Continuity and Logical Completeness

An application of sheaf theory and topoi

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Abstract

The notion of a continuously variable quantity can be regarded as a generalization of that of a particular (constant) quantity, and the properties of such quantities are then akin to, and derived from, the properties of constants. For example, the continuous, real-valued functions on a topological space behave like the field of real numbers in many ways, but instead form a ring.

Topos theory permits one to apply this same idea to logic, and to consider continuously variable sets (sheaves). In this expository paper, such applications are explained to the non-specialist. Some recent results are mentioned, including a new completeness theorem for higher-order logic.

The *main argument* of this paper is as follows:

1. The distinction between the Particular and the Abstract General is present in that between the Constant and the Continuously Variable. More specially, *continuous variation is a form of abstraction.*

2. Higher-order logic (HOL) can be presented algebraically. As a consequence of this fact, it has *continuously variable models.*

3. Variable models are classical mathematical objects; namely, *sheaves.*

4. HOL is *complete* with respect to such continuously variable models. Standard semantics appears thereby as the constant case of "no variation." In this sense, *HOL is the logic of continuous variation.*

The argument will be developed in four sections: (i) the algebraic formulation of HOL is given; (ii) rings of real-valued functions are considered as an example of variable structure; (iii) the idea of continuously variable sets is then discussed; and finally, (iv) it is explained how HOL is the logic of continuous variation

1 Algebraic logic

Categorical logic can be seen as the successful completion of the program of "algebraicizing" logic begun in the 19-century. Everyone is familiar with the boolean algebra approach to propositional logic, but the treatment of quantification in particular has posed a serious obstacle to extending the algebraic treatment. The categorical treatment of quantifiers as adjoint functors—due to F.W. Lawvere in the 1960s—solved this problem, although it has been little appreciated until very recently.

Category theory is of course a branch of abstract algebra, but the sense in which the categorical treatment of logic is "algebraic" is deeper than just that. Rather, it is the recognition of the quantifiers—and indeed all of the logical operations—as *adjoint functors* that makes logic algebraic. For it is a general fact about adjoints that they always admit an algebraic description, in a definite, technical sense. This is the same fact that makes possible the equational description of e.g. cartesian products and pairing. Figure 1 shows the (two-way) rules of inference for the first-order logical operations expressed as adjoints.[1]

HOL also includes quantification over "higher types" of relations, functions, properties of functions, and so on. Figure 2 indicates the basic ingredients of algebraic HOL, as it results from the adjoint analysis of these operations. The *axioms* consist of a handful of equations of the sort indicated, and the *rules of inference* are essentially substitution of equals for equals, as in elementary algebra.

It may be noted that these are *all* of the logical operations required; the first-order operations are definable from these, as suggested in figure 3 (which also indicates how even fewer would still suffice). The adjoint rules of figure 1 can then be proven.

In categorical logic we extend the treatment of propositional logic as a

[1]The quantifier rules require the variable x not occur freely in ϑ. For a full statement see [4, 2]

Figure 1: Adjoint rules for FOL

$$\bot \vdash \varphi \qquad\qquad \varphi \vdash \top$$

$$\frac{\varphi \vdash \vartheta \text{ and } \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta} \qquad\qquad \frac{\vartheta \vdash \varphi \text{ and } \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi}$$

$$\frac{\vartheta \wedge \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

$$\frac{\varphi(x) \vdash \vartheta}{\exists x.\varphi(x) \vdash \vartheta} \qquad\qquad \frac{\vartheta \vdash \varphi(x)}{\vartheta \vdash \forall x.\varphi(x)}$$

Figure 2: Algebraic formulation of HOL

Types: $X \times Y,\ Y^X,\ \mathcal{P}(X),\ \mathcal{P}$
Terms: $\langle s, t\rangle,\ \pi_1 t,\ \pi_2 t,\ \lambda x.t,\ t(s),\ \{x \mid \varphi\}$
Formulas: $s = t,\ s \in t$
Axioms: equations such as:

$$\begin{aligned} \pi_1\langle s, t\rangle &= s \\ \lambda x.t(x) &= t \\ x \in \{x \mid \varphi\} &= \varphi \end{aligned}$$

