1. *Modalities and Intensional Languages*

The source of this paper was published in *Synthese,* XIII, 4 (December 1961): 303–322. It was presented in February 1962 at the Boston Colloquium for the Philosophy of Science in conjunction with a commentary by W. V. Quine. Quine's comments were also published in *Synthese,* XIII, and later republished under the title ''Reply to Prof. Marcus.'' My paper, the comments, and a discussion in *Synthese* XIV listing R. B. Marcus, W. V. Quine, S. Kripke, J. McCarthy, and D. Føllesdal as discussants also appeared in *Boston Studies in the Philosophy of Science,* ed. Marx Wartofsky (Dordrecht: Reidel, 1963). The discussion is included here as an appendix. The present printing of my paper contains some editorial corrections and nonsubstantive changes, some of which were included in the reprinting in *Contemporary Readings in Logical Theory,* ed. I. Copi and J. Gould (New York: Macmillan, 1967).

I was especially appreciative of the presence of Saul Kripke and of his participation in the discussion at the Boston Colloquium since Kripke was clearly open to my interests at a time when concern with modal issues was viewed askance. It is worth noting that in saying, in the text that follows, that ''I have never appreciated the force of the original argument'' about failures of substitutivity in modal contexts, I had assumed that Arthur Smullyan's paper ''Modality and Description,'' *Journal of Symbolic Logic,* XIII (1948): 31–37, which I reviewed in the same journal during that year, pp. 149–150, was fully appreciated. Smullyan had shown that Russell's theory of descriptions, properly employed with attention to scope, dispelled the puzzles. Smullyan's solution was also elaborated and extended by F. B. Fitch in ''The Problem of the Morning Star and the Evening Star,'' *Philosophy of Science,* XVI (1949). However, the absence of appreciation of Smullyan's account persisted, and I have therefore included my review of his paper as a second appendix.

It was not unusual at the time of presentation of this paper, even among the most rigorous of logicians, to use 'tautology' and its cognates for 'logically valid' or 'valid' and sometimes for 'analytic'; similarly, 'value of a variable' was used as an alternative to 'substituend of a variable'. 'Implication' was sometimes used for 'consequence' and sometimes for 'conditional'. Such alternative usage did not always signify use/mention confusions. I have made a few replacements in this volume, consistent with present more standardized practices but only if such replacements made no substantive difference. ■

There is a normative sense in which it has been claimed that modal logic is without foundation. W. V. Quine, in *Word and Object*, seems to believe that it was conceived in sin: the sin of confusing use and mention. The original transgressors were Russell and Whitehead. C. I. Lewis followed suit and constructed a logic in which an operator corresponding to 'necessarily' operates on sentences, whereas 'is necessary' ought to be viewed as a predicate of sentences. As Quine reconstructs the history of the enterprise,[1] the operational use of modalities promised only one advantage: the possibility of quantifying into modal contexts. This some of us[2] were enticed into doing. But the evils of the modal sentential calculus were found out in the functional calculus, and with them—to quote from *Word and Object*—"the varied sorrows of modality."

I do not intend to claim that modal logic is wholly without sorrows. I do claim that its sorrows are not those that Quine describes, and that modal logic is worthy of defense, for it is useful in connection with many interesting and important questions, such as the analysis of causation, entailment, obligation, and belief statements, to name only a few.

If we insist on equating formal logic with strongly extensional functional calculi, then P. F. Strawson[3] is correct in saying that "the analytical equipment (of the formal logician) is inadequate for the dissection of most ordinary types of empirical statement."

Intensional Languages

I will begin with the notion of an intensional language. I will make a further distinction between those languages that are explicitly intensional and those that are implicitly so. Our notion of intensionality does not divide languages into mutually exclusive classes but, rather, orders them loosely as strongly or weakly intensional. A language is explicitly intensional to the degree to which it does not equate the identity relation with some weaker form of equivalence. We will as-

1. *Word and Object* (Cambridge: Harvard University Press, 1960), pp. 195–96.
2. a. R. C. Barcan (Marcus), "A Functional Calculus of First Order Based on Strict Implication," *Journal of Symbolic Logic,* XI (1946): 1–16.
   b. R. C. Barcan (Marcus), "The Identity of Individuals in a Strict Functional Calculus of First Order," *Journal of Symbolic Logic,* XII (1947): 12–15.
   c. R. Carnap, "Modalities and Quantification," *Journal of Symbolic Logic,* XI (1946): 33–64.
   d. F. B. Fitch, *Symbolic Logic* (New York: Ronald Press, 1952).
3. P. F. Strawson, *Introduction to Logical Theory* (London: Methuen, 1952), p. 216.

