.a) &= \lambda q. e(\lambda_.a, q) * ((f(a, q))) \\ &= \lambda q. f(a, q) = f(a) \end{aligned}$$

by function extensionality, since $e(\lambda_{_}. a, q) = \text{refl}$. Finally, if $x, y : Q \rightarrow A$, then $(x = y) \simeq \prod_{(q:Q)} x(q) = y(q)$, and the map

$$\left(\prod_{(q:Q)} x(q) = y(q) \right) \rightarrow \left(Q \rightarrow \prod_{(q:Q)} x(q) = y(q) \right)$$

is (by currying) essentially precomposition with a product projection $Q \times Q \rightarrow Q$, and that is an equivalence since Q is a mere proposition.

Example 5.6.8. Again, let Q be a mere proposition. The **closed modality** determined by Q is defined by $\text{Cl}_Q A = Q * A$, the *join* of Q and A (the pushout of Q and A under $Q \times A$). We show that this is a Σ -closed reflective subuniverse. Define a type B to be modal if $Q \rightarrow \text{is_contr}(B)$, and note that it is indeed the case that $Q \rightarrow \text{is_contr}(Q * A)$, for any type A . By the universal property of pushouts, a map $Q * A \rightarrow B$ consists of a map $f : A \rightarrow B$ and a map $g : Q \rightarrow B$ and for any $a : A$ and $q : Q$ an identification $p : f(a) = g(q)$. But if $Q \rightarrow \text{is_contr}(B)$, then g and p are uniquely determined, so this is just a map $A \rightarrow B$. Thus $(\text{Cl}_Q A \rightarrow B) \rightarrow (A \rightarrow B)$ is an equivalence, so we have a reflective subuniverse. It is Σ -closed since the dependent sum of a contractible family of types over a contractible base is contractible.

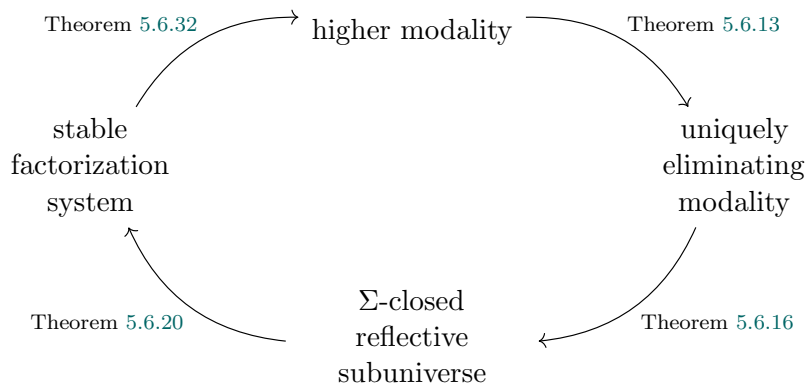
Example 5.6.9. The **double negation modality** is defined by $A \mapsto \neg\neg A$, i.e. $(A \rightarrow \mathbf{0}) \rightarrow \mathbf{0}$, with $\eta(a) = \lambda g. g(a)$. We show that this is a uniquely eliminating modality. Since the map $\lambda f. f \circ \eta_A$ that must be an equivalence has mere propositions as domain and codomain, it suffices to give a map in the other direction. Thus, let $P : \neg\neg A \rightarrow \mathcal{U}$ and $f : \prod_{(a:A)} \neg\neg P(\lambda g. g(a))$; given $z : \neg\neg A$ we must derive a contradiction from $g : \neg P(z)$. Since we are proving a contradiction, we can strip the double negation from z and assume given an $a : A$. And since $\neg\neg A$ is a mere proposition, we have $z = \lambda g. g(a)$, so that we can transport $f(a)$ to get an element of $\neg\neg P(z)$, contradicting g .

Example 5.6.10. The **trivial modality** is the identity function on \mathcal{U} . It coincides with Op_{\top} and with Cl_{\perp} .

Dually, the **zero modality** sends all types to $\mathbf{1}$. It is equivalently the (-2) -truncation, and coincides with Op_{\perp} and with Cl_{\top} .

Summary. In each of Definitions 5.6.1 to 5.6.4 we have defined what it means for a type to be modal. In each case, being modal is a family of mere propositions indexed by the universe, i.e. a subuniverse. We will show in Theorems 5.1.8, 5.6.12, 5.6.15 and 5.6.31 that each kind of structure is completely determined by this subuniverse. (Theorem 5.1.8 is more general, not requiring Σ -closedness.)

It follows that the type of all modalities of each kind is a subset of the set $\mathcal{U} \rightarrow \text{Prop}$ of all subuniverses, and in particular is a set. This makes it easier to establish the equivalences of the different kinds of modalities. It suffices to show that any modality of one kind determines a modality of the next kind with the same modal types, which we will do as follows:



Before Theorem 5.6.20 we take the opportunity to develop a bit more theory of reflective subuniverses, including closure under identity types (Corollary 5.1.21) and dependent products (Proposition 5.1.22), along with several equivalent characterizations of Σ -closedness (Theorem 5.6.19).

Of these equivalences, the most surprising is that a stable factorization system is uniquely determined by its underlying reflective subuniverse of types. This is false for stable factorization systems on arbitrary categories; the reason it holds here is that we are talking *in type theory* about factorization systems *on the category of types*. An analogous fact is true in classical set-based mathematics for stable factorization systems on the category of sets (although in that case there are much fewer interesting examples). We [28] we also observe that when type theory is interpreted in a higher category, the data of a reflective subuniverse or modality has to be interpreted “fiberwise”, giving a richer structure than a single reflective subcategory.

5.6.1 Higher modalities

We start by showing that a higher modality is determined by its modal types, and gives rise to a uniquely eliminating modality.

Lemma 5.6.11. *If \circ is a higher modality, then any type of the form $\circ X$ is modal.*

Proof. We want to show that the modal unit $\eta_{\circ X} : \circ X \rightarrow \circ \circ X$ is an equivalence. By the induction principle and the computation rule for higher modalities, we find a function $f : \circ \circ X \rightarrow \circ X$ with the property that $f \circ \eta_{\circ X} \sim \text{id}_{\circ X}$. We wish to show that we also have $\eta_{\circ X} \circ f \sim \text{id}$. Since identity types of types of the form $\circ Y$ are declared to be modal, it is equivalent to find a term of type

$$\prod_{(x:\circ \circ X)} \circ(\eta_{\circ X}(f(x)) = x).$$

Now we are in the position to use the induction principle of higher modalities again, so it suffices to show that $\eta(f(\eta(x))) = \eta(x)$ for any $x : \circ X$. This follows from the fact that $f \circ \eta = \text{id}$. \square

Theorem 5.6.12. *The data of two higher modalities \circ and \circ' are identical if and only if they have the same modal types.*

Proof. Another way of stating this, is that the function from the type of *all* modalities on \mathcal{U} to the type $\mathcal{U} \rightarrow \mathbf{Prop}$ of predicates on \mathcal{U} , given by mapping a modality to the predicate `is_modal`, is an embedding. Thus, we need to show that for any predicate $\mathcal{M} : \mathcal{U} \rightarrow \mathbf{Prop}$, we can find at most one modality for which \mathcal{M} is the class of modal types. This follows, once we demonstrate that,

for any $\mathcal{M} : \mathcal{U} \rightarrow \mathbf{Prop}$ closed under identity types, and for any type X , the type of tuples (Y, p, π, I, C) — consisting of a type Y with p witnessing that Y satisfies \mathcal{M} , a function $\pi : X \rightarrow Y$, a function

$$I_P : (\prod_{(x:X)} P(\pi(x))) \rightarrow (\prod_{(y:Y)} P(y))$$

for every $P : Y \rightarrow \mathcal{U}_{\mathcal{M}}$, which is a right inverse of precomposing with π , as is witnessed by the term C — is a mere proposition.

We prove this statement in two parts. First, we show that the type of pairs (I, C) , with I and C of the indicated types, is a mere proposition for any (Y, p, π) . After that, we show that the type of triples (Y, p, π) is also a mere proposition. These two facts combined prove the statement.

Consider a type Y satisfying \mathcal{M} , and a function $\pi : X \rightarrow Y$, and let (I, C) and (I', C') be two terms witnessing that Y satisfies an induction principle with a computation rule. We want to show that $(I, C) = (I', C')$, and of course it suffices to show that $(I(s), C(s)) = (I'(s), C'(s))$ for any $P : Y \rightarrow \mathcal{U}_{\mathcal{M}}$ and $s : \prod_{(x:X)} P(\pi(x))$.

To show that $I(s, y) = I'(s, y)$ for any $y : Y$, we use the induction principle (I, C) . So it suffices to show that $I(s, \pi(x)) = I'(s, \pi(x))$. Both of these terms are equal to $s(x)$. Thus, we obtain a proof $J(s, y)$ that $I(s, y) = I'(s, y)$, with the property that $J(s, \pi(x)) = C(s, x) \cdot C'(s, x)^{-1}$. Now we need to show that $J(s)_*(C(s)) = C'(s)$, which is equivalent to the property we just stated. This finishes the proof that the type of the induction principle and computation rule is a mere proposition.

It remains to show that $(Y, \pi) = (Y', \pi')$, provided that Y and Y' are both in \mathcal{M} , and that both sides satisfy the induction principle and computation rule. It suffices to find an equivalence $f : Y \rightarrow Y'$ such that $f \circ \pi = \pi'$.

From the induction principles of Y resp. Y' , we obtain a function $f : Y \rightarrow Y'$ with the property that $f \circ \pi = \pi'$, and a function $f' : Y' \rightarrow Y$ with the property that $f' \circ \pi' = \pi$. To show that $f' \circ f = \text{id}$ we use the induction principle of Y . Since the type $f'(f(y)) = y$ is in \mathcal{M} , it suffices to show that $f'(f(\pi(y))) = \pi(y)$. This readily follows from the defining properties of f and f' . Similarly, we have $f \circ f' = \text{id}$. \square

Theorem 5.6.13. *A higher modality is a uniquely eliminating modality, with the same modal types.*

Proof. Let \circ be a modality with modal units η_A . Our goal is to show that the precomposition map

$$\lambda s. s \circ \eta_A : (\prod_{(x:\circ A)} \circ(P(x))) \rightarrow (\prod_{(a:A)} \circ(P(\eta_A(a))))$$

is an equivalence for each $A : \mathcal{U}$ and $P : \circ A \rightarrow \mathcal{U}$. By the given induction principle and computation rule, we obtain a right inverse $\text{ind}_{\circ A}$ of $- \circ \eta_A$.

To show that it is a left inverse, consider $s : \prod_{(x:\circ A)} \circ(P(x))$. We need to find a homotopy

$$\prod_{(x:\circ A)} s(x) = \text{ind}_{\circ A}(s \circ \eta_A)(x).$$

By assumption we have that $P(x)$ is modal for each $z : \circ A$ and hence it follows that $s(x) = \text{ind}_{\circ A}(s \circ \eta_A)(x)$ is modal for each x . Hence it suffices to find a function of type

$$\prod_{(a:A)} s(\eta_A(a)) = \text{ind}_{\circ A}(s \circ \eta_A)(\eta_A(a)).$$

This follows straight from the computation rule of higher modalities. \square

5.6.2 Uniquely eliminating modalities

Next, we show that a uniquely eliminating modality is determined by its modal types, and gives rise to a Σ -closed reflective subuniverse.

Lemma 5.6.14. *Given a uniquely eliminating modality, $\circ X$ is modal for any type X .*

Proof. Using the elimination principle of $\circ \circ X$, we find a function $f : \circ \circ X \rightarrow \circ X$ and an identification $f \circ \eta_{\circ X} = \text{id}_{\circ X}$. By the uniqueness property, the type

$$\sum_{(g:\circ \circ X \rightarrow \circ \circ X)} g \circ \eta_{\circ X} = \eta_{\circ X}$$

is contractible. Since both $\text{id}_{\circ \circ X}$ and $\eta_{\circ X} \circ f$ are in this type (with suitable identifications), we find that f is also the right inverse of $\eta_{\circ X}$. This shows that $\eta_{\circ X}$ is an equivalence, so $\circ X$ is modal. \square

Theorem 5.6.15. *The data of two uniquely eliminating modalities \circ and \circ' are equivalent if and only if both have the same modal types.*

Proof. We need to show that the type of uniquely eliminating modalities with a given class $\mathcal{M} : \mathcal{U} \rightarrow \text{Prop}$ of modal types is a mere proposition. Since the types of the form $\circ X$ are modal, it suffices to show for any class $\mathcal{M} : \mathcal{U} \rightarrow \text{Prop}$ and any type X , that

the type of tuples (Y, p, π, H) — consisting of a type Y with p witnessing that Y is in \mathcal{M} , a function $\pi : X \rightarrow Y$, and for each $P : Y \rightarrow \mathcal{U}$ a term H_P witnessing that the function

$$\lambda s. s \circ \pi : \left(\prod_{(y:Y)} \circ(P(y)) \right) \rightarrow \left(\prod_{(x:X)} \circ(P(\pi(x))) \right)$$

is an equivalence — is a mere proposition.

Let (Y, p, π, H) and (Y', p', π', H') be such tuples. To show that they are equal, it suffices to show that $(Y, \pi) = (Y', \pi')$ because the other things in the list are terms of mere propositions. Furthermore, showing that $(Y, \pi) = (Y', \pi')$ is equivalent to finding an equivalence $f : Y \simeq Y'$ with the property that $f \circ \pi = \pi'$. By H , there is such a function, and by H' there is a function $f' : Y' \rightarrow Y$ such that $f' \circ \pi' = \pi$. Now the uniqueness gives that $f' \circ f$ is the only function from Y to Y such that $f' \circ f \circ \pi = \pi$ and of course id_Y is another such function. Therefore it follows that $f' \circ f = \text{id}$, and similarly it follows that $f \circ f' = \text{id}$. \square

Theorem 5.6.16. *Any uniquely eliminating modality determines a Σ -closed reflective subuniverse with the same modal types.*

Proof. It is immediate from the definition of uniquely eliminating modalities that every map $f : A \rightarrow B$ into a modal type B has a homotopy unique extension to $\circ A$ along the modal unit:

$$\begin{array}{ccc} A & & \\ \eta_A \downarrow & \searrow f & \\ \circ A & \xrightarrow{\quad \tilde{f} \quad} & B \end{array}$$

Since the types of the form $\circ X$ are modal, we obtain a reflective subuniverse. It remains to verify that the type $\sum_{(x:\circ X)} \circ(P(x))$ is modal for any type X and $P : X \rightarrow \mathcal{U}$. We have the function

$$\varphi \equiv \lambda m. (f(m), g(m)) : \circ(\sum_{(x:\circ X)} \circ(P(x))) \rightarrow \sum_{(x:\circ X)} \circ(P(x)),$$

where

$$\begin{aligned} f &\equiv \text{ind}_{\circ}(\lambda x. \lambda u. x) && : \circ(\sum_{(x:\circ X)} \circ(P(x))) \rightarrow \circ X \\ g &\equiv \text{ind}_{\circ}(\lambda x. \lambda u. u) && : \prod_{(w:\circ(\sum_{(x:\circ X)} \circ(P(x))))} \circ(P(f(w))) \end{aligned}$$

Our goal is to show that φ is an inverse to the modal unit.

Note that

$$\varphi(\eta(x, y)) \equiv (f(\eta(x, y)), g(\eta(x, y))) \equiv (x, y),$$

so we see immediately that φ is a left inverse of η .

To show that φ is a right inverse of η , note that the type of functions h fitting in a commuting square of the form

$$\begin{array}{ccc} \circ(\sum_{(x:\circ X)} \circ(P(x))) & \xrightarrow{\quad h \quad} & \circ(\sum_{(x:\circ X)} \circ(P(x))) \\ \swarrow \eta & & \nearrow \eta \\ & \sum_{(x:\circ X)} \circ(P(x)) & \end{array}$$

is contractible, and it contains the identity function. Therefore, it suffices to show that $(\eta \circ \varphi) \circ \eta = \eta$, but this follows from the fact that φ is a left inverse of the modal unit. \square

5.6.3 Σ -closed reflective subuniverses

Now we study reflective subuniverses in a bit more detail, and end by showing that Σ -closed ones give rise to stable factorization systems. Σ -closure is used in Theorem 5.6.20 to show that left maps and right maps are closed under composition.

5.6.3.1 Σ -closed reflective subuniverses

Definition 5.6.17. Let $\mathcal{M} : \mathcal{U} \rightarrow \text{Prop}$ be a reflective subuniverse with modal operator \circ . We say that a type X is **\circ -connected** if $\circ X$ is contractible, and we say that a function $f : X \rightarrow Y$ is **\circ -connected** if each of its fibers is. Similarly, we say that f is **modal** if each of its fibers is.

Note that a type X is modal or \circ -connected just when the map $X \rightarrow \mathbf{1}$ is.

Example 5.6.18. Recall from Example 5.6.7 that the open modality associated to a proposition Q is defined by $\text{Op}_Q(A) \equiv (Q \rightarrow A)$. We claim that A is Op_Q -connected if and only if $Q \rightarrow \text{is_contr}(A)$. In other words, $(Q \rightarrow \text{is_contr}(A)) \simeq \text{is_contr}(Q \rightarrow A)$. For on the one hand, if $Q \rightarrow \text{is_contr}(A)$, then $Q \rightarrow A$; while any two $f, g : Q \rightarrow A$ can be shown equal by function extensionality, since if Q then A is contractible. But on the other hand, if $\text{is_contr}(Q \rightarrow A)$ and Q , then $(Q \rightarrow A) \simeq A$, hence $\text{is_contr}(A)$.

Note that $Q \rightarrow \text{is_contr}(A)$ is also the defining condition for the Cl_Q -modal types from Example 5.6.8. That is, the Op_Q -connected types coincide with the Cl_Q -modal types.

The following theorem combines Lemma 7.5.7 and Theorem 7.7.4 of [33].

Theorem 5.6.19. *Given a reflective universe with modal operator \circ , the following are equivalent:*

- (i) *It is Σ -closed.*
- (ii) *It is uniquely eliminating.*
- (iii) *The modal units are \circ -connected.*

Proof. Suppose first that \circ is Σ -closed, let X be a type and let $P : \circ X \rightarrow \mathcal{U}_\circ$, i.e. $P(x)$ is modal for each $x : \circ X$. To show that \circ is uniquely eliminating, we want

$$\lambda s. s \circ \eta_X : \left(\prod_{(x:\circ X)} P(x) \right) \rightarrow \left(\prod_{(x:X)} P(\eta(x)) \right)$$

to be an equivalence. Since the type $\prod_{(a:A)} B(a)$ is equivalent to the type of functions $f : A \rightarrow \sum_{(a:A)} B(a)$ such that $\text{pr}_1 \circ f = \text{id}_A$, we get the desired equivalence if the pre-composition map $\lambda j. j \circ \eta$ gives an equivalence from diagonal fillers of the square

$$\begin{array}{ccc} X & \dashrightarrow & \sum_{(x:\circ X)} P(x) \\ \eta \downarrow & \nearrow j & \downarrow \text{pr}_1 \\ \circ X & \xlongequal{\quad} & \circ X \end{array}$$

to the type of maps $X \rightarrow \sum_{(x:\circ X)} P(x)$ such that the indicated square commutes. But this is true by the universal property of η , since $\sum_{(x:\circ X)} P(x)$ is modal by Σ -closedness.