Rules: substitution of equals for equals

Figure 3: Logical operations defined

$$
\begin{aligned}
\top &=_{df} \{x \mid x = x\} = \{x \mid x = x\} \\
\varphi \wedge \psi &=_{df} \langle \varphi, \psi \rangle = \langle \top, \top \rangle \\
\forall x.\, \varphi(x) &=_{df} \{x \mid \varphi(x)\} = \{x \mid \top\} \\
Y^X &=_{df} \{f \in \mathcal{P}(X \times Y) \mid \forall x \exists y. \langle x, y \rangle \in f\} \\
\mathcal{P}(X) &=_{df} \mathcal{P}^X
\end{aligned}
$$

boolean algebra to HOL, by introducing the new notion of a *topos*. A topos is a certain kind of algebraic object (a category equipped with a certain adjoint structure) that bears the same relation to HOL as does a boolean algebra to propositional logic:

$$\frac{\text{propositional logic}}{\text{boolean algebra}} = \frac{\text{higher-order logic}}{\textit{topos}}$$

It should be emphasized that this reformulation is still equivalent to standard deductive HOL with respect to the logical formulas and consequences. We do not change the "logical theorems" but only the presentation of the logical system, replacing the machinery of formal deductive systems with elementary algebraic manipulations.

It also should be noted that we are making no use of either what the logician calls *standard* or *Henkin* semantics. Instead, from a logical point of view, we are going to specify a *new kind of semantics*. Indeed, the algebraic formulation just given admits continuously variable models, resulting in so-called *topological semantics*. This possibility results from general facts about algebraic objects and continuous variation; so it may be useful to briefly recall how it works in the familiar case of rings, before considering the new one of algebraic logic.

2 Rings of $\mathbb{R}$-valued functions

The real numbers $\mathbb{R}$ form a topological space, an abelian group, a commutative ring, a complete ordered field, and much more.

We shall consider just the properties expressed in the *language of rings:*

$$0, 1, a + b, a \cdot b, -a$$

and first-order logic. For example, $\mathbb{R}$ is a field:

$$\mathbb{R} \models \forall x(x = 0 \vee \exists y.\ x \cdot y = 1)$$

Now consider the *product ring* $\mathbb{R} \times \mathbb{R}$, with elements of the form

$$r = (r_1, r_2)$$

and the *product operations*:

$$\begin{aligned} 0 &= (0, 0) \\ 1 &= (1, 1) \\ (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ (x_1, x_2) \cdot (y_1, y_2) &= (x_1 \cdot y_1, x_2 \cdot y_2) \\ -(x_1, x_2) &= (-x_1, -x_2) \end{aligned}$$

Since these operations are still associative, commutative, and distributive, $\mathbb{R} \times \mathbb{R}$ is still a ring.

But the element $(1, 0) \neq 0$ cannot have an inverse, since $(1, 0)^{-1}$ would have to be $(1^{-1}, 0^{-1})$. Therefore $\mathbb{R} \times \mathbb{R}$ *is not a field.*

In a similar way, one can form the more general product rings $\mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$, or $\mathbb{R}^I$ for any index-set I. Elements have the form:

$$r = (r_i)_{i \in I}$$

the *pointwise operations* are defined by:

$$\begin{aligned} 0 &= (0)_i \\ 1 &= (1)_i \\ (x_i) + (y_i) &= (x_i + y_i) \\ (x_i) \cdot (y_i) &= (x_i \cdot y_i) \\ -(x_i) &= (-x_i) \end{aligned}$$

$\mathbb{R}^I$ is again a ring, but with still fewer properties of $\mathbb{R}$. Product rings $\mathbb{R}^I$ are, however, always (von Neumann) *regular:*

$$\mathbb{R}^I \models \forall x \exists y.\ x \cdot y \cdot x = x.$$

For, given x, we can take $y = (y_i)$ with:

$$y_i = \begin{cases} x_i^{-1}, & \text{if } x_i \neq 0 \\ 0, & \text{if } x_i = 0 \end{cases}$$

Then:

$$(x \cdot y \cdot x)_i = x_i \cdot y_i \cdot x_i = \begin{cases} x_i \cdot x_i^{-1} \cdot x_i = x_i, & \text{if } x_i \neq 0 \\ 0 \cdot 0 \cdot 0 = x_i, & \text{if } x_i = 0 \end{cases}$$

The *main point* of these examples is that one can produce rings that violate even more properties of $\mathbb{R}$ by passing to "continuously varying reals". But what is a "continuously varying real number"?

Let X be a topological space. A "real number r_x varying continuously over X" is just a continuous function:

$$r : X \to \mathbb{R}$$

We equip these functions with the pointwise operations, as before:

$$(f + g)(x) = f(x) + g(x), \quad \text{etc.}$$

The set $\mathcal{C}(X)$ of all such functions then forms a *subring* of the product ring over the index set of points $|X|$:

$$\mathcal{C}(X) \subseteq \mathbb{R}^{|X|}$$

But unlike the product ring, $\mathcal{C}(X)$ is in general not regular:

$$\mathcal{C}(X) \not\models \forall f \exists g.\ f \cdot g \cdot f = f$$

For take e.g. $X = \mathbb{R}$ and $f(x) = x^2$, then we must have:

$$g(x) = \frac{1}{x^2}, \qquad \text{if } x \neq 0$$

but of course:

$$g(0) = \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$$

so there can be no *continuous* g satisfying $f \cdot g \cdot f = f$.