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sume that every language must have some constant objects of reference (things), ways of classifying and ordering them, ways of making statements, and ways of separating true statements from false ones. We will not go into the question of how we come to regard some elements of experience as things, but one criterion for sorting out the elements of experience that we regard as things is that things may enter into the identity relation. In a formalized language, those symbols that name things will be those for which it is meaningful to assert that 'I' names the identity relation.

Ordinarily, and in the familiar constructions of formal systems, the identity relation must be held appropriate for individuals. If 'x' and 'y' are individual names, then

(1) $x\mathrm{I}y$

is a sentence, and, if they are individual variables, then (1) is a sentential function. Whether a language confers thinghood on attributes, classes, or propositions is not so much a matter of whether variables appropriate to them can be quantified over (and we will return to this later), but rather of whether (1) is meaningful where 'x' and 'y' may take as substitution instances names of attributes, classes, propositions. We note in passing that the meaningfulness of (1) with respect to attributes and classes is more frequently allowed than the meaningfulness of (1) in connection with propositions.

Returning now to the notion of explicit intensionality, if identity is appropriate to propositions, attributes, classes, as well as lowest-type individuals, then any weakening of the identity relation with respect to any of these entities may be thought of as an extensionalizing of the language. By a weakening of the identity relation is meant equating it with some weaker equivalence relation.

On the level of individuals, one or perhaps two equivalence relations are customarily present: identity and indiscernibility. This does not preclude the introduction of others, such as congruence, but the strongest of these is identity. Where identity is defined rather than taken as primitive, it is customary to define it in terms of indiscernibility. Indiscernibility may in turn be defined as

(2) $x \text{ Ind } y =_{\mathrm{df}} (\varphi)(\varphi x \text{ eq } \varphi y)$

for some equivalence relation eq.

In a system of material implication (Sm), eq is taken as $\equiv$ (material equivalence). In modal systems, eq may be taken as ≣ (strict equivalence). In more strongly intensional systems eq may be taken as the strongest equivalence relation appropriate to such expressions as 'φx'.

In separating (1) and (2) I should like to suggest the possibility that to equate identity and Ind may already be an explicit weakening of the identity relation and, consequently, an extensionalizing principle. This was first suggested to me by a paper of F. P. Ramsey.[4] Though I now regard his particular argument in support of the distinction as unconvincing, I am reluctant to reject the possibility. I suppose that at bottom my appeal is to ordinary language, since, although it is obviously absurd to talk of two things being the same thing, it seems not quite so absurd to talk of two things being indiscernible from each other. In equating I and Ind we are saying that to be distinct is to be discernibly distinct in the sense of there being one property not common to both. Perhaps it is unnecessary to mention that, if we confine things to objects with spatiotemporal histories, it makes no sense to distinguish (1) and (2). And indeed, in my extensions of modal logic, I have chosen to define identity in terms of (2). However, the possibility of such a distinction ought to be mentioned before it is obliterated. Except for the weakening of (1) by equating it with (2), extensionality principles are absent on the level of individuals.

Proceeding now to functional calculi with theory of types, an extensionality principle is of the form

(3) $x \text{ eq } y \rightarrow x\mathrm{I}y$

The arrow may represent one of the implication relations present in the system or some metalinguistic conditional; eq is one of the equivalence relations appropriate to x and y, but not identity. Within the system of material implication, 'x' and 'y' may be taken as symbols for classes, 'eq' as class equality (in the sense of having the same members); or 'x' and 'y' may be taken as symbols for propositions and 'eq' as the triple bar for material equivalence. In extended modal systems 'eq' may be taken as the quadruple bar where 'x' and 'y' are symbols for propositions. If the extended modal system has symbols for classes, 'eq' may be taken as 'having the same members' or, alternatively, 'necessarily having the same members', which can be expressed within such a language. If we wish to distinguish classes from attributes in such a system 'eq' may be taken as 'necessarily applies to the same thing', which is directly expressible within the system. In a language that permits epistemic contexts such as belief contexts, an equivalence relation even stronger than either material or strict equivalence may have to be present. Taking that stronger relation as

4. F. P. Ramsey, *The Foundations of Mathematics* (London and New York: Methuen, 1931), pp. 30–32.

eq, (3) would still be an extensionalizing principle in such a strongly intensional language.