Now suppose that \circ is uniquely eliminating. To show that the modal units are connected, we want a term of type

$$\prod_{(x:\circ X)} \text{is_contr}(\circ(\text{fib}_\eta(x))).$$

Using the dependent eliminators, it is easy to find a term $s : \prod_{(x:\circ X)} \circ(\text{fib}_\eta(x))$ with the property that $s \circ \eta(x) = \eta(x, \text{refl}_{\eta(x)})$. Now we need to show that

$$\prod_{(x:\circ X)} \prod_{(w:\circ(\text{fib}_\eta(x)))} w = s(x).$$

Since the type $w = s(x)$ is modal, this is equivalent to

$$\prod_{(x:\circ X)} \prod_{(x':X)} \prod_{(p:\eta(x')=x)} \eta(x', p) = s(x).$$

Moreover, the type $\sum_{(x:\circ X)} \eta(x') = x$ is contractible, so this is equivalent to

$$\prod_{(x':X)} \eta(x', \text{refl}_{\eta(x')}) = s(\eta(x')),$$

of which we have a term by the defining property of s .

Finally, suppose that all the modal units are \circ -connected, let X be modal and let $P : X \rightarrow \mathcal{U}_{\circ}$. To show that $\sum_{(x:X)} P(x)$ is modal, we show that $\eta : (\sum_{(x:X)} P(x)) \rightarrow \circ(\sum_{(x:X)} P(x))$ is an equivalence. Since X is modal, we can extend pr_1 along η as indicated in the diagram

$$\begin{array}{ccc} \sum_{(x:X)} P(x) & & \\ \downarrow \eta & \searrow \text{pr}_1 & \\ \circ(\sum_{(x:X)} P(x)) & \xrightarrow{p} & X \end{array}$$

The type of maps

$$f : \circ(\sum_{(x:X)} P(x)) \rightarrow \sum_{(x:X)} P(x)$$

such that $\text{pr}_1 \circ f = p$ is equivalent to the type $\prod_{(z:\circ(\sum_{(x:X)} P(x)))} P(p(z))$. Using the assumption that η is connected, we calculate

$$\begin{aligned} \prod_{(z:\circ(\sum_{(x:X)} P(x)))} P(p(z)) &\simeq \prod_{(z:\circ(\sum_{(x:X)} P(x)))} \circ(\text{fib}_{\eta}(z)) \rightarrow P(p(z)) \\ &\simeq \prod_{(z:\circ(\sum_{(x:X)} P(x)))} \text{fib}_{\eta}(z) \rightarrow P(p(z)) \\ &\simeq \prod_{((x,y):\sum_{(x:X)} P(x))} P(x) \end{aligned}$$

We have the second projection pr_2 of the latter type. We obtain a term

$$q : \prod_{(z:\circ(\sum_{(x:X)} P(x)))} P(p(z))$$

such that $q(\eta(x, y)) = y$. Therefore, we get the map $(p, q) : \circ(\sum_{(x:X)} P(x)) \rightarrow \sum_{(x:X)} P(x)$ for which the diagram

$$\begin{array}{ccc} \sum_{(x:X)} P(x) & \xrightarrow{\text{id}} & \sum_{(x:X)} P(x) \\ \eta \searrow & & \nearrow (p,q) \\ & \circ(\sum_{(x:X)} P(x)) & \\ \text{pr}_1 \swarrow & \downarrow p & \swarrow \text{pr}_1 \\ & X & \end{array}$$

commutes. In particular, (p, q) is a left inverse of the modal unit. To see that it is also a right inverse, note that $\eta \circ (p, q) \circ \eta = \eta = \text{id} \circ \eta$; thus $\eta \circ (p, q) = \text{id}$ follows by uniqueness. Hence $\sum_{(x:X)} P(x)$ is modal. \square

Theorem 5.6.20. *A Σ -closed reflective subuniverse determines a stable orthogonal factorization system with the same modal types.*

Proof. Define \mathcal{L} to be the class of \circ -connected maps and \mathcal{R} to be the class of modal maps. We first show that both \mathcal{L} and \mathcal{R} are closed under composition. Recall that for $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, one has $\text{fib}_{g \circ f}(z) = \sum_{(p: \text{fib}_g(z))} \text{fib}_f(\text{pr}_1(p))$. Thus, by Σ -closedness, if f and g are both in \mathcal{R} then so is $g \circ f$, so \mathcal{R} is closed under composition; while Proposition 5.5.7 implies that \mathcal{L} is closed under composition. And since the fibers of an identity map are contractible, and contractible types are both modal and \circ -connected, both \mathcal{L} and \mathcal{R} contain all identities.

To obtain a factorization system, it remains to show that the type of $(\mathcal{L}, \mathcal{R})$ -factorizations of any function f is contractible. Since $(X, f) = (\sum_{(y:Y)} \text{fib}_f(y), \text{pr}_1)$, it is sufficient to show that $\text{fact}_{\mathcal{L}, \mathcal{R}}(\text{pr}_1)$ is contractible for any $\text{pr}_1 : \sum_{(y:Y)} P(y) \rightarrow Y$. But pr_1 factors as

$$\sum_{(y:Y)} P(y) \xrightarrow{p_{\mathcal{L}}} \sum_{(y:Y)} \circ(P(y)) \xrightarrow{p_{\mathcal{R}}} Y$$

where $p_{\mathcal{L}} := \text{tot}(\eta_{P(-)})$ and $p_{\mathcal{R}} := \text{pr}_1$. The fibers of $p_{\mathcal{R}}$ are $\circ(P(-))$, so it follows immediately that $p_{\mathcal{R}}$ is in \mathcal{R} . Moreover, since $\text{fib}_{\text{tot}(\eta)}((y, u)) \simeq \text{fib}_{\eta_{P(y)}}(u)$ and each η is \circ -connected, it follows that $p_{\mathcal{L}}$ is in \mathcal{L} .

Now consider any other factorization (g, h, H) of pr_1 into an \mathcal{L} -map $g : (\sum_{(y:Y)} P(y)) \rightarrow I$ followed by an \mathcal{R} -map $h : I \rightarrow Y$. Since $I = \sum_{(y:Y)} \text{fib}_h(y)$, we have a commuting square

$$\begin{array}{ccc} \sum_{(y:Y)} P(y) & \xrightarrow{g} & I \\ \text{tot}(\gamma) \downarrow & \swarrow & \downarrow h \\ \sum_{(y:Y)} \text{fib}_h(y) & \xrightarrow{\text{pr}_1} & Y \end{array}$$

in which $\gamma(y, u) := (g(y, u), H(y, u))$. It follows that $(g, h, H) = (\text{tot}(\gamma), \text{pr}_1, _)$. Thus suffices to show that there is a commuting triangle

$$\begin{array}{ccc} & P(y) & \\ \eta \swarrow & & \searrow \gamma_y \\ \circ(P(y)) & \text{=====} & \text{fib}_h(y) \end{array}$$

We will do this using Proposition 5.1.2, by showing that γ_y has the same universal property as $\eta_{P(y)}$. This follows from the following calculation:

$$\begin{aligned} (\text{fib}_h(y) \rightarrow Z) &\simeq ((\sum_{(w: \text{fib}_h(y))} \circ(\text{fib}_g(\text{pr}_1(w)))) \rightarrow Z) \\ &\simeq ((\sum_{(w: \text{fib}_h(y))} \text{fib}_g(\text{pr}_1(w))) \rightarrow Z) \\ &\simeq (\text{fib}_{h \circ g}(y) \rightarrow Z) \\ &\simeq (P(y) \rightarrow Z). \end{aligned}$$

which we can verify is given by precomposition with γ_y .

It remains to show that our orthogonal factorization system is stable. Consider a pullback diagram

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ k \downarrow & & \downarrow l \\ B' & \xrightarrow{g} & B \end{array}$$

in which l is in \mathcal{L} . By the pasting lemma for pullbacks, it follows that $\text{fib}_k(b) = \text{fib}_l(g(b))$ for each $b : B'$. Thus, it follows that k is in \mathcal{L} . \square

5.6.4 Stable orthogonal factorization systems

To complete §5.6, we will show that stable orthogonal factorization systems are also determined by their modal types, and give rise to higher modalities.

5.6.4.1 Orthogonal factorization systems

In classical category theory, orthogonal factorization systems are equivalently characterized by a unique lifting property. We begin with the analogue of this in our context.

Definition 5.6.21. Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorization system, and consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ l \downarrow & S & \downarrow r \\ B & \xrightarrow{g} & Y \end{array}$$

(i.e. paths $S : r \circ f = g \circ l$) for which l is in \mathcal{L} and r is in \mathcal{R} . We define $\text{fill}_{\mathcal{L}, \mathcal{R}}(S)$ to be the type of **diagonal fillers** of the above diagram, i.e. the type of tuples (j, H_f, H_g, K) consisting of $j : B \rightarrow X$, $H_f : j \circ l = f$ and $H_g : r \circ j = g$ and an equality $K : r \circ H_f = S \cdot (H_g \circ l)$.

Lemma 5.6.22. Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorization system, and consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ l \downarrow & S & \downarrow r \\ B & \xrightarrow{g} & Y \end{array}$$

for which l is in \mathcal{L} and r is in \mathcal{R} . Then the type $\text{fill}_{\mathcal{L}, \mathcal{R}}(S)$ of diagonal fillers is contractible.

Proof. By the fact that every morphism factors uniquely as a left map followed by a right map, we may factorize f and g in $(\mathcal{L}, \mathcal{R})$ as $H_f : f = f_{\mathcal{R}} \circ f_{\mathcal{L}}$ and $H_g : g = g_{\mathcal{R}} \circ g_{\mathcal{L}}$, obtaining the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f_{\mathcal{L}}} & \text{im}(f) & \xrightarrow{f_{\mathcal{R}}} & X \\ l \downarrow & & & & \downarrow r \\ B & \xrightarrow{g_{\mathcal{L}}} & \text{im}(g) & \xrightarrow{g_{\mathcal{R}}} & Y \end{array}$$

Now both $(r \circ f_{\mathcal{R}}) \circ f_{\mathcal{L}}$ and $g_{\mathcal{R}} \circ (g_{\mathcal{L}} \circ l)$ are factorizations of the same function $r \circ f : A \rightarrow Y$. Since $\text{fact}_{\mathcal{L}, \mathcal{R}}(r \circ f)$ is contractible, so is its identity type

$$(\text{im}(f), f_{\mathcal{L}}, r \circ f_{\mathcal{R}}, r \circ H_f) = (\text{im}(g), g_{\mathcal{L}} \circ l, g_{\mathcal{R}}, S \cdot (H_g \circ l)).$$

This identity type is equivalent to

$$\sum_{(e:\text{im}(f)\simeq\text{im}(g))} \sum_{(H_{\mathcal{L}}:g_{\mathcal{L}}\circ l=e\circ f_{\mathcal{L}})} \sum_{(H_{\mathcal{R}}:r\circ f_{\mathcal{R}}=g_{\mathcal{R}}\circ e)} \\ ((r \circ H_f) \cdot (H_{\mathcal{R}} \circ f_{\mathcal{L}}) = S \cdot (H_g \circ l) \cdot (g_{\mathcal{R}} \circ H_{\mathcal{L}}))$$

Now since $\text{fact}_{\mathcal{L},\mathcal{R}}(f)$ and $\text{fact}_{\mathcal{L},\mathcal{R}}(g)$ are also contractible, we can sum over them to get that the following type is contractible:

$$\sum_{(\text{im}(f):\mathcal{U})} \sum_{(f_{\mathcal{L}}:A\rightarrow\text{im}(f))} \sum_{(f_{\mathcal{R}}:\text{im}(f)\rightarrow X)} \sum_{(H_f:f=f_{\mathcal{R}}\circ f_{\mathcal{L}})} \\ \sum_{(\text{im}(g):\mathcal{U})} \sum_{(g_{\mathcal{L}}:B\rightarrow\text{im}(g))} \sum_{(g_{\mathcal{R}}:\text{im}(g)\rightarrow Y)} \sum_{(H_g:g=g_{\mathcal{R}}\circ g_{\mathcal{L}})} \\ \sum_{(e:\text{im}(f)\simeq\text{im}(g))} \sum_{(H_{\mathcal{L}}:g_{\mathcal{L}}\circ l=e\circ f_{\mathcal{L}})} \sum_{(H_{\mathcal{R}}:r\circ f_{\mathcal{R}}=g_{\mathcal{R}}\circ e)} \\ ((r \circ H_f) \cdot (H_{\mathcal{R}} \circ f_{\mathcal{L}}) = S \cdot (H_g \circ l) \cdot (g_{\mathcal{R}} \circ H_{\mathcal{L}}))$$

(omitting the hypotheses that $f_{\mathcal{L}}, g_{\mathcal{L}} \in \mathcal{L}$ and $f_{\mathcal{R}}, g_{\mathcal{R}} \in \mathcal{R}$). Reassociating and removing the contractible type $\sum_{(\text{im}(g):\mathcal{U})}(\text{im}(f) \simeq \text{im}(g))$, and renaming $\text{im}(f)$ as simply I , this is equivalent to

$$\sum_{(I:\mathcal{U})} \sum_{(f_{\mathcal{L}}:A\rightarrow I)} \sum_{(f_{\mathcal{R}}:I\rightarrow X)} \sum_{(H_f:f=f_{\mathcal{R}}\circ f_{\mathcal{L}})} \\ \sum_{(g_{\mathcal{L}}:B\rightarrow I)} \sum_{(g_{\mathcal{R}}:I\rightarrow Y)} \sum_{(H_g:g=g_{\mathcal{R}}\circ g_{\mathcal{L}})} \sum_{(H_{\mathcal{L}}:g_{\mathcal{L}}\circ l=f_{\mathcal{L}})} \sum_{(H_{\mathcal{R}}:r\circ f_{\mathcal{R}}=g_{\mathcal{R}})} \\ ((r \circ H_f) \cdot (H_{\mathcal{R}} \circ f_{\mathcal{L}}) = S \cdot (H_g \circ l) \cdot (g_{\mathcal{R}} \circ H_{\mathcal{L}}))$$

Removing the contractible $\sum_{(f_{\mathcal{L}}:A\rightarrow I)}(g_{\mathcal{L}} \circ l = f_{\mathcal{L}})$ and $\sum_{(g_{\mathcal{R}}:I\rightarrow Y)}(r \circ f_{\mathcal{R}} = g_{\mathcal{R}})$, this becomes

$$\sum_{(I:\mathcal{U})} \sum_{(f_{\mathcal{R}}:I\rightarrow X)} \sum_{(g_{\mathcal{L}}:B\rightarrow I)} \sum_{(H_f:f=f_{\mathcal{R}}\circ g_{\mathcal{L}}\circ l)} \sum_{(H_g:g=r\circ f_{\mathcal{R}}\circ g_{\mathcal{L}})} \\ (r \circ H_f = S \cdot (H_g \circ l))$$

Inserting a contractible $\sum_{(j:B\rightarrow X)}(f_{\mathcal{R}} \circ g_{\mathcal{L}} = j)$, and reassociating more, we get

$$\sum_{(j:B\rightarrow X)} \sum_{(I:\mathcal{U})} \sum_{(f_{\mathcal{R}}:I\rightarrow X)} \sum_{(g_{\mathcal{L}}:B\rightarrow I)} \sum_{(H_j:f_{\mathcal{R}}\circ g_{\mathcal{L}}=j)} \\ \sum_{(H_f:f=f_{\mathcal{R}}\circ g_{\mathcal{L}}\circ l)} \sum_{(H_g:g=r\circ f_{\mathcal{R}}\circ g_{\mathcal{L}})} (r \circ H_f = S \cdot (H_g \circ l))$$

But now $\sum_{(I:\mathcal{U})} \sum_{(f_{\mathcal{R}}:I\rightarrow X)} \sum_{(g_{\mathcal{L}}:B\rightarrow I)} \sum_{(H_j:f_{\mathcal{R}}\circ g_{\mathcal{L}}=j)}$ is just $\text{fact}_{\mathcal{L},\mathcal{R}}(j)$, hence contractible. Removing it, we get

$$\sum_{(j:B\rightarrow X)} \sum_{(H_f:f=j\circ l)} \sum_{(H_g:g=r\circ j)} (r \circ H_f = S \cdot (H_g \circ l))$$

which is just $\text{fill}_{\mathcal{L},\mathcal{R}}(S)$. Therefore, this is also contractible. \square

Definition 5.6.23. For any class $\mathcal{C} : \prod_{(A,B:\mathcal{U})}(A \rightarrow B) \rightarrow \text{Prop}$ of maps, we define

- (i) ${}^{\perp}\mathcal{C}$ to be the class of maps with **(unique) left lifting property** with respect to all maps in \mathcal{C} : the mere proposition ${}^{\perp}\mathcal{C}(l)$ asserts that for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \iota \downarrow & S & \downarrow r \\ B & \xrightarrow{g} & Y \end{array}$$

with r in \mathcal{C} , the type $\text{fill}_{\mathcal{L},\mathcal{R}}(S)$ of diagonal fillers is contractible.

- (ii) \mathcal{C}^\perp to be the class of maps with the dual **(unique) right lifting property** with respect to all maps in \mathcal{C} .

Lemma 5.6.24. *In an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$, one has $\mathcal{L} = {}^\perp\mathcal{R}$ and $\mathcal{L}^\perp = \mathcal{R}$.*

Proof. We first show that $\mathcal{L} = {}^\perp\mathcal{R}$, i.e. we show that $\mathcal{L}(f) \leftrightarrow {}^\perp\mathcal{R}(f)$ for any map f . Note that the implication $\mathcal{L}(f) \rightarrow {}^\perp\mathcal{R}(f)$ follows from Lemma 5.6.22.