Summarizing the lesson of these examples, we've seen that the "continuously varying reals" $\mathcal{C}(X)$ have even fewer properties of the field of "constant" reals $\mathbb{R}$ than do the product rings $\mathbb{R}^I$. In that sense, they are closer to a general notion of "quantity". Passing from constants to continuous variation therefore "abstracts away" some properties of the constants. We note by the way that it does so without introducing any new "abstract entities".

Figure 4: Analogy

Real numbers	Sets
Algebraic operations $x + y, x \cdot y, x^{-1}, 0, 1$	Algebraic operations $X \times Y, Y^X, \mathcal{P}(X), \emptyset$
Algebraic condition (formula in ring operations) $\forall x \exists y.\ x \cdot y \cdot x = x$	Algebraic condition (formula in HOL operations) $\forall f \in A^B \exists g \in B^A.\ f \circ g \circ f = f$
Variable real number (continuous $\mathbb{R}$-valued function)	Variable set (sheaf)

3 Continuously variable sets

Now let us take stock: we have an algebraic presentation of HOL, and we have seen how continuously varying algebraic structures like rings can violate some properties of the constant ones. We now proceed according to the analogy indicated in figure 3, comparing real numbers to sets. The similarity rests on regarding reals as *linear* magnitudes, while sets are *extensive* magnitudes.

The condition $\forall f \in A^B \exists g \in B^A.\ f \circ g \circ f = f$ on (non-empty) sets corresponding to regularity is actually a form of the axiom of choice (see [5]). The notion of a "continuously variable set" that we seek will turn out to be that of a *sheaf*.

First, observe that the (algebraically specified) logical operations can be interpreted in other "universes" of sets, e.g. in the universe of "pairs of sets":

$$\mathbf{Sets} \times \mathbf{Sets}$$

The elements have the form:

$$A = (A_1, A_2)$$

and the operations are defined componentwise:

$$(A_1, A_2) \times (B_1, B_2) = (A_1 \times B_1, A_2 \times B_2)$$
$$\mathcal{P}(A_1, A_2) = (\mathcal{P}(A_1), \mathcal{P}(A_2))$$
$$(a_1, a_2) \in (A_1, A_2) \Leftrightarrow a_1 \in A_1 \text{ and } a_2 \in A_2$$
$$\text{etc.}$$

This interpretation models HOL, but it doesn't satisfy *all* the properties of **Sets**. For example:

$$\mathbf{Sets} \models A \cong 0 \vee \exists x.\ x \in A$$

But in **Sets** $\times$ **Sets** we can take $(1, 0) \ncong 0$ as A, and then $a \in (1, 0)$ means $a = (a_1, a_2)$ with $a_1 \in 1$ and $a_2 \in 0$, which is impossible.

Next, just as in the case of rings, we can generalize to $\mathbf{Sets} \times \ldots \times \mathbf{Sets} = \mathbf{Sets}^n$, and indeed to $\mathbf{Sets}^I$ for any index set I, to get the "universe" of I-indexed families of sets:

$$A = (A_i)_{i \in I}$$
$$(A_i) \times (B_i) = (A_i \times B_i)$$
$$\text{etc.}$$

These families again model HOL, but they have still fewer properties of **Sets**.

However, all such "product universes" do satisfy e.g. the axiom of choice. To find even more general "universes" that violate it, we can consider even more general families of sets:

$$(F_x)_{x \in X}$$

varying continuously over an arbitrary space X. (This generalizes the case where the set I in the previous example is regarded as a discrete space). But what should a "continuously varying set" be? The problem is that we cannot simply take a "continuous set-valued function"

$$F : X \to \mathbf{Sets}$$

as we did for rings of real-valued functions, since **Sets** is not a topological space!

Of course, there are people who already know how to do this sort of thing, so let us look at what the topologists and algebraists do when they need *continuously varying structures.*

A "continuously varying space" $(Y_x)_{x\in X}$ over a space X is called a *fiber bundle.* It consists of a space $Y = \sum_{x\in X} Y_x$ and a continuous "indexing" projection $\pi : Y \to X$, with $\pi^{-1}\{x\} = Y_x$, as indicated below.

$$\begin{array}{l} Y = \displaystyle\sum_{x\in X} Y_x \\ \Big\downarrow \pi \\ X \end{array}$$

A "continuously varying group" $(A_x)_{x\in X}$ is a *sheaf of groups.* It consists essentially of a fiber bundle $\pi : A \to X$ as indicated below,

$$\begin{array}{l} A = \displaystyle\sum_{x\in X} A_x \\ \Big\downarrow \pi \\ X \end{array}$$

satisfying the additional requirements:

1. π is a local homeomorphism (see below),

2. each A_x is a group,

3. the operations in the fibers A_x "fit together continuously".

Now, what should a "continuously varying set" be? Clearly, it should be a *sheaf of sets:* an indexed family $(F_x)_{x\in X}$ as indicated below,

$$\begin{array}{l} F = \displaystyle\sum_{x\in X} F_x \\ \Big\downarrow \pi \\ X \end{array}$$

such that each fiber $F_x = \pi^{-1}(x)$ is discrete, and moreover π is a *local homeomorphism*: each point $y \in F$ has some neighborhood U on which π is a homeomorphism $U \xrightarrow{\sim} \pi(U)$. This ensures that the variation over the space is continuous.

Some of the logical operations on sheaves can be defined pointwise:

$$(F \times G)_x \cong (F_x \times G_x)$$

Others, however, cannot. The exponential G^F of sheaves F, G is the "sheaf-valued hom" $\hom(F, G)$, defined in terms of *germs of continuous maps* $F \to G$:

$$\begin{aligned} (G^F)_x &\cong \hom(F, G)_x \quad \text{(germs of maps } F \to G) \\ &\ncong G_x^{F_x} \end{aligned}$$

The "universe" $\mathrm{Sh}(X)$ of all sheaves on a space X models HOL, but in general it *violates the axiom of choice*:

$$\mathrm{Sh}(X) \nvDash \mathrm{AC}$$

Indeed, one can find sheaf models of HOL that violate many other properties of sets.

Thus we've seen that HOL can be modeled in various "universes" other than **Sets**. In particular, the "universe" of all sets varying continuously over a space models HOL, where the notion of a continuously varying set is reasonably taken as that of a sheaf. Moreover, sheaves violate some properties of sets.

4 The logic of continuous variation

It's time to be more precise about the notion of a "universe". We've seen that only a few constructions are required to model HOL:

$$0,\ A \times B,\ \mathcal{P}(A),\ a \in A, \ldots$$

A *topos* is defined as a category equipped with adjoint structure corresponding to these operations (see [6]). In this sense, a topos is a "universe of abstract sets". It's worth noting the following theorem, which just says that we have the definition right.

Theorem. *(Topos completeness of HOL)*
A sentence of HOL is provable iff it is true in every topos model.

Given the foregoing discussion, it should come as no surprise to learn that the categories **Sets**, **Sets** $\times$ **Sets**, $\mathbf{Sets}^I$ are toposes. Moreover, the category $\mathrm{Sh}(X)$ of all sheaves of sets on a space X is also a topos.

The topos $\mathrm{Sh}(X)$ of sheaves consists of sets F_x varying continuously in a parameter $x \in X$. The logic of the constant sets is quite strong; the logic of variable sets is much weaker. Fewer things are true of variable sets in general than are true of constant ones (think of the difference between the field of real numbers and the ring of real-valued functions)

What is the logic of continuously varying sets? That is, which formulas of HOL are true in all sheaf models? The answer is given by the following theorem from [2]:

Theorem. *Logic of sheaves = classical deductive HOL.*

The proof of this fact uses recent, non-trivial results in topos theory.[2] The sheaf-theory on which it rests [3] is rooted in geometry, not logic. It is worth emphasizing that, unlike the preceding theorem, there is no obvious reason why this one needs to be true. Sheaves are classical mathematical objects, and their logical properties depend on continuous variation, not deduction. HOL is a classical deductive system going back to Frege and Russell and having nothing to do with continuity. That these things should coincide is remarkable.

Note that the Gödel incompleteness of deductive higher-order logic can be easily understood in these terms:

> Gödel's "true but unprovable"
> involves only "true of all *constant* sets"
> but not "true of all *variable* sets"

A "true but unprovable" Gödel sentence is therefore true only of constant sets, not of all variable ones.

Thus, summing up, we see that fewer things are true of all continuously varying sets than of all constant ones. HOL captures just those statements that are "variably true". Precisely: *HOL is deductively complete with respect to topological semantics*, which is the real statement of the second theorem mentioned above.

[2]The treatment of classical logic (with the law of excluded middle) is also somewhat delicate, requiring a different interpretation than the usual one in topos theory.

References

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