Let $f : A \rightarrow B$ be a map in ${}^\perp\mathcal{R}$. We wish to show that $\mathcal{L}(f)$. Consider the factorization $(f_{\mathcal{L}}, f_{\mathcal{R}})$ of f . Then the square

$$\begin{array}{ccc} A & \xrightarrow{f_{\mathcal{L}}} & \text{im}_{\mathcal{L}, \mathcal{R}}(f) \\ f \downarrow & & \downarrow f_{\mathcal{R}} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

commutes. Since f has the left lifting property, the type of diagonal fillers of this square is contractible. Thus we have a section j of $f_{\mathcal{R}}$. The map $j \circ f_{\mathcal{R}}$ is then a diagonal filler of the square

$$\begin{array}{ccc} A & \xrightarrow{f_{\mathcal{L}}} & \text{im}_{\mathcal{L}, \mathcal{R}}(f) \\ f_{\mathcal{L}} \downarrow & & \downarrow f_{\mathcal{R}} \\ \text{im}_{\mathcal{L}, \mathcal{R}}(f) & \xrightarrow{f_{\mathcal{R}}} & B \end{array}$$

Of course, the identity map $\text{id}_{\text{im}_{\mathcal{L}, \mathcal{R}}(f)}$ is also a diagonal filler for this square, so the fact that the type of such diagonal fillers is contractible implies that $j \circ f_{\mathcal{R}} = \text{id}$. Thus, j and $f_{\mathcal{R}}$ are inverse equivalences, and so the pair (B, f) is equal to the pair $(\text{im}_{\mathcal{L}, \mathcal{R}}(f), f_{\mathcal{L}})$. Hence f , like $f_{\mathcal{L}}$, is in \mathcal{L} .

Similarly, Lemma 5.6.22 also implies that $\mathcal{R}(f) \rightarrow \mathcal{L}^\perp(f)$ for any map f , while we can prove $\mathcal{L}^\perp(f) \rightarrow \mathcal{R}(f)$ analogously to ${}^\perp\mathcal{R}(f) \rightarrow \mathcal{L}(f)$. \square

Corollary 5.6.25. *The data of two orthogonal factorization systems $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}', \mathcal{R}')$ are identical if and only if $\mathcal{R} = \mathcal{R}'$.*

Proof. “Only if” is obvious. Conversely, if $\mathcal{R} = \mathcal{R}'$, then by Lemma 5.6.24 we have $\mathcal{L} = \mathcal{L}'$, and the remaining data of an orthogonal factorization system is a mere proposition. \square

Lemma 5.6.26. *Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorization system. Then the class \mathcal{R} is stable under pullbacks.*

Proof. Consider a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ k \downarrow & & \downarrow h \\ B & \xrightarrow{f} & Y \end{array}$$

where $h : X \rightarrow Y$ is assumed to be in \mathcal{R} , and let $k = k_{\mathcal{R}} \circ k_{\mathcal{L}}$ be a factorization of h . Then the outer rectangle in the diagram

$$\begin{array}{ccccc} A & \xlongequal{\quad} & A & \xrightarrow{g} & X \\ k_{\mathcal{L}} \downarrow & & k \downarrow & & \downarrow h \\ \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k) & \xrightarrow{k_{\mathcal{R}}} & B & \xrightarrow{f} & Y \end{array}$$

commutes, so by the universal property of pullbacks we obtain a unique map $j : \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k) \rightarrow A$ such that $j \circ k_{\mathcal{L}} = \mathrm{id}$ and $k \circ j = k_{\mathcal{R}}$. It suffices to show that $k_{\mathcal{L}}$ is an equivalence, and since we already have that $j \circ k_{\mathcal{L}} = \mathrm{id}$ we only need to show that $k_{\mathcal{L}} \circ j = \mathrm{id}$.

We do this using the contractibility of the type of diagonal fillers. Consider the square

$$\begin{array}{ccc} A & \xrightarrow{k_{\mathcal{L}}} & \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k) \\ k_{\mathcal{L}} \downarrow & & \downarrow k_{\mathcal{R}} \\ \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k) & \xrightarrow{k_{\mathcal{R}}} & B, \end{array}$$

for which $\mathrm{id} : \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k) \rightarrow \mathrm{im}_{\mathcal{L}, \mathcal{R}}(k)$ (with the trivial homotopies) is a diagonal filler. However, we also have the homotopies $k_{\mathcal{L}} \circ j \circ k_{\mathcal{L}} \sim k_{\mathcal{L}}$ and $k_{\mathcal{R}} \circ k_{\mathcal{L}} \circ j \sim k \circ j \sim k_{\mathcal{R}}$. This shows that we have a second diagonal filler, of which the underlying map is $k_{\mathcal{L}} \circ j$. Since the type of diagonal fillers is contractible, it follows that $k_{\mathcal{L}} \circ j = \mathrm{id}$, as desired. \square

5.6.4.2 Stable orthogonal factorization systems

Lemma 5.6.27. *Given l, r, f, g and a homotopy $S : r \circ f = g \circ l$, consider as $b : B$ varies all the diagrams of the form*

$$\begin{array}{ccccc} \mathrm{fib}_l(b) & \xrightarrow{i_b} & A & \xrightarrow{f} & X \\ \downarrow ! & & \downarrow l & S & \downarrow r \\ \mathbf{1} & \xrightarrow{b} & B & \xrightarrow{g} & Y \end{array}$$

and write $S_b : r \circ (f \circ i_b) = (g \circ b) \circ !$ for the induced commutative square. (Note that the square on the left commutes judgmentally.) Then the map

$$\mathrm{fill}_{\mathcal{L}, \mathcal{R}}(S) \rightarrow \prod_{(b:B)} \mathrm{fill}_{\mathcal{L}, \mathcal{R}}(S_b),$$

defined by precomposition with b , is an equivalence.

Proof. It suffices to show that the map on total spaces

$$\left(\sum_{(S:r \circ f = g \circ l)} \mathrm{fill}_{\mathcal{L}, \mathcal{R}}(S) \right) \rightarrow \left(\sum_{(S:r \circ f = g \circ l)} \prod_{(b:B)} \mathrm{fill}_{\mathcal{L}, \mathcal{R}}(S_b) \right) \quad (5.1)$$

is an equivalence. The domain of (5.1) can be computed as

$$\begin{aligned} & \sum_{(S:r \circ f = g \circ l)} \sum_{(j:B \rightarrow X)} \sum_{(H_f:j \circ l = f)} \sum_{(H_g:r \circ j = g)} (r \circ H_f) \cdot (H_g \circ l)^{-1} = S \\ & \simeq \sum_{(j:B \rightarrow X)} (j \circ l = f) \times (r \circ j = g) \end{aligned}$$

by contracting a based path space. On the other hand, note that

$$\begin{aligned}
(r \circ f = g \circ l) &\simeq \prod_{(a:A)} r(f(a)) = g(l(a)) \\
&\simeq \prod_{(a:A)} \prod_{(b:B)} \prod_{(l(a)=b)} r(f(a)) = g(l(a)) \\
&\simeq \prod_{(b:B)} \prod_{(u:\text{fib}_l(b))} r(f(i_b(a))) = g(l(i_b(a))) \\
&\simeq \prod_{(b:B)} \prod_{(u:\text{fib}_l(b))} r(f(i_b(a))) = g(b) \\
&\simeq \prod_{(b:B)} (r \circ (f \circ i_b) = (g \circ b) \circ !)
\end{aligned}$$

That is, to give S is the same as to give each S_b . Thus the codomain of (5.1) can be computed as

$$\begin{aligned}
&\sum_{(S:r \circ f = g \circ l)} \prod_{(b:B)} \text{fill}_{\mathcal{L}, \mathcal{R}}(S_b) \\
&\simeq \sum_{(S:\prod_{(b:B)} (r \circ (f \circ i_b) = (g \circ b) \circ !))} \prod_{(b:B)} \text{fill}_{\mathcal{L}, \mathcal{R}}(S_b) \\
&\simeq \prod_{(b:B)} \sum_{(S_b:r \circ (f \circ i_b) = (g \circ b) \circ !)} \text{fill}_{\mathcal{L}, \mathcal{R}}(S_b) \\
&\simeq \prod_{(b:B)} \sum_{(j_b: \mathbf{1} \rightarrow X)} (j_b = f \circ i_b) \times (r \circ j_b = g(b))
\end{aligned}$$

using the same argument as above for S . Now we can compute

$$\begin{aligned}
&\prod_{(b:B)} \sum_{(j_b: \mathbf{1} \rightarrow X)} (j_b = f \circ i_b) \times (r \circ j_b = g \circ b) \\
&\simeq \prod_{(b:B)} \sum_{(j_b: X)} (\lambda x. j_b = f \circ i_b) \times (r(j_b) = g(b)) \\
&\simeq \sum_{(j: B \rightarrow X)} \prod_{(b:B)} (\lambda _ . j(b) = f \circ i_b) \times (r(j(b)) = g(b)) \\
&\simeq \sum_{(j: B \rightarrow X)} (\prod_{(b:B)} \lambda _ . j(b) = f \circ i_b) \times (\prod_{(b:B)} r(j(b)) = g(b)) \\
&\simeq \sum_{(j: B \rightarrow X)} (\prod_{(b:B)} \lambda _ . j(b) = f \circ i_b) \times (r \circ j = g) \\
&\simeq \sum_{(j: B \rightarrow X)} (\prod_{(b:B)} \prod_{(a:A)} \prod_{(p:l(a)=b)} j(b) = f(a)) \times (r \circ j = g) \\
&\simeq \sum_{(j: B \rightarrow X)} (\prod_{(a:A)} j(l(a)) = f(a)) \times (r \circ j = g) \\
&\simeq \sum_{(j: B \rightarrow X)} (j \circ l = f) \times (r \circ j = g)
\end{aligned}$$

which is what we computed as the domain of (5.1) above. \square

Corollary 5.6.28. *In any orthogonal factorization system $(\mathcal{L}, \mathcal{R})$, if $l : A \rightarrow B$ is a map such that $\text{fib}_l(b) \rightarrow \mathbf{1}$ is in \mathcal{L} for each $b : B$, then also l itself is in \mathcal{L} .*

Proof. By Lemma 5.6.24, l is in \mathcal{L} iff $\text{fill}_{\mathcal{L}, \mathcal{R}}(S)$ is contractible for each $r \in \mathcal{R}$ and S as in Lemma 5.6.27, while similarly $\text{fib}_l(b) \rightarrow \mathbf{1}$ is in \mathcal{L} iff $\text{fill}_{\mathcal{L}, \mathcal{R}}(S_b)$ is contractible. But the product of contractible types is contractible. \square

Corollary 5.6.29. *In any stable orthogonal factorization system, if $l \perp r$ for all maps $l \in \mathcal{L}$ of the form $l : A \rightarrow \mathbf{1}$, then $r \in \mathcal{R}$. In particular, for any modality \circ , if $X \rightarrow (A \rightarrow X)$ is an equivalence for all \circ -connected types A , then X is modal.*

Proof. By Lemma 5.6.27, for any $l \in \mathcal{L}$ and commutative square S from l to r , we have $\text{fill}_{\mathcal{L}, \mathcal{R}}(S) \simeq \prod_{(b:B)} \text{fill}_{\mathcal{L}, \mathcal{R}}(S_b)$. Since $(\mathcal{L}, \mathcal{R})$ is stable, each map $!_b : \text{fib}_l(b) \rightarrow \mathbf{1}$ is also in \mathcal{L} , so that $!_b \perp r$ by assumption. Thus $\text{fill}_{\mathcal{L}, \mathcal{R}}(S_b)$ is contractible for all b , hence so is $\text{fill}_{\mathcal{L}, \mathcal{R}}(S)$.

For the second statement, the type $f : A \rightarrow X$ is equivalent to the type of commutative squares

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

and the type of fillers for such a square is equivalent to the type of $x : X$ such that $f(a) = x$ for all $a : A$, i.e. the fiber of $X \rightarrow (A \rightarrow X)$ over f . Thus, the assumption ensures that all such types of fillers are contractible, i.e. $l \perp r$ for all \circ -connected maps of the form $l : A \rightarrow \mathbf{1}$, so the first statement applies. \square

Lemma 5.6.30. *Let $(\mathcal{L}, \mathcal{R})$ be a stable orthogonal factorization system. Then a map $r : X \rightarrow Y$ is in \mathcal{R} if and only if $\text{fib}_r(y)$ is $(\mathcal{L}, \mathcal{R})$ -modal for each $y : Y$.*

Proof. The class of right maps is stable under pullbacks by Lemma 5.6.26, so it suffices to show that any map with modal fibers is in \mathcal{R} .

Let $r : X \rightarrow Y$ be a map with modal fibers. Our goal is to show that r is in \mathcal{R} . By Lemma 5.6.24 it suffices to show that r has the right lifting property with respect to the left maps. Consider a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ l \downarrow & & \downarrow r \\ B & \xrightarrow{g} & Y \end{array}$$

in which l is a map in \mathcal{L} . We wish to show that the type of diagonal fillers is contractible. By Lemma 5.6.27, the type of diagonal fillers of the above diagram is equivalent to the dependent product of the types of fillers of

$$\begin{array}{ccc} \text{fib}_l(b) & \xrightarrow{f \circ i_b} & X \\ \downarrow & & \downarrow r \\ \mathbf{1} & \xrightarrow{g(b)} & Y \end{array}$$

indexed by $b : B$. Thus, it suffices that the type of diagonal fillers for this square is contractible for each $b : B$. Since any filler factors uniquely through the pullback $\mathbf{1} \times_Y X$, which is $\text{fib}_r(g(b))$, the type of diagonal fillers of the above square is equivalent to the type of diagonal fillers of the square

$$\begin{array}{ccc} \text{fib}_l(b) & \dashrightarrow & \text{fib}_r(g(b)) \\ \downarrow & & \downarrow \\ \mathbf{1} & \xlongequal{\quad} & \mathbf{1} \end{array}$$

where the dotted map, is the unique map into the pullback $\text{fib}_r(g(b))$. In this square, the left map is in \mathcal{L} because \mathcal{L} is assumed to be stable under pullbacks, and the right map is in \mathcal{R} by assumption, so the type of diagonal fillers is contractible. \square

Theorem 5.6.31. *Any two stable orthogonal factorization systems with the same modal types are equal.*

Proof. By Corollary 5.6.25 it follows that any orthogonal factorization system is completely determined by the class of right maps. By Lemma 5.6.30 it follows that in a stable orthogonal factorization system, the class of right maps is completely determined by the modal types. \square

Theorem 5.6.32. *Any stable orthogonal factorization system determines a higher modality with the same modal types.*

Proof. For every type X we have the $(\mathcal{L}, \mathcal{R})$ -factorization $X \rightarrow \circ X \rightarrow \mathbf{1}$ of the unique map $X \rightarrow \mathbf{1}$. This determines the modal unit $\eta : X \rightarrow \circ X$ which is in \mathcal{L} , and the unique map $\circ X \rightarrow \mathbf{1}$ is in \mathcal{R} , i.e. $\circ X$ is $(\mathcal{L}, \mathcal{R})$ -modal.

To show the induction principle, let $P : \circ X \rightarrow \mathcal{U}$ and $f : \prod_{(x:X)} \circ(P(\eta(x)))$. Then we have a (judgmentally) commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & \sum_{(x:\circ X)} \circ(P(x)) \\ \eta \downarrow & & \downarrow \text{pr}_1 \\ \circ X & \xlongequal{\quad} & \circ X. \end{array}$$

Note that by Lemma 5.6.30, the projection $\text{pr}_1 : (\sum_{(x:\circ X)} \circ(P(x))) \rightarrow \circ X$ is in \mathcal{R} because its fibers are modal. Also, the modal unit $\eta : X \rightarrow \circ X$ is in \mathcal{L} . Thus, by Definition 5.6.23, the type of fillers of this square is contractible. Such a filler consists of a function s and homotopies filling the two triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & \sum_{(x:\circ X)} \circ(P(x)) \\ \eta \downarrow & \nearrow & \downarrow \text{pr}_1 \\ \circ X & \xlongequal{\quad} & \circ X \end{array}$$

whose composite is reflexivity, i.e. the type

$$\sum_{(s:\circ X \rightarrow \sum_{(x:\circ X)} \circ(P(x)))} \sum_{(H:\prod_{(x:\circ X)} \text{pr}_1(s(x))=x)} \sum_{(K:\prod_{(x:X)} s(\eta(x))=f(x))} \prod_{(x:X)} \text{pr}_1(K(x)) = H(\eta(x)).$$

If we decompose s , f , and K by their components, we get

$$\begin{aligned} & \sum_{(s_1:\circ X \rightarrow \circ X)} \sum_{(s_2:\prod_{(x:\circ X)} \circ(P(s_1(x))))} \sum_{(H:\prod_{(x:\circ X)} s_1(x)=x)} \\ & \quad \sum_{(K_1:\prod_{(x:X)} s_1(\eta(x))=f_1(x))} \sum_{(K_2:\prod_{(x:X)} s_2(\eta(x))=K_1(x)f_2(x))} \\ & \quad \quad \quad \prod_{(x:X)} K_1(x) = H(\eta(x)). \end{aligned}$$

Now we can contract s_1 and H , and also K_1 with the final unnamed homotopy, to get

$$\sum_{(s_2:\prod_{(x:\circ X)} \circ(P(x)))} \prod_{(x:X)} s_2(\eta(x)) =_{K_1(x)} f_2(x).$$

But this is just the type of extensions of f along η , i.e. the fiber of precomposition by η . Thus, precomposition by η is an equivalence, so in fact that we have a uniquely eliminating modality. By Corollary 5.1.21, the identity types of $\circ X$ are modal, so we have a higher modality as well. \square

5.7 Accessible reflective subuniverses

Definition 5.7.1. Given a family $f : \prod_{(i:I)} A_i \rightarrow B_i$ of maps, a type X is said to be **f -local** if the precomposition map

$$f_i^* : (B_i \rightarrow X) \rightarrow (A_i \rightarrow X)$$

is an equivalence, for each $i : I$. The family f is said to be a **presentation** of a reflective subuniverse L if the subuniverses of f -local types and L -local types coincide. A reflective subuniverse is said to be **accessible** if there exists a presentation for it.

In [28] it is shown that the subuniverse of f -local types is always a reflective subuniverse, provided that sufficiently many higher inductive types are available. However, it is not clear whether their construction is possible in our current setting, where the only higher inductive types that are assumed to exist are homotopy pushouts. In this section we will establish general properties of accessible reflective subuniverses. We will show in Chapter 7 that for any family f of maps between *compact* types, the subuniverse of f -local types is indeed reflective.

Remark 5.7.2. Note that being accessible is structure; different families can present the same reflective subuniverse or modality. As a trivial example, note that localizing at the empty type, and localizing at the type family on $\mathbf{2}$ defined by $0_2 \mapsto \mathbf{0}$ and $1_2 \mapsto \mathbf{1}$ both map all types to contractible types.

However, we are usually only interested in properties of presentations insofar as they determine properties of subuniverses. For instance, by Lemma 5.7.8, a reflective subuniverse is a modality exactly when it has a presentation in which each $C(a) = \mathbf{1}$.

Example 5.7.3. The trivial modality $\|- \|_{(-2)}$ is presented by $\mathbf{0}$, while the propositional truncation modality $\|- \|_{(-1)}$ is presented by $\mathbf{2}$. More generally, the n -truncation modality $\|- \|_n$ is presented by the $(n+1)$ -sphere \mathbb{S}^{n+1} .

Example 5.7.4. For every mere proposition P , the open modality $\text{Op}_P(X) := (P \rightarrow X)$ from Example 5.6.7 is presented by the singleton type family P . To see this, note that $\eta_X : X \rightarrow (P \rightarrow X)$ is the same as the map in the definition of locality, so that X is modal for the open modality on P if and only if it is P -local. (If P is not a mere proposition, however, then $X \mapsto (P \rightarrow X)$ is not a modality, and in particular does not coincide with localization at P .)

Example 5.7.5. The closed modality Cl_P from Example 5.6.8 associated to a mere proposition P is presented by the type family $\lambda x. \mathbf{0} : P \rightarrow \mathcal{U}$. For by definition, A is null for this family if and only if for any $p : P$ the map $A \rightarrow (\mathbf{0} \rightarrow A)$ is an equivalence. But $\mathbf{0} \rightarrow P$ is contractible, so this says that $P \rightarrow \text{is_contr}(A)$, which was the definition of Cl_P -modal types from Example 5.6.8.

Lemma 5.7.6. *Let $f : \prod_{(i:I)} A_i \rightarrow B_i$ be a family of maps. Denote the family consisting of the suspensions of the functions by $\Sigma f : \prod_{(i:I)} \Sigma A_i \rightarrow \Sigma B_i$. A type X is Σf -local if and only if for every $x, y : X$, the type $x =_X y$ is f -local. In other words, $L_{\Sigma f} = (L_f)'$.*

Proof. By the induction principle for suspension and naturality, we obtain for each $i : I$ a commutative square

$$\begin{array}{ccc} (\Sigma B_i \rightarrow X) & \xrightarrow{\cong} & \left(\sum_{(x,y:X)} (B_i \rightarrow x = y) \right) \\ \downarrow & & \downarrow \\ (\Sigma A_i \rightarrow X) & \xrightarrow{\cong} & \left(\sum_{(x,y:X)} (A_i \rightarrow x = y) \right) \end{array}$$

in which the horizontal maps are equivalences. So X is Σf -local if and only if the right vertical map is an equivalence for every $i : I$, if and only if for each $x, y : X$, the type $x = y$ is f_i -local for every $i : I$. \square

A general localization is only a reflective subuniverse, but there is a convenient sufficient condition for it to be a modality: if each $C(a) = \mathbf{1}$. A localization modality of this sort is called *nullification*.

Theorem 5.7.7. *If $F : \prod_{(a:A)} B(a) \rightarrow C(a)$ is such that each $C(a) = \mathbf{1}$, then localization at F is a modality, called **nullification at B** .*

Proof. It suffices to show that for any $B : A \rightarrow \mathcal{U}$, the B -null types are Σ -closed. Thus, let $X : \mathcal{U}$ and $Y : X \rightarrow \mathcal{U}$ be such that X and each $Y(x)$ are B -null. Then

$$\begin{aligned} (B \rightarrow \sum_{(x:X)} Y(x)) &\simeq \sum_{(g:B \rightarrow X)} \prod_{(b:B)} Y(g(b)) \\ &\simeq \sum_{(x:X)} B \rightarrow Y(x) \\ &\simeq \sum_{(x:X)} Y(x) \end{aligned}$$

with the inverse equivalence being given by constant maps. Thus, $\sum_{(x:X)} Y(x)$ is B -null. \square

Of course, it might happen that \mathcal{L}_F is a modality even if F doesn't satisfy the condition of Theorem 5.7.7. For instance, if $B : A \rightarrow \mathcal{U}$ has a section $s : \prod_{(a:A)} B(a)$, then localizing at the family $s' : \prod_{(a:A)} \mathbf{1} \rightarrow B(a)$ is equivalent to nullifying at B , since in a section-retraction pair the section is an equivalence if and only if the retraction is. However, we can say the following.

Lemma 5.7.8. *If $F : \prod_{(a:A)} B(a) \rightarrow C(a)$ is such that \mathcal{L}_F is a modality, then there exists a family $E : D \rightarrow \mathcal{U}$ such that \mathcal{L}_F coincides with nullification at E .*

Proof. Write $\circ \equiv \mathcal{L}_F$ and η for its modal unit. Define $D = \sum_{(a:A)} (\circ(B(a)) + \circ(C(a)))$, and $E : D \rightarrow \mathcal{U}$ by

$$\begin{aligned} E(a, \text{inl}(b)) &\equiv \text{fib}_{\eta_{B(a)}}(b) \\ E(a, \text{inr}(c)) &\equiv \text{fib}_{\eta_{C(a)}}(c). \end{aligned}$$

Then since η is \circ -connected, each $E(d)$ is \circ -connected, and hence every F -local type is E -null.

On the other hand, suppose X is an E -null type. Each $\eta_{B(a)}$ and $\eta_{C(a)}$ is \mathcal{L}_E -connected, since their fibers are \mathcal{L}_E -connected (by definition); thus X is also $\eta_{B(a)}$ -local and $\eta_{C(a)}$ -local. But we have the following commutative square:

$$\begin{array}{ccc} B(a) & \xrightarrow{\eta_{B(a)}} & \circ(B(a)) \\ F(a) \downarrow & & \downarrow \circ(F(a)) \\ C(a) & \xrightarrow{\eta_{C(a)}} & \circ(C(a)) \end{array}$$

and $\circ(F(a))$ is an equivalence; thus X is also $F(a)$ -local. So the F -local types coincide with the E -null types. \square

This shows that the following definition of accessible modality is consistent with our terminology of accessible reflective subuniverse.

Definition 5.7.9. A modality \circ on \mathcal{U} is said to be **accessible** if it is the nullification at a family of types in \mathcal{U} , indexed by a type in \mathcal{U} . A **presentation** of a modality \circ consists of a family of types $B : A \rightarrow \mathcal{U}$, where $A : \mathcal{U}$, such that the subuniverse of modal types coincides with the subuniverse of B -null types.

Examples 5.7.10. Our characterizations of the truncation and open and closed modalities in Examples 5.7.3 to 5.7.5 made no reference to the ambient universe.

Example 5.7.11. By contrast, the double-negation modality $\neg\neg$ is defined in a polymorphic way on all universes, but in general there seems no reason for it to be accessible on any of them. However, if propositional resizing holds, then it is the nullification at $\mathbf{2}$ together with all propositions P such that $\neg\neg P$ holds, and hence accessible.

Whether or not any inaccessible modalities remain after imposing propositional resizing may depend on large-cardinal principles. It is shown in [10] that this is the case for the analogous question about reflective sub- $(\infty, 1)$ -categories of the $(\infty, 1)$ -category of ∞ -groupoids.

Remark 5.7.12. It is tempting to think that *any* reflective subuniverse \circ on \mathcal{U} could be extended to an accessible one on \mathcal{U}' by localizing at the family of *all* functions in \mathcal{U} that are inverted by \circ (or nullifying at the family of all \circ -connected types in \mathcal{U} , in the case of modalities), which is a \mathcal{U}' -small family though not a \mathcal{U} -small one. This does produce an accessible reflective subuniverse \circ' of \mathcal{U}' such that the \circ' -modal types in \mathcal{U} coincide with the \circ -modal ones, but there seems no reason why the modal operators \circ' and \circ should agree on types in \mathcal{U} .

Chapter 6

The equifibrant replacement operation

We begin this chapter with a generalization of the descent theorem for reflexive coequalizers Theorem 3.4.11: the *modal descent theorem*. Any modality \circ gives rise to a class of maps $f : A \rightarrow B$ satisfying the condition, due to Wellen [38], that the naturality square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta \downarrow & & \downarrow \eta \\ \circ A & \xrightarrow{\circ f} & \circ B \end{array}$$

is a pullback square. Following [38] we call such maps \circ -étale maps. The modal descent theorem asserts that a \circ -étale map into A is equivalently described as a modal map into $\circ A$. Every \circ -étale map is certainly modal, but the condition of being \circ -étale is slightly stronger than the condition of being modal.

The difference between the notions of \circ -étale maps and modal maps becomes perhaps most visible when we look at the left orthogonal classes of the \circ -étale maps and modal maps. The \circ -connected maps are left orthogonal to the modal maps, whereas a map is left orthogonal to the \circ -étale maps if and only if it is a \circ -equivalence (i.e. a map f such that $\circ f$ is an equivalence). In the case of the n -truncation there is a clear difference: a map $f : X \rightarrow Y$ is an n -equivalence if and only if it induces an isomorphism of homotopy groups $\pi_i(X) \rightarrow \pi_i(Y)$ for any $i \leq n$, whereas an n -connected map is an n -equivalence satisfying the further condition that $\pi_{n+1}(X) \rightarrow \pi_{n+1}(Y)$ is surjective.

In Theorem 6.2.11 we show that the \circ -equivalences and the \circ -étale maps form an orthogonal factorization system. We call this factorization system the *reflective* factorization system of a modality. The reflective factorization system is not stable, so it does not form a new modality. What does follow is that for the unique factorization $f = f_r \circ f_l$ of $f : A \rightarrow X$ as a \circ -equivalence f_l followed by a \circ -étale map f_r , the canonical map

$$\mathrm{hom}_X(f_r, e) \rightarrow \mathrm{hom}_X(f, e)$$

is an equivalence for any \circ -étale map e into X .

We then proceed to apply these ideas to the case of reflexive graphs and the reflexive coequalizer. The class of Δ -étale maps is introduced as the class of morphisms $f :$

$\mathbf{rGph}(\mathcal{B}, \mathcal{A})$ of reflexive graphs satisfying the condition that the square

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f} & \mathcal{A} \\ \text{constr} \downarrow & & \downarrow \text{constr} \\ \Delta(\mathbf{rcoeq}(\mathcal{B})) & \xrightarrow{\Delta(\mathbf{rcoeq}(f))} & \Delta(\mathbf{rcoeq}(\mathcal{A})) \end{array}$$

is a pullback square, and we show that the class of Δ -étale morphisms is precisely the class of fibrations of reflexive graphs, as defined in Definition 3.4.1.

In the case of sequential colimits, another way of obtaining a cartesian transformation from an arbitrary one is to take the sequential colimit fiberwise. We establish this result in Theorem 6.4.3. It has many important consequences. First of all, we show in Corollary 6.4.5 that sequential colimits commute with identity types. Second, we show in Corollary 6.4.7 that the sequential colimit of a fiber sequence is again a fiber sequence. Third, we show in Proposition 6.4.9 that truncation levels are closed under sequential colimits, and that sequential colimits commute with truncations. Finally, we show in Theorem 6.4.11 that sequential colimits commute with homotopy groups.

6.1 Modal descent

6.1.1 \circ -étale maps

Definition 6.1.1 (Definition 4.4.1 of [38]). We say that a map $f : A \rightarrow B$ is \circ -étale if the square

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \circ A \\ f \downarrow & & \downarrow \circ f \\ B & \xrightarrow{\eta} & \circ B \end{array}$$

is a pullback square. We will write

$$\text{is_etale}(f) := \text{is_pullback}(\eta, f, \text{nat_unit}_{\circ}(f)).$$

It is immediate from the definition that any equivalence is \circ -étale, and that the \circ -étale maps are closed under composition, and that every equivalence is \circ -étale.

Example 6.1.2. We claim that a map $f : A \rightarrow B$ is $\|_-\|$ -étale if and only if $A \rightarrow \text{is_equiv}(f)$. Examples of maps that satisfy this condition include equivalences, maps between propositions, and any map of the form $\mathbf{0} \rightarrow B$.

To see that if $f : A \rightarrow B$ is \circ -étale, then $A \rightarrow \text{is_equiv}(f)$, consider the pullback square

$$\begin{array}{ccc} A & \longrightarrow & \|A\| \\ f \downarrow & & \downarrow \|f\| \\ B & \longrightarrow & \|B\|, \end{array}$$

and let $a : A$. Then both $\|A\|$ and $\|B\|$ are contractible, so $\|f\| : \|A\| \rightarrow \|B\|$ is an equivalence. Since equivalences are stable under pullback it follows that f is an equivalence.

Now suppose that $A \rightarrow \text{is_equiv}(f)$. Since $\text{is_equiv}(f)$ is a proposition, we also have $\|A\| \rightarrow \text{is_equiv}(f)$. To see that the gap map

$$A \rightarrow B \times_{\|B\|} \|A\|$$

is an equivalence, we will show that its fibers are contractible. Let $b : B$, $x : \|A\|$ and $p : |b| = \|f\|(x)$. Since $\|A\| \rightarrow \text{is_equiv}(f)$, it follows that f is an equivalence. Then $\|f\|$ is also an equivalence, from which it follows that the naturality square is a pullback square. We conclude that the fibers of the gap map are contractible.

Lemma 6.1.3. *Any map between \circ -modal types is \circ -étale.*

Proof. Suppose $f : X \rightarrow Y$ is a map between \circ -modal types. Then the top and bottom maps in the square

$$\begin{array}{ccc} X & \longrightarrow & \circ X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \circ Y \end{array}$$

are equivalences. Therefore this square is a pullback square, so f is \circ -étale. \square

For the following lemma, recall that for a modality all propositions are modal if and only if all units $\eta : X \rightarrow \circ X$ are surjective.

Lemma 6.1.4. *Let \circ be a modality for which all propositions are modal, and consider a map $f : A \rightarrow B$. The following are equivalent:*

(i) f is \circ -étale.

(ii) The commuting square

$$\begin{array}{ccc} A \times_{\circ A} A & \xrightarrow{f \times_{\circ} f} & B \times_{\circ B} B \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square.

Remark 6.1.5. In the special case of (-1) -truncation, the characterization of Lemma 6.1.4 asserts that a map $f : A \rightarrow B$ is (-1) -étale if and only if the square

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square.

Proof. Suppose first that f is \circ -étale, and consider the commuting cube

$$\begin{array}{ccccc}
 & & A \times_{\circ A} A & & \\
 & \swarrow & \downarrow & \searrow & \\
 A & & B \times_{\circ B} B & & A \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 B & & \circ A & & B \\
 & \swarrow & \downarrow & \searrow & \\
 & & \circ B & &
 \end{array}$$

Since the top, bottom, and both front squares are pullback squares, it follows that both back squares are pullback. This proves that (i) implies (ii).

Now suppose that (ii) holds. Then the map

$$\text{fib}_\eta(\eta(a)) \rightarrow \text{fib}_\eta(\eta(f(a)))$$

is an equivalence for every $a : A$. Since all propositions are assumed to be modal, it follows that

$$\text{fib}_\eta(t) \rightarrow \text{fib}_\eta(\circ f(t))$$

is an equivalence for every $t : \circ A$. Thus it follows that the square

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \eta \downarrow & & \downarrow \eta \\
 \circ A & \longrightarrow & \circ B
 \end{array}$$

is a pullback square. □

Corollary 6.1.6. *If $f : A \rightarrow B$ is \circ -étale, then the square*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \delta_\eta \downarrow & & \downarrow \delta_\eta \\
 A \times_{\circ A} A & \xrightarrow{f \times_{\circ f} f} & B \times_{\circ B} B
 \end{array}$$

is a pullback square.

Proof. Consider the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \delta_\eta \downarrow & & \downarrow \delta_\eta \\
 A \times_{\circ A} A & \xrightarrow{f \times_{\circ f} f} & B \times_{\circ B} B \\
 \pi_1 \downarrow & & \downarrow \pi_1 \\
 A & \xrightarrow{f} & B
 \end{array}$$

The bottom square is a pullback square by Lemma 6.1.4, and the outer rectangle is a pullback since both vertical composites are homotopic to the respective identity functions. Therefore the top square is a pullback. □

Theorem 6.1.7. *A map $f : A \rightarrow B$ is 0-étale if and only if for each $a : A$ the restriction*

$$\begin{array}{ccc} \sum_{(x:A)} \|a = x\| & \xrightarrow{f} & \sum_{(y:B)} \|f(a) = y\| \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ A & \xrightarrow{f} & B \end{array}$$

of f to the connected component at a of A is an equivalence.

Proof. By Lemma 6.1.4 and the fact that $(|a|_0 = |x|_0) \simeq \|a = x\|$, it follows that f is 0-étale if and only if the square

$$\begin{array}{ccc} \sum_{(a,x:A)} \|a = x\| & \xrightarrow{\text{tot}(\|ap_f\|)} & \sum_{(b,y:B)} \|b = y\| \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square. Furthermore, this square is a pullback if and only if the induced map

$$\left(\sum_{(x:A)} \|a = x\| \right) \rightarrow \left(\sum_{(y:B)} \|f(a) = y\| \right)$$

is an equivalence, for each $a : A$. □

6.1.2 Modal descent

The following theorem can be seen as a ‘modal flattening lemma’, since it is analogous to Theorem 3.4.11.

Theorem 6.1.8. *Consider a pullback square*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

with $H : f \circ p' \sim p \circ g$, where E and B are modal types. Then the square

$$\begin{array}{ccc} \circ E' & \xrightarrow{\tilde{g}} & E \\ \circ p' \downarrow & & \downarrow p \\ \circ B & \xrightarrow{\tilde{f}} & B \end{array}$$

is a pullback square, where \tilde{f} and \tilde{g} are the unique extensions of f and g along the modal units of B' and E' , respectively.

Proof. Consider the diagram

$$\begin{array}{ccccc}
 E' & \xrightarrow{\text{gap}(p',g,H)} & \circ B' \times_B E & \xrightarrow{\pi_2} & E \\
 p' \downarrow & & \pi_1 \downarrow & & \downarrow p \\
 B' & \xrightarrow{\eta} & \circ B' & \xrightarrow{\tilde{f}} & B
 \end{array}$$

In this diagram, the square on the right is a pullback by definition, and the outer rectangle is a pullback by assumption, so the square on the left is also a pullback. Therefore the gap map $E' \rightarrow \circ B' \times_B E$ is \circ -connected. Moreover, since the modal types are closed under pullbacks it follows that $\circ B' \times_B E$ is modal, and therefore it follows that $\pi_2 : \circ B' \times_B E \rightarrow E$ is a modal map. Therefore the composite

$$E' \xrightarrow{\text{gap}(p',g,H)} \circ B' \times_B E \xrightarrow{\pi_2} E$$

factors g as a \circ -connected map followed by a \circ -modal map. Of course, another such factorization is the composite $g \sim \tilde{g} \circ \eta$. Since factorizations are unique, the claim follows. \square

Using modal flattening we establish partial left exactness of the modality.

Corollary 6.1.9. *Consider a pullback square*

$$\begin{array}{ccc}
 A' & \longrightarrow & A \\
 f' \downarrow & & \downarrow f \\
 B' & \longrightarrow & B,
 \end{array}$$

where f is assumed to be \circ -étale. Then the square

$$\begin{array}{ccc}
 \circ A' & \longrightarrow & \circ A \\
 \circ f' \downarrow & & \downarrow \circ f \\
 \circ B' & \longrightarrow & \circ B,
 \end{array}$$

is again a pullback square.

Proof. Since f is assumed to be \circ -étale, the square on the right in the diagram

$$\begin{array}{ccccc}
 A' & \longrightarrow & A & \longrightarrow & \circ A \\
 f' \downarrow & & f \downarrow & & \downarrow \circ f \\
 B' & \longrightarrow & B & \longrightarrow & \circ B
 \end{array}$$

is a pullback square. Therefore the outer rectangle is a pullback square by the pullback pasting lemma. Now the claim follows from modal flattening Theorem 6.1.8, using the outer rectangle. \square

Corollary 6.1.10. *Consider a pullback square*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

and suppose that $p : E \rightarrow B$ is \circ -étale. Then $p' : E' \rightarrow B'$ is \circ -étale.

Proof. Consider the commuting cube

$$\begin{array}{ccccc} & & E' & & \\ & \swarrow & \downarrow & \searrow & \\ \circ E' & & B' & & E \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \circ B' & & \circ E & & B \\ & \swarrow & \downarrow & \searrow & \\ & & \circ B & & \end{array}$$

The vertical squares on the back right and front right are pullback squares by assumption. Then it follows from Corollary 6.1.9 that the vertical square on the front left is a pullback square. Therefore the square on the back left is a pullback square by the pullback pasting property. \square

Definition 6.1.11. Let X be a type. We will define an operation

$$\text{etale_map} : \left(\sum_{(A:\mathcal{U}_\circ)} A \rightarrow \circ X \right) \rightarrow \left(\sum_{(Y:\mathcal{U})} \sum_{(g:Y \rightarrow X)} \text{is_etale}(g) \right)$$

Construction. Given a map $f : A \rightarrow \circ X$ we take the pullback

$$\begin{array}{ccc} X \times_{\circ X} A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ X & \xrightarrow{\eta} & \circ X. \end{array}$$

Then the map $\pi_1 : X \times_{\circ X} A \rightarrow X$ is \circ -étale by Lemma 6.1.3 and Corollary 6.1.10. \square

The following is a descent theorem for \circ -étale maps.

Theorem 6.1.12 (Modal descent). *For any modality \circ , and any type X , the operation*

$$\text{etale_map} : \left(\sum_{(A:\mathcal{U}_\circ)} A \rightarrow \circ X \right) \rightarrow \left(\sum_{(Y:\mathcal{U})} \sum_{(g:Y \rightarrow X)} \text{is_etale}(g) \right)$$

is an equivalence.

Proof. If $g : Y \rightarrow X$ is \circ -étale, then the square

$$\begin{array}{ccc} Y & \xrightarrow{\eta} & \circ Y \\ g \downarrow & & \downarrow \circ g \\ X & \xrightarrow{\eta} & \circ X \end{array}$$

is a pullback square. Therefore $g : Y \rightarrow X$ is in the fiber of `etale_map` at $\circ g : \circ Y \rightarrow \circ X$.

It remains to show that for any map $f : A \rightarrow \circ X$ with modal domain, there is an equivalence $A \simeq \circ(X \times_{\circ X} A)$ such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{\simeq} & \circ(X \times_{\circ X} A) \\ f \searrow & & \swarrow \circ(\text{etale_map}(f)) \\ & \circ X & \end{array}$$

commutes. To see this, note that both $f \circ \pi_2$ and $\circ(\text{etale_map}(f)) \circ \eta$ factor the same map as a \circ -connected map followed by a modal map, so the claim follows from uniqueness of factorizations. \square

Corollary 6.1.13. *Suppose $P : X \rightarrow \mathcal{U}_{\circ}$ is a family of modal types such that the projection map $\text{pr}_1 : (\sum_{(x:X)} P(x)) \rightarrow X$ is \circ -étale. Then P has a unique extension*

$$\begin{array}{ccc} X & \xrightarrow{P} & \mathcal{U}_{\circ} \\ \eta \downarrow & \nearrow \tilde{P} & \\ \circ X & & \end{array}$$

It follows that the square commuting square

$$\begin{array}{ccc} \sum_{(x:X)} P(x) & \longrightarrow & \sum_{(t:\circ X)} \tilde{P}(t) \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ X & \xrightarrow{\eta} & \circ X \end{array}$$

is a pullback square. In particular the top map is \circ -connected, so this square is in fact a \circ -naturality square.

We conclude with a slightly more economic rephrasing of Theorem 6.1.8, which is an easy corollary of the results in this section.

Theorem 6.1.14. *Consider a commuting square*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

with $H : f \circ p' \sim p \circ g$, where E and B are modal types, where $p' : E' \rightarrow B'$ is \circ -étale. Then the following are equivalent:

(i) The square is a pullback square.

(ii) The square

$$\begin{array}{ccc} \circ E' & \xrightarrow{\tilde{g}} & E \\ \circ p' \downarrow & & \downarrow p \\ \circ B & \xrightarrow{\tilde{f}} & B \end{array}$$

is a pullback square, where \tilde{f} and \tilde{g} are the unique extensions of f and g along the modal units of B' and E' , respectively.

6.1.3 The reflective factorization system of a modality

In this subsection we investigate the reflective factorization system associated to a modality, of which the right class is the class of \circ -étale maps. The left class is the class of \circ -equivalences.

Definition 6.1.15. We say that a map $f : A \rightarrow B$ is an \circ -equivalence if $Lf : \circ A \rightarrow \circ B$ is an equivalence.

Remark 6.1.16. The difference between the notions of \circ -equivalences and \circ -connected maps is best explained by an example. In the case of n -truncation, the n -equivalences are precisely the maps that induce isomorphisms on the first n homotopy groups. The n -connected maps are the maps that induce isomorphisms on the first n homotopy groups, and moreover induce an epimorphism on the $(n + 1)$ -st homotopy group.

We also note that the n -equivalences are not stable under pullbacks, whereas the n -connected maps are. Consider for instance the pullback square

$$\begin{array}{ccc} \Omega(\mathbb{S}^{n+1}) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbb{S}^{n+1} \end{array}$$

Here the map on the right is an n -equivalence, since \mathbb{S}^{n+1} is n -connected. However, the map on the left is not an n -equivalence, since the n -th homotopy group of $\Omega(\mathbb{S}^{n+1})$ is not trivial: it is the $(n + 1)$ -st homotopy group of \mathbb{S}^{n+1} , which is \mathbb{Z} .

Definition 6.1.17. The **reflective factorization system** associated to a modality \circ consists of the \circ -equivalences as the left class, and the \circ -étale maps as the right class.

Our goal in this section is to show that the reflective factorization system associated to a modality is an orthogonal factorization system.

Lemma 6.1.18. *The \circ -equivalences satisfy the 3-for-2 property: given a commuting triangle*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & C, \end{array}$$

if any two of f , g , and h are \circ -equivalences, then so is the third.

Proof. Apply \circ to the commuting triangle, and use the 3-for-2 property of equivalences. \square

Lemma 6.1.19. *For a map $f : A \rightarrow B$ the following are equivalent:*

- (i) f is an \circ -equivalence.
- (ii) For any modal type X , the precomposition map

$$f^* : (B \rightarrow X) \rightarrow (A \rightarrow X)$$

is an equivalence.

Proof. Suppose first that f is an \circ -equivalence, and let X be \circ -modal. Then the square

$$\begin{array}{ccc} X^B & \xrightarrow{f^*} & X^A \\ \eta^* \downarrow & & \downarrow \eta^* \\ X^{\circ B} & \xrightarrow{\circ f^*} & X^{\circ A} \end{array}$$

commutes. In this square the two vertical maps are equivalences by the universal property of modalization, and the bottom map is an equivalence since $\circ f$ is an equivalence. Therefore the map $f^* : X^B \rightarrow X^A$ is an equivalence, as desired.

Conversely, assume that $f^* : X^B \rightarrow X^A$ is an equivalence for every \circ -modal type X . By the square above it follows that $\circ f^* : X^{\circ B} \rightarrow X^{\circ A}$ is an equivalence for every \circ -modal type X . The fiber of $\circ A^{\circ B} \rightarrow \circ A^{\circ A}$ at $\text{id} : \circ A \rightarrow \circ A$ is contractible, so we obtain a retraction g of $\circ f$. To see that g is also a section observe that the fiber of $\circ B^{\circ B} \rightarrow \circ B^{\circ A}$ at $\circ f$ is contractible. This fiber contains $(\text{id}_{\circ B}, \text{refl}_{\circ f})$. However, we also have an identification $p : \circ f^*(\circ f \circ g) = \circ f$, since

$$\circ f^*(\circ f \circ g) \equiv (\circ f \circ g) \circ \circ f \equiv \circ f \circ (g \circ \circ f) = \circ f.$$

Therefore $(\circ f \circ g, p)$ is in the fiber of $\circ f^* : \circ B^{\circ B} \rightarrow \circ B^{\circ A}$ at $\circ f$. By the contractibility of the fibers it follows that $(\circ f \circ g, p) = (\text{id}_{\circ B}, \text{refl}_{\circ f})$, so it follows that $\circ f \circ g = \text{id}_{\circ B}$. In other words, g is both a retraction and a section of $\circ f$, so $\circ f$ is an equivalence. \square

Corollary 6.1.20. *Every \circ -connected map is a \circ -equivalence.*

Definition 6.1.21. Let $f : A \rightarrow B$ be a map. We define

$$[A]_f := B \times_{\circ B} \circ A$$

and we define the maps

$$\begin{aligned} \text{et}(f) : [A]_f &\rightarrow B \\ \bar{\eta} : A &\rightarrow [A]_f \end{aligned}$$

by the universal property of pullbacks, as indicated in the following diagram

$$\begin{array}{ccccc} A & & & & \\ & \searrow \eta & & & \\ & & [A]_f & \xrightarrow{\pi_2} & \circ A \\ & \searrow \bar{\eta} & \downarrow \text{et}(f) & & \downarrow \circ f \\ & & B & \xrightarrow{\eta} & \circ B. \end{array}$$

Lemma 6.1.22. *For every map $f : A \rightarrow B$, the map $\bar{\eta} : A \rightarrow [A]$ is a \circ -equivalence, and the map $\text{et}(f)$ is \circ -étale.*

Proof. The map $\text{et}(f)$ is a pullback of a map between modal types, so it is \circ -étale by Corollary 6.1.10. Furthermore, the map $\pi_2 : [A] \rightarrow \circ A$ is a pullback of a \circ -connected map, so it is \circ -connected. It follows from Corollary 6.1.20 that it is a \circ -equivalence. Since the modal unit $\eta : A \rightarrow \circ A$ is also \circ -connected, and therefore a \circ -equivalence, we obtain by the 3-for-2 property of \circ -equivalences established in Lemma 6.1.18 that the gap map is also a \circ -equivalence. \square

Lemma 6.1.23. *The class of \circ -equivalences is left orthogonal to the class of \circ -étale maps.*

Proof. We have to show that for every \circ -equivalence $i : A \rightarrow B$, and every \circ -étale map $f : X \rightarrow Y$, the square

$$\begin{array}{ccc} X^B & \longrightarrow & Y^B \\ \downarrow & & \downarrow \\ X^A & \longrightarrow & Y^A \end{array}$$

is a pullback square. Consider the commuting cube

$$\begin{array}{ccccc} & & X^B & & \\ & \swarrow & \downarrow & \searrow & \\ (\circ X)^B & & X^A & & Y^B \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ (\circ X)^A & & (\circ Y)^B & & Y^A \\ & \swarrow & \downarrow & \searrow & \\ & & (\circ Y)^A & & \end{array}$$

In this cube the top and bottom squares are pullback by the assumption that f is \circ -étale and the fact that exponents of pullback squares are again pullback squares. Furthermore, the square in the front left is pullback, because the two vertical maps are equivalences by the assumption that $i : A \rightarrow B$ is a \circ -equivalence. Therefore we conclude that the square in the back right is also a pullback square, as desired. \square

Corollary 6.1.24. *For any map $f : X \rightarrow Y$, the type of factorizations into a \circ -connected map followed by a \circ -étale map is contractible.*

The class of \circ -étale morphisms into a given type A , thought of as objects of the slice category \mathcal{U}/A , form a reflective subuniverse in the following sense.

Theorem 6.1.25. *Let $f : A \rightarrow X$ be a map. Then the pre-composition function*

$$\text{hom}_X(\text{et}(f), e) \rightarrow \text{hom}_X(f, e)$$

is an equivalence for every étale map $e : B \rightarrow X$.

Proof. Let $e : B \rightarrow X$ be a \circ -étale map. Then the square

$$\begin{array}{ccc} B^{[A]} & \xrightarrow{-\circ\eta_f} & B^A \\ e\circ- \downarrow & & \downarrow e\circ- \\ X^{[A]} & \xrightarrow{-\circ\eta_f} & X^A \end{array}$$

is a pullback square by Lemmas 6.1.22 and 6.1.23. Therefore we have a fiberwise equivalence

$$\prod_{(i:[A] \rightarrow X)} \mathbf{fib}_{e\circ-}(i) \rightarrow \mathbf{fib}_{e\circ-}(i \circ \eta_f)$$

by Corollary 1.4.13. Now the claim follows, since we have a commuting square

$$\begin{array}{ccc} \mathbf{fib}_{e\circ-}(i) & \longrightarrow & \mathbf{fib}_{e\circ-}(i \circ \eta_f) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbf{hom}_X(\mathbf{et}(f), e) & \longrightarrow & \mathbf{hom}_X(f, e) \end{array}$$

with equivalences on both sides, for each $i : [A] \rightarrow X$. □

6.2 The reflective factorization system for the reflexive coequalizer

6.2.1 Δ -étale maps

Definition 6.2.1. Let $f : \mathbf{rGph}(\mathcal{A}, \mathcal{B})$ be a morphism of reflexive graphs. We say that f is Δ -étale if the square

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \Delta(\mathbf{rcoeq}(\mathcal{A})) \\ f \downarrow & & \downarrow \Delta(\mathbf{rcoeq}(f)) \\ \mathcal{B} & \longrightarrow & \Delta(\mathbf{rcoeq}(\mathcal{B})) \end{array}$$

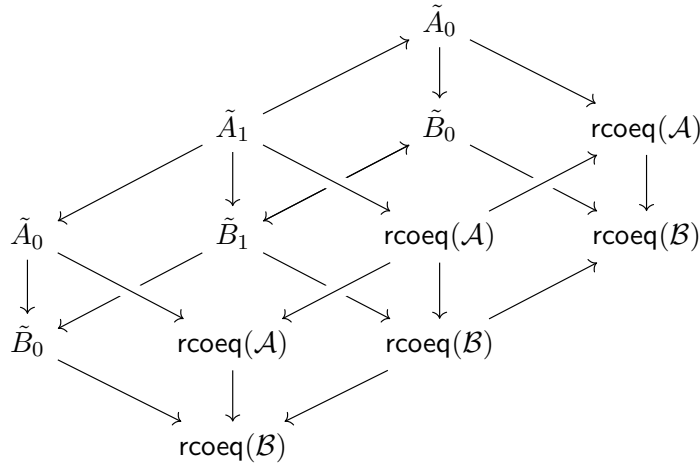
is a pullback square of reflexive graphs. We write $\mathbf{is_etale}_\Delta(f)$ for the proposition that f is Δ -étale, and we also write \mathcal{R}^Δ for the class of Δ -étale morphisms of reflexive graphs.

Theorem 6.2.2. *Consider a morphism $f : \mathbf{rGph}(\mathcal{A}, \mathcal{B})$ of reflexive graphs. The following are equivalent:*

(i) *The morphism f is a fibration in the sense of Definition 3.4.1.*

(ii) *The morphism f is Δ -étale.*

Proof. Consider the commuting diagram



If f is Δ -étale, then the three parallel vertical squares are pullback squares, hence so are the two squares on the back left side. This shows that (ii) implies (i).

Now suppose that f is a fibration, or equivalently, that f is cartesian. Then the map $\text{rcoeq}(f)$ is the unique map such that the naturality squares are pullback squares. In particular, f is a Δ -étale map. \square

Corollary 6.2.3. *Let \mathcal{B} be a family of reflexive graphs over \mathcal{A} . The following are equivalent:*

- (i) *The family \mathcal{B} is equifibered.*
- (ii) *The morphism $\text{pr}_1 : \text{rGph}(\Sigma(\mathcal{A}, \mathcal{B}), \mathcal{A})$ is Δ -étale.*

The following proposition is analogous to Corollary 6.1.9.

Proposition 6.2.4. *Consider a pullback square*

$$\begin{array}{ccc} \mathcal{A}' & \longrightarrow & \mathcal{A} \\ f' \downarrow & & \downarrow f \\ \mathcal{B}' & \longrightarrow & \mathcal{B} \end{array}$$

of reflexive graphs, and suppose that f is a fibration. Then the square

$$\begin{array}{ccc} \text{rcoeq}(\mathcal{A}') & \longrightarrow & \text{rcoeq}(\mathcal{A}) \\ \text{rcoeq}(f') \downarrow & & \downarrow \text{rcoeq}(f) \\ \text{rcoeq}(\mathcal{B}') & \longrightarrow & \text{rcoeq}(\mathcal{B}) \end{array}$$

is again a pullback square.

Proof. Consider the diagram

$$\begin{array}{ccccc}
 \mathcal{A}' & \longrightarrow & \mathcal{A} & \xrightarrow{\text{constr}} & \Delta(\text{rcoeq}(\mathcal{A})) \\
 f' \downarrow & & \downarrow f & & \downarrow \Delta(\text{rcoeq}(f)) \\
 \mathcal{B}' & \longrightarrow & \mathcal{B} & \xrightarrow{\text{constr}} & \Delta(\text{rcoeq}(\mathcal{B}))
 \end{array}$$

of reflexive graphs. Since f is assumed to be a fibration, we obtain by Theorem 6.2.2 that the square on the right is a pullback square. Furthermore, since the left square is a pullback square by assumption, it follows that the outer rectangle is again a pullback square. Hence the assertion follows from Theorem 3.4.11. \square

The following proposition is analogous to Corollary 6.1.10.

Proposition 6.2.5. *Consider a pullback square*

$$\begin{array}{ccc}
 \mathcal{A}' & \longrightarrow & \mathcal{A} \\
 f' \downarrow & & \downarrow f \\
 \mathcal{B}' & \longrightarrow & \mathcal{B}
 \end{array}$$

of reflexive graphs, and suppose that f is a fibration. Then f' is a fibration.

Proof. Consider the cube

$$\begin{array}{ccccc}
 & & \mathcal{A}' & & \\
 & \swarrow & \downarrow & \searrow & \\
 \Delta(\text{rcoeq}(\mathcal{A}')) & & \mathcal{B}' & & \mathcal{A} \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \Delta(\text{rcoeq}(\mathcal{B}')) & & \Delta(\text{rcoeq}(\mathcal{A})) & & \mathcal{B} \\
 & \swarrow & \downarrow & \searrow & \\
 & & \Delta(\text{rcoeq}(\mathcal{B})) & &
 \end{array}$$

Then the two squares in the front and the square in the back right are pullback squares, so it follows that the square in the back left is a pullback square. \square

6.2.2 The reflective factorization system of discrete graphs

Definition 6.2.6. Let $f : \text{rGph}(\mathcal{B}, \mathcal{A})$ be a morphism of reflexive graphs. We say that f is a Δ -equivalence if the map

$$\text{rcoeq}(f) : \text{rcoeq}(\mathcal{B}) \rightarrow \text{rcoeq}(\mathcal{A})$$

is an equivalence. We also write \mathcal{L}^Δ for the class of Δ -equivalences of reflexive graphs.

Lemma 6.2.7. *The Δ -equivalences satisfy the 3-for-2 property.* \square

Proposition 6.2.8. *Let $f : \mathbf{rGph}(\mathcal{B}, \mathcal{A})$ be a morphism of reflexive graphs. The following are equivalent:*

(i) f is a Δ -equivalence.

(ii) For any type X , the map

$$\mathbf{rGph}(\mathcal{A}, \Delta(X)) \rightarrow \mathbf{rGph}(\mathcal{B}, \Delta(X))$$

is an equivalence.

Proof. We have a commuting square

$$\begin{array}{ccc} \mathbf{rGph}(\mathbf{rcoeq}(\mathcal{A}), X) & \xrightarrow{-\circ \mathbf{rcoeq}(f)} & \mathbf{rGph}(\mathbf{rcoeq}(\mathcal{B}), X) \\ \downarrow & & \downarrow \\ \mathbf{rGph}(\mathcal{A}, \Delta(X)) & \xrightarrow{-\circ f} & \mathbf{rGph}(\mathcal{B}, \Delta(X)) \end{array}$$

in which the two vertical maps are equivalences. Therefore it follows that if $\mathbf{rcoeq}(f)$ is an equivalence, then so is the bottom map in the square. Conversely, if the bottom map in the square is an equivalence for every type X , then $-\circ \mathbf{rcoeq}(f)$ is an equivalence for any type X , which implies by Proposition 1.3.7 that $\mathbf{rcoeq}(f)$ is an equivalence. \square

Definition 6.2.9. Let $f : \mathbf{rGph}(\mathcal{B}, \mathcal{A})$ be a morphism of reflexive graphs. We define the morphisms

$$\begin{aligned} \mathbf{et}_\Delta(f) &: \mathbf{rGph}(\mathcal{B}^\simeq, \mathcal{A}) \\ \eta_f &: \mathbf{rGph}(\mathcal{B}, \mathcal{B}^\simeq) \end{aligned}$$

by the universal property of pullbacks, as indicated in the following diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad} & \Delta(\mathbf{rcoeq}(\mathcal{B})) \\ \eta_f \downarrow \swarrow & & \downarrow \\ \mathcal{B}^\simeq & \xrightarrow{\quad} & \Delta(\mathbf{rcoeq}(\mathcal{B})) \\ \mathbf{et}_\Delta(f) \downarrow \lrcorner & & \downarrow \\ \mathcal{A} & \xrightarrow{\text{constr}} & \Delta(\mathbf{rcoeq}(\mathcal{A})) \end{array}$$

The morphism $\mathbf{et}_\Delta(f)$ is called the **étale factor** of f , and the morphism η_f is called the **unit** of the étale factor.

Lemma 6.2.10. *For any morphism $f : \mathbf{rGph}(\mathcal{B}, \mathcal{A})$, the unit η_f of the étale factor of f is a Δ -equivalence.*

Proof. By Theorem 3.4.11, it follows that the square

$$\begin{array}{ccc} \mathbf{rcoeq}(\mathcal{B}^\simeq) & \xrightarrow{\quad} & \mathbf{rcoeq}(\Delta(\mathbf{rcoeq}(\mathcal{B}))) \\ \downarrow & & \downarrow \\ \mathbf{rcoeq}(\mathcal{A}) & \xrightarrow{\quad} & \mathbf{rcoeq}(\Delta(\mathbf{rcoeq}(\mathcal{A}))) \end{array}$$

is a pullback square. Since the bottom map is an equivalence of reflexive graphs, it follows that the top map is an equivalence of reflexive graphs. In other words, we have shown that the map

$$\mathcal{B}^{\simeq} \longrightarrow \Delta(\text{rcoeq}(\mathcal{B}))$$

is a Δ -equivalence. Of course, the morphism $\text{constr} : \mathcal{B} \rightarrow \Delta(\text{rcoeq}(\mathcal{B}))$ is also a Δ -equivalence, so the claim follows by the 3-for-2 property of Δ -equivalences. \square

Theorem 6.2.11. *The pair $(\mathcal{L}^\Delta, \mathcal{R}^\Delta)$ forms an orthogonal factorization system of reflexive graphs.*

Proof. Since any pullback of a morphism between discrete reflexive graphs is a fibration of reflexive graphs, it follows from Lemma 6.2.10 that every morphism of reflexive graphs factors as a Δ -equivalence followed by a Δ -étale map. Therefore it remains to show that the class of Δ -equivalences is left orthogonal to the class of Δ -étale maps.

We have to show that for every Δ -equivalence $i : \mathcal{A} \rightarrow \mathcal{B}$, and every Δ -étale morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$, the square

$$\begin{array}{ccc} \mathcal{X}^{\mathcal{B}} & \longrightarrow & \mathcal{Y}^{\mathcal{B}} \\ \downarrow & & \downarrow \\ \mathcal{X}^{\mathcal{A}} & \longrightarrow & \mathcal{Y}^{\mathcal{A}} \end{array}$$

is a pullback square. Consider the commuting cube

$$\begin{array}{ccccc} & & \mathcal{X}^{\mathcal{B}} & & \\ & \swarrow & \downarrow & \searrow & \\ \Delta(\text{rcoeq}(\mathcal{X}))^{\mathcal{B}} & & \mathcal{X}^{\mathcal{A}} & & \mathcal{Y}^{\mathcal{B}} \\ & \swarrow & \downarrow & \searrow & \downarrow \\ \Delta(\text{rcoeq}(\mathcal{X}))^{\mathcal{A}} & & \Delta(\text{rcoeq}(\mathcal{Y}))^{\mathcal{B}} & & \mathcal{Y}^{\mathcal{A}} \\ & \swarrow & \downarrow & \searrow & \\ & & \Delta(\text{rcoeq}(\mathcal{Y}))^{\mathcal{A}} & & \end{array}$$

In this cube the top and bottom squares are pullback by the assumption that f is Δ -étale and the fact that exponents of pullback squares are again pullback squares. Furthermore, the square in the front left is pullback, because the two vertical maps are equivalences by the assumption that $i : \mathcal{A} \rightarrow \mathcal{B}$ is a Δ -equivalence. Therefore we conclude that the square in the back right is also a pullback square, as desired. \square

Theorem 6.2.12. *For any morphism $f : \text{rGph}(\mathcal{B}, \mathcal{A})$, the pre-composition map*

$$\text{rGph}_{\mathcal{A}}(\text{et}_{\Delta}(f), e) \rightarrow \text{hom}_{\mathcal{A}}(f, e)$$

is an equivalence for every Δ -étale map $e : \mathcal{C} \rightarrow \mathcal{A}$.

Proof. Analogous to Theorem 6.1.25. \square

6.2.3 Equifibrant replacement

Definition 6.2.13. Let \mathcal{B} and \mathcal{C} be families of reflexive graphs over \mathcal{A} . We define the type

$$\mathbf{rGph}_{\mathcal{A}}(\mathcal{B}, \mathcal{C})$$

of morphisms of reflexive graphs **over** \mathcal{A} to consist of triples $(f_0, f_1, \mathbf{rfx}_f)$ consisting of

$$\begin{aligned} f_0(x) &: B_0(x) \rightarrow C_0(x) \\ f_1(e) &: \prod_{(u: B_0(x))} \prod_{(v: B_0(y))} B_1(e, u, v) \rightarrow C_1(e, f_0(u), f_0(v)) \\ \mathbf{rfx}_f(x) &: \prod_{(u: B_0(x))} f_1(\mathbf{rfx}_{\mathcal{B}}(x, u)) = \mathbf{rfx}_{\mathcal{C}}(x, f_0(u)) \end{aligned}$$

Lemma 6.2.14. *Let \mathcal{B} and \mathcal{C} be families of reflexive graphs over \mathcal{A} . Then there is an equivalence*

$$\mathbf{rGph}_{\mathcal{A}}(\mathcal{B}, \mathcal{C}) \simeq \sum_{(f: \mathbf{rGph}(\Sigma(\mathcal{A}, \mathcal{B}), \Sigma(\mathcal{A}, \mathcal{C})))} \mathbf{pr}_1 = \mathbf{pr}_1 \circ f.$$

Note that, given a reflexive graph \mathcal{A} , any family $B : \mathbf{rcoeq}(\mathcal{A}) \rightarrow \mathcal{U}$ determines an equifibered family $\mathbf{equifib_fam}(B)$ over \mathcal{A} given by

$$\begin{aligned} \mathbf{equifib_fam}(B)_0(x) &:\equiv B(\mathbf{constr}_0(x)) \\ \mathbf{equifib_fam}(B)_1(e) &:\equiv \mathbf{tr}_B(\mathbf{constr}_1(e)) \\ \mathbf{rfx}_{\mathbf{equifib_fam}(B)}(x, y) &:\equiv \mathbf{htpy_eq}(\mathbf{ap}_{\mathbf{tr}_B}(\mathbf{rfx}_{\mathbf{constr}}(x)), y). \end{aligned}$$

In other words, $\mathbf{equifib_fam}$ is an operation that takes a family $B : \mathbf{rcoeq}(\mathcal{A}) \rightarrow \mathcal{U}$ to an equifibered family over \mathcal{A} . Moreover, it is not hard to see that

$$\mathbf{equifib_fam} : (\mathbf{rcoeq}(\mathcal{A}) \rightarrow \mathcal{U}) \rightarrow \mathbf{equifib}(\mathcal{A})$$

is in fact an equivalence, where $\mathbf{equifib}(\mathcal{A})$ is the type of all (small) equifibered families over \mathcal{A} . In the following definition we observe that any family over \mathcal{A} induces an equifibered family over \mathcal{A} .

Definition 6.2.15. Let \mathcal{B} be a family of reflexive graphs over \mathcal{A} . We define the **equifibrant replacement** $\mathbf{EqF}(\mathcal{B})$ of \mathcal{B} by

$$\mathbf{EqF}(\mathcal{B}) :\equiv \mathbf{equifib_fam}(\mathbf{fib}_{\mathbf{rcoeq}(\mathbf{pr}_1)}),$$

and we define the morphism $\eta_{\mathcal{B}} : \mathbf{rGph}_{\mathcal{A}}(\mathcal{B}, \mathbf{EqF}(\mathcal{B}))$.

Theorem 6.2.16. *Consider a family \mathcal{B} of reflexive graphs over \mathcal{A} , and let \mathcal{E} be an equifibered family over \mathcal{A} . Then the pre-composition operation*

$$\mathbf{rGph}_{\mathcal{A}}(\mathbf{EqF}(\mathcal{B}), \mathcal{E}) \rightarrow \mathbf{rGph}_{\mathcal{A}}(\mathcal{B}, \mathcal{E})$$

is an equivalence for every equifibered family \mathcal{E} over \mathcal{A} .

Proof. First we observe that the square

$$\begin{array}{ccc} \Sigma(\mathcal{A}, \mathbf{EqF}(\mathcal{B})) & \longrightarrow & \Delta(\mathbf{rcoeq}(\Sigma(\mathcal{A}, \mathcal{B}))) \\ p \downarrow & & \downarrow \mathbf{rcoeq}(\mathbf{pr}_1) \\ \mathcal{A} & \xrightarrow{\mathbf{constr}} & \Delta(\mathbf{rcoeq}(\mathcal{A})) \end{array}$$

is a pullback square of reflexive graphs. Thus, the map p is the étale factor of the morphism $\mathbf{pr}_1 : \Sigma(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{A}$. Now the claim follows by Theorem 6.2.12. \square

Remark 6.2.17. By the universal property of the equifibrant replacement it follows that the family $\text{EqF}(\mathcal{B})_0$ of vertices of the equifibrant replacement of \mathcal{B} can be seen as a ‘recursive’ higher inductive family of types. It comes equipped with

$$\begin{aligned} \zeta &: \prod_{(x,y:A_0)} A_1(x,y) \rightarrow \text{EqF}(\mathcal{B})_0(x) \simeq \text{EqF}(\mathcal{B})_0(y) \\ \epsilon &: \prod_{(x:A_0)} \zeta(\text{rfx}_{\mathcal{A}}(x)) \sim \text{id}_{\text{EqF}(\mathcal{B})_0(x)} \\ \eta_0 &: \prod_{(x:A_0)} B_0(x) \rightarrow \text{EqF}(\mathcal{B})_0(x) \\ \eta_1 &: \prod_{(x,y:A_0)} \prod_{(e:A_1(x,y))} \prod_{(u:B_0(x))} \prod_{(v:B_0(y))} B_1(e,u,v) \rightarrow \zeta(e,u) = v \\ \text{rfx}_{\zeta} &: \prod_{(x:A_0)} \prod_{(u:B_0(x))} \eta_1(\text{rfx}_{\mathcal{A}}(x), \text{rfx}_{\mathcal{B}}(u)) = \epsilon(x,u). \end{aligned}$$

In other words, the equifibrant replacement is the homotopy initial family of types over A_0 with equivalences over the edges of \mathcal{A} (coherent with reflexivity) and a morphism $\eta : \text{rGph}_{\mathcal{A}}(\mathcal{B}, \text{EqF}(\mathcal{B}))$.

6.2.4 Identity types of reflexive coequalizers

Definition 6.2.18. Consider a reflexive graph \mathcal{A} with a base point $a : A_0$. We define the **universal Δ -bundle** $\mathcal{E}(a)$ over \mathcal{A} at a to be the equifibered family over \mathcal{A} corresponding to the equifibrant replacement of the morphism $a : \text{rGph}(\mathbf{1}, \mathcal{A})$ corresponding to $a : A_0$.

Proposition 6.2.19. *Consider a reflexive graph \mathcal{A} with a base point $a : A_0$. Then there are equivalences*

$$\mathcal{E}(a)_0(x) \simeq (\text{constr}_0(a) = \text{constr}_0(x))$$

In particular, we have an equivalence $\mathcal{E}(a)_0(a) \simeq \Omega(\text{rcoeq}(\mathcal{A}))$.

Proof. Immediate from the definition of the equifibrant replacement. \square

In the following theorem we establish the universal property of the identity type of $\text{rcoeq}(\mathcal{A})$ as the initial reflexive relation R on A_0 with certain extra structure.

Theorem 6.2.20. *Consider a reflexive graph \mathcal{A} , and let*

$$\begin{aligned} R &: A_0 \rightarrow A_0 \rightarrow \mathcal{U}, \\ \rho &: \prod_{(x:A_0)} R(x,x) \end{aligned}$$

be a reflexive relation on A_0 equipped with a ‘composition’ operation

$$\mu : \prod_{(a,x,y:A_0)} A_1(x,y) \rightarrow (R(a,x) \simeq R(a,y)).$$

satisfying the unit law

$$\text{left_unit}_{\mu} : \prod_{(a,x:A_0)} \prod_{(r:R(a,x))} \mu(\text{rfx}_{\mathcal{A}}(x), r) = r.$$

Then there is a unique extension

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\bar{\mu}} & (A_0, R, \rho) \\ \downarrow & \nearrow & \\ k(\text{constr}_0) & & \end{array}$$

of the morphism $\bar{\mu} : \mathbf{rGph}(\mathcal{A}, (A_0, R, \rho))$ consisting of

$$\begin{aligned}\bar{\mu}_0 &::= \mathbf{constr}_0 \\ \bar{\mu}_1(e) &::= \mu(e, \rho(x)) \\ \mathbf{rfx}_{\bar{\mu}}(x) &::= \mathbf{left_unit}_{\mu}(\rho(x)),\end{aligned}$$

where $k(\mathbf{constr}_0)$ is the pre-kernel of the function $\mathbf{constr}_0 : A_0 \rightarrow \mathbf{rcoeq}(\mathcal{A})$.

Proof. The data $(R, \mu, \mathbf{left_unit}_{\mu})$ equivalently gives for every $a : A$ an equifibered family $\mathcal{B}(a)$ over \mathcal{A} consisting of

$$\begin{aligned}\mathcal{B}(a)_0(x) &::= R(a, x) \\ \mathcal{B}(a)_1(e) &::= \mu(e) \\ \mathbf{rfx}_{\mathcal{B}(a)}(x, r) &::= \mathbf{left_unit}_{\mu}(r),\end{aligned}$$

and the reflexivity term ρ provides for every $a : A_0$ a term of type $\mathcal{B}(a)_0$. Thus we have by Theorem 6.2.16 for every $a : A_0$ a unique morphism from the equifibrant replacement of $a : \mathbf{1} \rightarrow \mathcal{A}$ to \mathcal{B} over \mathcal{A} . Equivalently, we have a unique extension of $\bar{\mu}$ along $\mathcal{A} \rightarrow k(\mathbf{constr}_0)$ as asserted. \square

Corollary 6.2.21. *The loop space of the suspension ΣX of a pointed type X is the initial pointed type Y equipped with a pointed map*

$$X \rightarrow_* (Y \simeq Y).$$

In particular, the loop space of the $(n+1)$ -sphere is the initial pointed type Y equipped with a pointed map $\mathbb{S}^n \rightarrow (Y \simeq Y)$, or equivalently, an $(n+1)$ -loop $\Omega^{n+1}(\mathbf{BAut}(Y))$. Even more in particular, the loop space of the 2-sphere is the initial pointed type Y equipped with a homotopy $\mathbf{id}_Y \sim \mathbf{id}_Y$.

Corollary 6.2.22. *The type \mathbb{Z} of integers is the initial pointed type equipped with an automorphism.*

6.3 Equifibrant replacement for other homotopy colimits

6.3.1 Equifibrant replacement for diagrams over graphs

Recall that the colimit of a diagram \mathcal{D} over a reflexive graph \mathcal{A} is simply the reflexive coequalizer of the reflexive graph $\Sigma(\mathcal{A}, \mathcal{D})$.

Definition 6.3.1. Let $\tau : \mathcal{D}' \rightarrow \mathcal{D}$ be a natural transformation of diagrams over a reflexive graph \mathcal{A} .

- (i) We say that τ is a **weak equivalence** if it induces an equivalence

$$\mathbf{colim}(\tau) : \mathbf{colim}(\mathcal{D}') \rightarrow \mathbf{colim}(\mathcal{D}).$$

We write \mathcal{W} for the class of weak equivalences.

(ii) We say that τ is **étale** if the square

$$\begin{array}{ccc} \Sigma(\mathcal{A}, \mathcal{D}') & \longrightarrow & \Delta(\operatorname{colim}(\mathcal{D}')) \\ \tau \downarrow & & \downarrow \operatorname{colim}(\tau) \\ \Sigma(\mathcal{A}, \mathcal{D}) & \longrightarrow & \Delta(\operatorname{colim}(\mathcal{D})) \end{array}$$

is a pullback square of reflexive graphs. We write \mathcal{R} for the class of étale maps.

Theorem 6.3.2. *A natural transformation $\tau : \mathcal{D}' \rightarrow \mathcal{D}$ is étale if and only if it is cartesian.*

Proof. This follows directly by Theorem 6.2.2 and Proposition 3.5.9. \square

Theorem 6.3.3. *The pair $(\mathcal{W}, \mathcal{R})$ forms an orthogonal factorization system.*

Proof. This follows from Theorem 6.2.11. \square

6.4 Equifibrant replacement for sequential colimits

Definition 6.4.1. Let \mathcal{B} be a sequential family over \mathcal{A} , and consider $x : A_n$. We write $\mathcal{B}_n[x]$ for the type sequence

$$B_n(x) \longrightarrow B_{n+1}(f_n(x)) \longrightarrow B_{n+2}(f_{n+1}(f_n(x))) \longrightarrow \dots$$

Definition 6.4.2. Let \mathcal{B} be a sequential family over \mathcal{A} . We will define an equifibered sequential family $\square\mathcal{B}$ over \mathcal{A} equipped with a morphism

$$\operatorname{hom}_{\mathcal{A}}(\mathcal{B}, \square\mathcal{B})$$

Construction. We define

$$\square B : \prod_{(n:\mathbb{N})} A_n \rightarrow \mathcal{U}$$

by $\square B_n(x) := \operatorname{colim} \mathcal{B}_n[x]$. Next, we have to construct a fiberwise equivalence

$$g_\infty : \prod_{(n:\mathbb{N})} \prod_{(x:A_n)} \square B_n(x) \simeq \square B_{n+1}(f_n(x))$$

Note that we have the natural transformation

$$\begin{array}{ccccccc} B_n(x) & \longrightarrow & \dots & \longrightarrow & B_{k+n}(f^k(x)) & \longrightarrow & B_{(k+1)+n}(f^{k+1}(x)) & \longrightarrow & \dots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ B_{n+1}(x) & \longrightarrow & \dots & \longrightarrow & B_{k+(n+1)}(f^k(f(x))) & \longrightarrow & B_{(k+1)+(n+1)}(f^{k+1}(f(x))) & \longrightarrow & \dots \end{array}$$

Now we observe that there are identifications

$$(k + (n + 1), f^k(f(x))) = ((k + 1) + n, f^{k+1}(x)),$$

which we may use to construct equivalences going up diagonally in each naturality square. An induction argument reveals that each square of the form

$$\begin{array}{ccc} B_{k+(n+1)}(f^k(f(x))) & \longrightarrow & B_{(k+1)+(n+1)}(f^{k+1}(f(x))) \\ \downarrow & & \downarrow \\ B_{(k+1)+n}(f^{k+1}(x)) & \longrightarrow & B_{(k+2)+n}(f^{k+2}(x)) \end{array}$$

commutes, from which we obtain the desired equivalence.¹ \square

Theorem 6.4.3. *The equifibered family $\square\mathcal{B}$ over \mathcal{A} satisfies the universal property that*

$$\text{hom}_{\mathcal{A}}(\square\mathcal{B}, \mathcal{E}) \rightarrow \text{hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{E})$$

is an equivalence for any equifibered family \mathcal{E} over \mathcal{A} .

Proof. This fact is formalized in [14]. An ‘informalized’ proof is work in progress. \square

Corollary 6.4.4. *Let \mathcal{B} be a sequential family over \mathcal{A} . Then we have a commuting triangle*

$$\begin{array}{ccc} \text{colim}(\Sigma(\mathcal{A}, \mathcal{B})) & \longrightarrow & \Sigma_{(x:A_\infty)} B_\infty(x) \\ & \searrow \text{colim}(\text{pr}_1) & \swarrow \text{pr}_1 \\ & & A_\infty \end{array}$$

in which the top map is an equivalence.

Proof. Since $\square\mathcal{B}$ and EqFB both satisfy the universal property of the equifibrant replacement of \mathcal{B} , they are the same sequences over \mathcal{A} . It follows that the square

$$\begin{array}{ccc} \Sigma(\mathcal{A}, \square\mathcal{B}) & \longrightarrow & \Delta(\text{colim}(\Sigma(\mathcal{A}, \mathcal{B}))) \\ \text{pr}_1 \downarrow & & \downarrow \Delta(\text{colim}(\mathcal{A})) \\ \mathcal{A} & \xrightarrow{\text{seq_in}} & \Delta(\text{colim}(\mathcal{A})) \end{array}$$

is a pullback square of type sequences. We conclude that the top arrow is a colimiting cocone, so the result follows. \square

Corollary 6.4.5. *Consider a type sequence*

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Then the canonical map

$$\text{colim}(f^n(x) = f^n(y)) \rightarrow (\text{seq_in}_0(x) = \text{seq_in}_0(y))$$

is an equivalence for every $x, y : A_0$.

¹A formalization of this argument appears in https://github.com/cmu-phil/Spectral/blob/master/colimit/seq_colim.olean

Theorem 6.4.6. *Consider a sequence*

$$\begin{array}{ccccccc}
 & & C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots \\
 & \swarrow & | & & \swarrow & | & \swarrow & | & \\
 A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\
 X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & &
 \end{array}$$

of pullback squares. Then the sequential colimit

$$\begin{array}{ccc}
 C_\infty & \longrightarrow & B_\infty \\
 \downarrow & & \downarrow \\
 A_\infty & \longrightarrow & X_\infty
 \end{array}$$

is again a pullback square.

Proof. Since sequential colimits commute with Σ and identity types, they commute with pullbacks. \square

Corollary 6.4.7. *Consider a sequence of fiber sequences*

$$\begin{array}{ccccccc}
 F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots
 \end{array}$$

in which all maps and homotopies are assumed to be pointed. Then the colimit

$$F_\infty \hookrightarrow E_\infty \twoheadrightarrow B_\infty$$

is again a fiber sequence.

Corollary 6.4.8. *Consider a type sequence*

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

of pointed types (and pointed maps between them). Then the canonical map

$$\text{colim}(\Omega(A_n)) \rightarrow \Omega(A_\infty)$$

is an equivalence.

Proposition 6.4.9. *Consider a type sequence*

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

If each A_n is k -truncated, then so is the sequential colimit A_∞ .

Proof. We prove the claim by induction on $k \geq -2$. The base case is trivial, and the inductive step follows since sequential colimits commute with identity types. \square

Theorem 6.4.10. *Consider a type sequence*

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Then the canonical map

$$\operatorname{colim}_n \|A_n\|_k \rightarrow \|A_\infty\|_k$$

is an equivalence.

Theorem 6.4.11. *Consider a type sequence*

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Then the canonical map

$$\operatorname{colim}_n (\pi_k(A_n)) \rightarrow \pi_k(A_\infty)$$

is a group isomorphism, for any $k \geq 1$.

Chapter 7

Compact types

In this chapter we introduce the notion of compact type, and we show that the compact types are closed under finite coproducts, Σ -types, finite products, pushouts, reflexive coequalizers, and retracts. In particular it follows that the spheres are compact.

In §7.2 we show that if P and Q are families of compact types, then the subuniverse of F -local types is reflective. Of course, one might impose stronger assumptions on the type theory so that the subuniverse of F -local types is reflective for any family F of maps (see for instance [28]). The setting of this dissertation is that of univalent type theory in which the universes are closed under homotopy pushouts, so we have to find a different way to define localizations.

The idea is to approximate the F -localization of a type X by iterated ‘quasi F -local extensions’ of X , by which we mean a map $l : X \rightarrow Y$ equipped with a diagonal filler for the square

$$\begin{array}{ccc} X^{Q_i} & \xrightarrow{l \circ -} & Y^{Q_i} \\ - \circ F_i \downarrow & & \downarrow - \circ F_i \\ X^{P_i} & \xrightarrow{l \circ -} & Y^{P_i}, \end{array}$$

for each $i : I$. We show in Proposition 7.2.4 that, provided that F is a family of maps between compact types, the sequential colimit of type sequence

$$X_0 \xrightarrow{l_0} X_1 \xrightarrow{l_1} X_2 \xrightarrow{l_2} \dots$$

is F -local if each l_n has the structure of a quasi F -local extension. We show that there is an initial quasi F -local extension $l : X \rightarrow QL_F X$, for every type X , and in Theorem 7.2.10 we show that the sequential colimit $X \rightarrow QL_F^\infty X$ is an F -localization. We establish in Theorem 7.2.10 that if F is a family of maps between compact types, then the subuniverse of F -local types is a reflective subuniverse. Furthermore, if A is a family of compact types, then the subuniverse of A -null types is a modality.

Dan Christensen pointed out that this construction of F -localization already appears in section 1.B of [18], where arbitrary localizations are constructed by iterating the initial quasi F -local extension transfinitely many times. We stick to the case ω since larger ordinals are not yet as well understood in homotopy type theory. A minor note is that in Theorem B.5 Dror Farjoun only establishes that the transfinite colimit $QL_F^\kappa X$ localizes the homotopy groups. Our proofs are therefore different than Dror Farjoun’s.

7.1 Compact types

Definition 7.1.1. Consider a type sequence $\mathcal{A} \equiv (A, f)$, and let X be a type. We define the type sequence \mathcal{A}^X to consist of

$$A_0^X \xrightarrow{f_0^{\circ-}} A_1^X \xrightarrow{f_1^{\circ-}} A_2^X \xrightarrow{f_2^{\circ-}} \dots$$

Furthermore, given a natural transformation $\tau : \text{Seq}(\mathcal{A}, \mathcal{B})$ of type sequences, we define the natural transformation τ^X to consist of

$$\begin{array}{ccccccc} A_0^X & \xrightarrow{f_0^{\circ-}} & A_1^X & \xrightarrow{f_1^{\circ-}} & A_2^X & \xrightarrow{f_2^{\circ-}} & \dots \\ \tau_0^{\circ-} \downarrow & & \tau_1^{\circ-} \downarrow & & \tau_2^{\circ-} \downarrow & & \\ B_0^X & \xrightarrow{g_0^{\circ-}} & B_1^X & \xrightarrow{g_1^{\circ-}} & B_2^X & \xrightarrow{g_2^{\circ-}} & \dots \end{array}$$

where the naturality squares commute by whiskering the naturality squares of τ .

In particular, if $\tau : \text{Seq}(\mathcal{A}, \Delta Y)$ is a cocone with vertex Y , then $\tau^X : \text{Seq}(\mathcal{A}^X, (\Delta Y)^X)$ is a cocone with vertex Y^X , since $(\Delta Y)^X \equiv \Delta(Y^X)$.

Definition 7.1.2. We call a type X (**sequentially**) **compact** if for any type sequence

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

the map

$$\text{seq_ind}(\text{constr}(\mathcal{A})^X) : \text{colim}(X \rightarrow A_n) \rightarrow (X \rightarrow A_\infty)$$

is an equivalence.

Example 7.1.3. The empty type is compact, simply because X^0 is contractible for any type X , and a sequential colimit of contractible types is contractible.

Example 7.1.4. The unit type is compact because $\lambda x. \text{const}_x : X \rightarrow X^1$ is an equivalence for any type X . Thus we obtain a natural equivalence

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ A_0^1 & \longrightarrow & A_1^1 & \longrightarrow & A_2^1 & \longrightarrow & \dots \end{array}$$

Thus we obtain a commuting triangle

$$\begin{array}{ccc} & A_\infty & \\ & \swarrow & \searrow \\ \text{colim}(A_n^1) & \longrightarrow & A_\infty^1 \end{array}$$

where two out of three maps are equivalences. Therefore the unit type is compact by the 3-for-2 property.

We will show in the remainder of this section that the compact types are closed under finite coproducts, Σ -types (and therefore also finite products), pushouts, reflexive coequalizers, and retracts. In the following example we note that compact types are not closed under pullbacks, and are not closed under exponentiation either.

Remark 7.1.5. The compact types are not closed under identity types, because \mathbb{Z} is not compact and $\mathbb{Z} \simeq \Omega(\mathbb{S}^1)$. To see that \mathbb{Z} is not compact, consider the sequence $A_n \equiv \mathbb{N}_{-n}$, where \mathbb{N}_{-n} is a version of the natural numbers of starting at $-n$ and $A_n \rightarrow A_{n+1}$ is the inclusion map. The colimit of A_n is \mathbb{Z} and the image of the map $\text{colim}(\mathbb{Z} \rightarrow A_n) \rightarrow (\mathbb{Z} \rightarrow \mathbb{Z})$ is the set of maps that are bounded from below. Thus, it is not surjective.

Also, the compact types are not closed under exponentiation. We have the equivalence $(\mathbb{S}^1 \rightarrow \mathbb{S}^1) \simeq \mathbb{S}^1 \times \mathbb{Z}$, which is not compact.

Lemma 7.1.6. *Finite coproducts of compact types are compact.*

Proof. We have already seen that the empty type is compact. Thus it remains to show that compact types are closed under disjoint sums.

Suppose that X and Y are compact. Then we have a commuting diagram of the form

$$\begin{array}{ccccc} \text{colim}(\mathcal{A}^{X+Y}) & \longrightarrow & \text{colim}(\mathcal{A}^X \times \mathcal{A}^Y) & \longrightarrow & \text{colim}(\mathcal{A}^X) \times \text{colim}(\mathcal{A}^Y) \\ \downarrow & & & & \downarrow \\ A_\infty^{X+Y} & \longrightarrow & & \longrightarrow & A_\infty^X \times A_\infty^Y \end{array}$$

where all but one of the maps are equivalences. Thus the remaining map is an equivalence too. \square

Definition 7.1.7. Let $\mathcal{A} : X \rightarrow \text{Seq}$ be an indexed type sequence. Then we define the type sequence $\prod_{(x:X)} \mathcal{A}(x)$ to consist of

$$\prod_{(x:X)} A_0(x) \xrightarrow{\lambda h. \lambda x. f_0(x, h(x))} \prod_{(x:X)} A_1(x) \xrightarrow{\lambda h. \lambda x. f_1(x, h(x))} \dots$$

Proposition 7.1.8. *Let $\mathcal{A} : X \rightarrow \text{Seq}$ be an indexed type sequence, indexed by a compact type X . Then the canonical map*

$$\text{colim}\left(\prod_{(x:X)} A_n(x)\right) \rightarrow \left(\prod_{(x:X)} A_\infty(x)\right)$$

is an equivalence.

Proof. Consider the commuting diagram

$$\begin{array}{ccccc} \text{colim}\left(\prod_{(x:X)} \mathcal{A}(x)\right) & \longrightarrow & \prod_{(x:X)} A_\infty(x) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow & & \downarrow \text{const}_{\text{id}_X} \\ \text{colim}\left(X \rightarrow \sum_{(x:X)} A_n(x)\right) & \longrightarrow & \left(X \rightarrow \sum_{(x:X)} A_\infty(x)\right) & \xrightarrow{\text{pr}_1 \circ -} & (X \rightarrow X). \end{array}$$

The square on the right and the outer rectangle are pullback squares. Hence the square on the left is a pullback square. Now observe that the type $\sum_{(x:X)} A_\infty(X)$ is the sequential colimit of the type sequence $\sum_{(x:X)} A_n(x)$, by Corollary 6.4.4. Therefore it follows by the assumption that X is compact, that the bottom map in the left square is an equivalence. We conclude that the top map in the left square is an equivalence, proving the claim. \square

Corollary 7.1.9. *Suppose X is compact, and $Y(x)$ is compact for each $x : X$. Then the total space $\sum_{(x:X)} Y(x)$ is also compact.*

Proof. Let \mathcal{A} be a type sequence. Then we have the following commuting pentagon

$$\begin{array}{ccc}
 \text{colim}\left(\mathcal{A}^{\sum_{(x:X)} Y(x)}\right) & \longrightarrow & A_\infty^{\sum_{(x:X)} Y(x)} \\
 \swarrow & & \searrow \\
 \text{colim}\left(\prod_{(x:X)} \mathcal{A}^{Y(x)}\right) & & \prod_{(x:X)} A_\infty^{Y(x)} \\
 \searrow & & \swarrow \\
 & \prod_{(x:X)} \text{colim}\left(\mathcal{A}^{Y(x)}\right) &
 \end{array}$$

in which all but the top map are known to be equivalences. Hence it follows that the top map is an equivalence, which shows that $\sum_{(x:X)} Y(x)$ is compact. \square

Corollary 7.1.10. *Compact types are closed under finite products.*

Proposition 7.1.11. *Compact types are closed under pushouts.*

Proof. Consider a span $\mathcal{S} \equiv (S, f, g)$ from X to Y , where X , Y , and S are assumed to be compact, and let \mathcal{A} be a type sequence. Then we have the commuting pentagon

$$\begin{array}{ccc}
 \text{colim}\left(\mathcal{A}^{X \sqcup^{\mathcal{S}} Y}\right) & \longrightarrow & A_\infty^{X \sqcup^{\mathcal{S}} Y} \\
 \swarrow & & \searrow \\
 \text{colim}\left(\mathcal{A}^X \times_{\mathcal{A}^{\mathcal{S}}} \mathcal{A}^Y\right) & & A_\infty^X \times_{A_\infty^{\mathcal{S}}} A_\infty^Y \\
 \searrow & & \swarrow \\
 & \text{colim}(\mathcal{A}^X) \times_{\text{colim}(\mathcal{A}^{\mathcal{S}})} \text{colim}(\mathcal{A}^Y) &
 \end{array}$$

In this diagram, the downwards maps from the top left and from the top right are equivalences by the universal property of pushouts. The downwards map to the left side to the bottom is an equivalence since sequential colimits preserve pullbacks by Theorem 6.4.6. The map from the bottom to the right side is an equivalence by the assumption that X , Y , and S are all compact types. Therefore we conclude that the top map is an equivalence, which proves the claim. \square

Corollary 7.1.12. *Compact types are closed under joins, suspensions, and wedges. Furthermore, the cofiber of a map between compact types is again compact. In particular, all the spheres and the finite dimensional real and complex projective spaces are compact, and the smash product of compact pointed types is again compact.*

Definition 7.1.13. A reflexive graph \mathcal{A} is said to be compact if its types of vertices and (the total space of) edges are compact.

Corollary 7.1.14. *The reflexive coequalizer of a compact reflexive graph is again compact.*

Proof. This follows immediately from Propositions 3.3.7 and 7.1.11. \square

Proposition 7.1.15. *Compact types are closed under retracts.*

Proof. Suppose that Y is a retract of a compact type X , i.e. we have $i : Y \rightarrow X$ and $r : X \rightarrow Y$ such that $r \circ i = \text{id}_Y$. Note that for any type Z , the type Z^Y is a retract of Z^X , since we have the section-retraction pair

$$Z^Y \xrightarrow{-\circ r} Z^X \xrightarrow{-\circ i} Z^Y.$$

It follows that for any type sequence \mathcal{A} we have a section-retraction pair

$$\begin{array}{ccccc} \text{colim}(\mathcal{A}^Y) & \longrightarrow & \text{colim}(\mathcal{A}^X) & \longrightarrow & \text{colim}(\mathcal{A}^Y) \\ \downarrow & & \downarrow & & \downarrow \\ A_\infty^Y & \longrightarrow & A_\infty^X & \longrightarrow & A_\infty^Y \end{array}$$

of morphisms. Note that the downward morphism in the middle is an equivalence by the assumption that X is compact. Recall from Theorem 4.7.4 of [33] that a retract of an equivalence is again an equivalence, so the claim follows. \square

7.2 Localizing at maps between compact types

In this section, we assume to have $P, Q : I \rightarrow \mathcal{U}$ and a family of maps $F : \prod_{(i:I)} P(i) \rightarrow Q(i)$.

Recall type of diagonal fillers for a commuting square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{j} & Y \end{array}$$

with $H : j \circ f \sim g \circ i$ is defined to be the fiber of the gap map

$$\text{gap} : B^X \rightarrow B^A \times_{Y^A} Y^X$$

at the point $(i, j, \text{eq_htpy}(H))$.

Definition 7.2.1. A **quasi F -local extension** of X consists of a map $l : X \rightarrow Y$ equipped with a diagonal filler for the commuting square

$$\begin{array}{ccc} X^{Q_i} & \xrightarrow{l \circ -} & Y^{Q_i} \\ - \circ F_i \downarrow & & \downarrow - \circ F_i \\ X^{P_i} & \xrightarrow{l \circ -} & Y^{P_i}, \end{array}$$

for each $i : I$.

Lemma 7.2.2. *For any map $l : X \rightarrow Y$ into an F -local type Y , the type of diagonal fillers for the commuting square*

$$\begin{array}{ccc} X^{Q_i} & \xrightarrow{l \circ -} & Y^{Q_i} \\ - \circ F_i \downarrow & & \downarrow - \circ F_i \\ X^{P_i} & \xrightarrow{l \circ -} & Y^{P_i}, \end{array}$$

is contractible. In particular, any such map is a quasi F -local extension.

Proof. If Y is F -local, then the map $F_i^* : Y^{Q_i} \rightarrow Y^{P_i}$ is an equivalence, for any $i : I$. Since equivalences are right orthogonal to any map, the claim follows. \square

Lemma 7.2.3. *Suppose $h : \text{Seq}(\mathcal{A}, \mathcal{B})$ is a natural transformation of type sequences equipped with the structure of a diagonal filler for each naturality square*

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & A_{n+1} \\ h_n \downarrow & \nearrow & \downarrow h_{n+1} \\ B_n & \xrightarrow{g_n} & B_{n+1}. \end{array}$$

Then the map $h_\infty : A_\infty \rightarrow B_\infty$ is an equivalence.

Proof. Let us write $j_n : B_n \rightarrow A_{n+1}$ for the diagonal fillers, which come equipped with homotopies

$$\begin{aligned} J_n &: f_n \sim j_n \circ h_n \\ K_n &: h_{n+1} \circ j_n \sim g_n \\ L_n &: (h_{n+1} \cdot J_n) \cdot (K_n \cdot h_n) \sim H_n \end{aligned}$$

Note that j is a natural transformation from \mathcal{B} to the shifted sequence $\text{shift}(\mathcal{A})$, with the homotopies $(f_{n+1} \cdot K_n) \cdot (J_{n+1} \cdot g_n) : f_{n+1} \circ j_n \sim j_{n+1} \circ g_n$ filling the naturality squares.

We claim that the composite $j \circ h$ is just the natural transformation $f : \mathcal{A} \rightarrow \text{shift}(\mathcal{A})$. Indeed, we have the homotopy $J_n : j_n \circ h_n \sim f_n$. Moreover, the type of $j_{n+1} \cdot L_n$

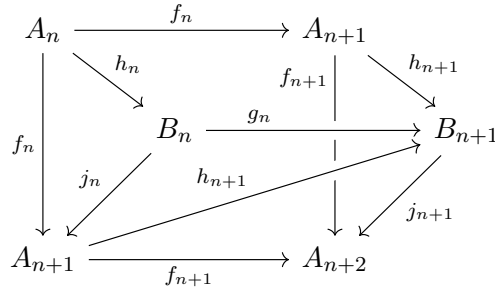
$$(j_{n+1} \cdot h_{n+1} \cdot J_n) \cdot (j_{n+1} \cdot K_n \cdot h_n) \sim j_{n+1} \cdot H_n$$

is equivalent to the type of homotopies

$$(J_{n+1} \cdot f_n) \cdot (j_{n+1} \cdot h_{n+1} \cdot J_n) \cdot (j_{n+1} \cdot K_n \cdot h_n) \cdot (j_{n+1} \cdot H_n)$$

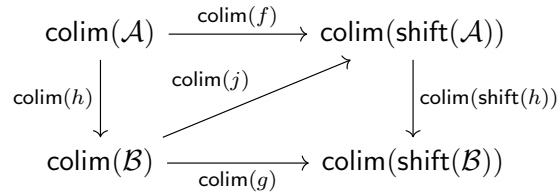
$$\sim \text{htpy_refl}_{(f_{n+1} \circ f_n)} \cdot (J_{n+1} \cdot f_n),$$

filling the diagram



Therefore we obtain the coherence of the naturality squares needed to conclude that $f = j \circ h$ as natural transformations.

Likewise, the composite $\text{shift}(h) \circ j$ is just the natural transformation $g : \mathcal{B} \rightarrow \text{shift}(\mathcal{B})$. Therefore we obtain a commuting diagram



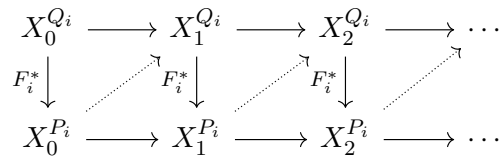
Since both $colim(f)$ and $colim(g)$ are equivalences, it follows by the 6-for-2 property of equivalences (see Exercise 4.5 of [33]), that the remaining maps are equivalences. In particular $colim(h)$ is an equivalence. \square

Proposition 7.2.4. *Suppose F is a family of maps between compact types, and let*

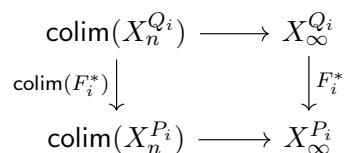
$$X_0 \xrightarrow{l} X_1 \xrightarrow{l} X_2 \xrightarrow{l} \dots$$

be a sequence of types in which each X_{n+1} is a quasi F -local extension of X_n . Then the sequential colimit X_∞ is F -local.

Proof. We have a natural transformation $F_i^* : \text{Seq}(\mathcal{X}^{Q_i}, \mathcal{X}^{P_i})$ for which each naturality square comes equipped with a diagonal filler



by the assumption that each $l_n : X_n \rightarrow X_{n+1}$ is a quasi F -local extension. Therefore $colim(F_i^*)$ is an equivalence. Note that we have the commuting square



in which the top and bottom maps are equivalences by the assumption that P_i and Q_i are compact. Therefore we conclude that $F_i^* : X_\infty^{Q_i} \rightarrow X_\infty^{P_i}$ is an equivalence. It follows that X_∞ is F -local. \square

Our goal is now to construct the initial quasi F -local extension of a type X , and use it to show that the subuniverse of F -local types is reflective. For any $i : I$, we have the maps

$$\begin{aligned} F_i &: P_i \rightarrow Q_i \\ F_i^* &: X^{Q_i} \rightarrow X^{P_i}. \end{aligned}$$

In the following characterization of quasi F -local extensions we use the pushout-product $F_i^* \square F_i$, which is defined as follows

$$\begin{array}{ccc} X^{Q_i} \times P_i & \longrightarrow & X^{Q_i} \times Q_i \\ \downarrow & & \downarrow \\ X^{P_i} \times P_i & \longrightarrow & (X^{P_i} \times P_i) \sqcup^{(X^{Q_i} \times P_i)} (X^{Q_i} \times Q_i) \\ & \searrow & \swarrow \\ & & X^{P_i} \times Q_i \end{array}$$

$F_i^* \square F_i$

by the universal property of pushouts. Also note that the outer square in the diagram

$$\begin{array}{ccc} X^{Q_i} \times P_i & \longrightarrow & X^{Q_i} \times Q_i \\ \downarrow & & \downarrow \\ X^{P_i} \times P_i & \longrightarrow & (X^{P_i} \times P_i) \sqcup^{(X^{Q_i} \times P_i)} (X^{Q_i} \times Q_i) \\ & \searrow & \swarrow \\ & & X \end{array}$$

ev ε_i

commutes, so we get a map ε_i as indicated.

Lemma 7.2.5. *For any type $l : X \rightarrow Y$, the type of I -indexed families of diagonal fillers for the squares*

$$\begin{array}{ccc} X^{Q_i} & \xrightarrow{l \circ -} & Y^{Q_i} \\ - \circ F_i \downarrow & & \downarrow - \circ F_i \\ X^{P_i} & \xrightarrow{l \circ -} & Y^{P_i} \end{array}$$

is equivalent to the type of morphisms $j : \left(\sum_{(i:I)} X^{P_i} \times Q_i \right) \rightarrow Y$ equipped with a homotopy witnessing that the square

$$\begin{array}{ccc} \sum_{(i:I)} \left((X^{P_i} \times P_i) \sqcup^{(X^{Q_i} \times P_i)} (X^{Q_i} \times Q_i) \right) & \xrightarrow{\text{tot}(\lambda i. F_i^* \square F_i)} & \sum_{(i:I)} X^{P_i} \times Q_i \\ \lambda(i,t). \varepsilon_i(t) \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes.

Proof. It suffices to show that for each $i : I$, the type of diagonal fillers in the first description is equivalent to the type of maps $j : X^{P_i} \times Q_i \rightarrow Y$ equipped with a homotopy witnessing that the square

$$\begin{array}{ccc} \left((X^{P_i} \times P_i) \sqcup^{(X^{Q_i} \times P_i)} (X^{Q_i} \times Q_i) \right) & \xrightarrow{F_i^* \square F_i} & X^{P_i} \times Q_i \\ \varepsilon_i \downarrow & & \downarrow \\ X & \xrightarrow{\quad\quad\quad} & Y \end{array}$$

commutes. The type $X^{P_i} \rightarrow Y^{Q_i}$ is equivalent to the type $X^{P_i} \times Q_i \rightarrow Y$ by adjointness.

By the dependent universal property of pushouts established in Proposition 2.1.6, the type $l \circ \varepsilon_i \sim j \circ (F_i^* \square F_i)$ is equivalent to the type of triples (K, L, M) consisting of

$$\begin{aligned} K &: \prod_{(f:P_i \rightarrow X)} \prod_{(p:P_i)} l(f(p)) = j(f, F_i(p)) \\ L &: \prod_{(g:Q_i \rightarrow X)} \prod_{(q:Q_i)} l(g(q)) = j(g \circ F_i, q) \\ M &: \prod_{(g:Q_i \rightarrow X)} \prod_{(p:P_i)} K(g \circ F_i, p) = L(g, F_i(p)). \end{aligned}$$

The type of such quadruples (j, K, L, M) is also equivalent to the fiber of the gap map

$$(X^{P_i} \rightarrow Y^{Q_i}) \rightarrow (X^{Q_i} \rightarrow Y^{Q_i}) \times_{(X^{Q_i} \rightarrow Y^{P_i})} (X^{P_i} \rightarrow Y^{P_i})$$

at $(l^{Q_i}, l^{P_i}, \text{htpy_refl})$. □

Definition 7.2.6. For any type X , we define the initial quasi F -local extension $l : X \rightarrow QL_F X$ of X to be the pushout

$$\begin{array}{ccc} \sum_{(i:I)} \left((X^{P_i} \times P_i) \sqcup^{(X^{Q_i} \times P_i)} (X^{Q_i} \times Q_i) \right) & \xrightarrow{F_i^* \square F_i} & \sum_{(i:I)} X^{P_i} \times Q_i \\ \lambda(i,t). \varepsilon_i(t) \downarrow & & \downarrow \\ X & \xrightarrow{\quad\quad\quad l \quad\quad\quad} & QL_F X \end{array}$$

Remark 7.2.7. The initial quasi F -local extension of a type X is initial by the universal property of the pushout and the characterization of Lemma 7.2.5.

Example 7.2.8. Consider the family of maps consisting just of the terminal projection $\mathbf{2} \rightarrow \mathbf{1}$. We sketch an argument that the quasi $\mathbf{2}$ -local extension of X is equivalent to $X * X$, i.e. that we have a pushout square

$$\begin{array}{ccc} (X^2 \times \mathbf{2}) \sqcup^{X \times \mathbf{2}} X & \longrightarrow & X^2 \\ \downarrow & & \downarrow \\ X & \longrightarrow & X * X \end{array}$$

To see this, we note first that the pushouts

$$\begin{array}{ccc} S & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \sqcup^S B \end{array} \quad \begin{array}{ccc} S + S & \longrightarrow & S \\ \downarrow & & \downarrow \\ A + B & \longrightarrow & (A + B) \sqcup^{(S+S)} S \end{array}$$

are equivalent, for any span $A \leftarrow S \rightarrow B$. Therefore it follows that the pushouts

$$\begin{array}{ccc} X & \longrightarrow & X^2 \\ \downarrow & & \downarrow \\ X^2 & \longrightarrow & X^2 \sqcup^X X^2 \end{array} \quad \begin{array}{ccc} X \times \mathbf{2} & \longrightarrow & X \\ \downarrow & & \downarrow \\ X^2 \times \mathbf{2} & \longrightarrow & (X^2 \times \mathbf{2}) \sqcup^{(X \times \mathbf{2})} X \end{array}$$

are equivalent. Therefore it suffices to compute the pushout

$$\begin{array}{ccc} X^2 \sqcup^X X^2 & \xrightarrow{\nabla_{\delta_X}} & X^2 \\ [\pi_1, \pi_2, \text{htpy_refl}_{\text{id}_X}] \downarrow & & \downarrow \\ X & \longrightarrow & X \sqcup^{(X^2 \sqcup^X X^2)} X^2. \end{array}$$

Now we observe that for any cube

$$\begin{array}{ccccc} & & A_{111} & & \\ & \swarrow & \downarrow & \searrow & \\ A_{110} & & A_{101} & & A_{011} \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ A_{100} & & A_{010} & & A_{001} \\ & \swarrow & \downarrow & \searrow & \\ & & A_{000} & & \end{array}$$

the following are equivalent:

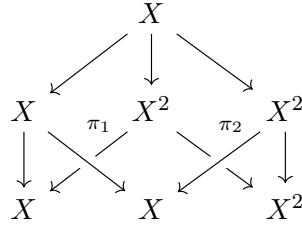
- (i) The cube is cocartesian.
- (ii) The the square

$$\begin{array}{ccc} A_{011} \sqcup^{A_{111}} A_{101} & \longrightarrow & A_{001} \\ \downarrow & & \downarrow \\ A_{010} \sqcup^{A_{110}} A_{100} & \longrightarrow & A_{000} \end{array}$$

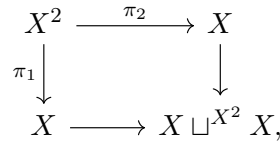
is cocartesian.

We cite **[Munson]** for this observation, and the analogous result in homotopy type theory is work in progress. Using the above equivalence, it follows that the pushout $X^2 \sqcup^X X^2$

is the colimit of the diagram



Furthermore, by the same equivalence we see that another way of computing this colimit is as the following pushout



which is the join $X * X$.

Proposition 7.2.9. *If X is F -local, then $l : X \rightarrow QL_F X$ is an equivalence.*

Proof. If X is F -local, then each F_i^* is an equivalence. It follows that the pushout-product $F_i^* \square F_i$ is an equivalence. Since pushouts of equivalences are equivalences, the claim follows. \square

Theorem 7.2.10. *For any type X and any family $F : \prod_{(a:A)} P(a) \rightarrow Q(a)$ of maps between compact types, the subuniverse of F -local types is reflective.*

Proof. We define the localization $\eta : X \rightarrow LX$ as the sequential colimit

$$X \longrightarrow QL_F X \longrightarrow QL_F^2 X \longrightarrow \dots$$

Since each $l : QL_F^n X \rightarrow QL_F^{n+1} X$ has the structure of a quasi F -local extension, it follows by Proposition 7.2.4 that LX is F -local. It remains to show that for any F -local type Y , the precomposition map

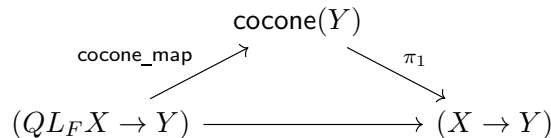
$$\eta^* : (LX \rightarrow Y) \rightarrow (X \rightarrow Y)$$

is an equivalence.

We first show that the map

$$l^* : (QL_F X \rightarrow Y) \rightarrow (X \rightarrow Y)$$

is an equivalence. To see this, note that we have a commuting triangle



We note that by Lemma 7.2.5, the fiber of $\pi_1 : \text{cocone}(Y) \rightarrow (X \rightarrow Y)$ is equivalent to the type of diagonal fillers of the square

$$\begin{array}{ccc} X^{Q_i} & \xrightarrow{l \circ -} & Y^{Q_i} \\ - \circ F_i \downarrow & & \downarrow - \circ F_i \\ X^{P_i} & \xrightarrow{l \circ -} & Y^{P_i}, \end{array}$$

This type of diagonal fillers is contractible by Lemma 7.2.2, since Y is assumed to be F -local. In other words, the map $\pi_1 : \text{cocone}(Y) \rightarrow (X \rightarrow Y)$ is an equivalence. Since $\text{cocone_map} : (QL_F X \rightarrow Y) \rightarrow \text{cocone}(Y)$ is an equivalence by the universal property of $QL_F X$, we conclude by the 3-for-2 property that the map

$$l^* : (QL_F X \rightarrow Y) \rightarrow (X \rightarrow Y)$$

is an equivalence.

We conclude that any map $g : X \rightarrow Y$ extends uniquely to a cocone on the type sequence

$$X \longrightarrow QL_F X \longrightarrow QL_F^2 X \longrightarrow \dots,$$

and therefore it follows that $\eta^* : (LX \rightarrow Y) \rightarrow (X \rightarrow Y)$ is an equivalence. \square

Corollary 7.2.11. *For any family $N : I \rightarrow \mathcal{U}$ of compact types, the subuniverse of N -null types is a modality.*

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