I should now like to turn to the notion of implicit extensionality, which is bound up with the kinds of substitution principles available in a language. If we confine ourselves for the sake of simplicity of exposition to a sentential calculus, one form of the substitution theorem is

(4) $x \text{ eq}_1 y \rightarrow z \text{ eq}_2 w$

where *x, y, z, w* are well formed, *w* is the result of replacing one or more occurrences of *x* by *y* in *z*, and '→' symbolizes an implication or a metalinguistic conditional. In the system of material implication (Sm or QSm), (4) is provable where eq_1 and eq_2 are both taken as material equivalence for appropriate values of *x, y, z, w*. That is,

(5) $(x \equiv y) \supset (z \equiv w)$

Now (5) is clearly false if we are going to allow contexts involving belief, logical necessity, physical necessity, and so on. We are familiar with the examples. If '*x*' is taken as 'John is a featherless biped', and '*y*' as 'John is a rational animal', then an unrestricted (5) fails. Our choice is either to reject (5) as it stands or to reject all contexts in which it fails. If the latter choice is made, the language is *implicitly* extensional, since it cannot countenance predicates or contexts that might be permissible in a more strongly intensional language. This is Quine's solution. All such contexts are assigned to a shelf labeled 'referential opacity' or, more precisely, 'contexts that confer referential opacity', and are disposed of. But the contents of that shelf are of enormous interest to some of us, and we would like to examine them in a systematic and formal manner. For this we need a language that is appropriately intensional.

In the modal sentential calculi, since there are two kinds of equivalence that may hold between *x* and *y*, (4) represents several possible substitution theorems, some of which are provable. We will return to this shortly. Similarly, if we are going to permit epistemic contexts, the modal analogue of (4) will fail in those contexts and a more appropriate principle will have to supplement it. A stronger eq_1 is required to support substitutivity in epistemic contexts.

Identity and Substitution in Quantified Modal Logic

In the light of the previous remarks I would like to turn specifically to the criticisms raised against extended modal systems in connection

with identity and substitution. In particular, I will refer to my[5] extension of Lewis's[6] S4, which consisted of introducing quantification in the usual manner and adding the axiom[7]

(6) $\Diamond(\exists x)A \prec (\exists x)\Diamond A$

I will call this system 'QS4'. QS4 does not have an explicit axiom of extensionality, although it does have an implicit weak extensionalizing principle in the form of a substitution theorem.

It would appear that, for many uses to which modal calculi may be put, S5 is to be preferred.[8] A. N. Prior[9] has shown that (6) is a theorem in an extended S4 (i.e., S5). My subsequent remarks, unless otherwise indicated, apply equally to QS5. In QS4 (1) is defined in terms of (2). (2), and consequently (1), admit of alternatives where 'eq' may be taken as material or strict equivalence: 'I_m' and 'I' respectively. The following are theorems of QS4:

(7) $(xI_m y) \equiv (xIy)$

(8) $(xIy) \equiv \Box(xIy)$

where '□' is the modal symbol for logical necessity. In (7) 'I_m' and 'I' are strictly equivalent; within such a modal language, they are therefore indistinguishable by virtue of the substitution theorem. Contingent identities are disallowed by (8).

(9) $(xIy) \cdot \Diamond \sim(xIy)$

is a contradiction.

Professor Quine[10] finds these results offensive, for he sees (8) as "purifying the universe." Concrete entities are said to be banished and replaced by pallid concepts. The argument is familiar:

(10) The evening star eq the morning star

5. Barcan (Marcus), "A Functional Calculus of First Order Based on Strict Implication"; "The Identity of Individuals in a Strict Functional Calculus of First Order."
6. C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York: Methuen, 1932).
7. See A. N. Prior, *Time and Modality* (New York: Oxford University Press, 1957), for an extended discussion of this controversial axiom, which has come to be known as the Barcan formula.
8. S5 results from adding to S4:

$p \prec \Box\Diamond p$

9. A. N. Prior, "Modality and Quantification in S5," *Journal of Symbolic Logic*, XXI (1956), pp. 60–62.
10. *From a Logical Point of View* (Cambridge: Harvard University Press, 1953), pp. 152–154.

is said to express a "true identity," yet the expressions on either side of 'eq' are not validly intersubstitutable in

(11) It is necessary that the evening star is the evening star.

The rebuttals are familiar, and I will try to state some of them. This is difficult, for I have never appreciated the force of the original argument. In restating the case, I would like to consider the following informal claim:

(12) If p is a tautology, and p eq q, then q is a tautology.

where 'eq' names some equivalence relation appropriate to p and q. In Sm, if 'eq' is taken as $\equiv$, then a restricted (12) is available where '$p \equiv q$' is provable.

One might say informally that, with respect to any language, if (12) is said to fail, then we must be using 'tautology' in a very peculiar way, or what is taken as 'eq' is not sufficient as an equivalence relation appropriate to p and q.

Consider by contrast the claim that

(13) $a\mathrm{I}b$

is a true identity. Now if (13) is such a true identity, then a and b are the same thing. (13) doesn't say that a and b are two things that happen, through some accident, to be one. True, we are using two different names for that same thing, but we must be careful about use and mention. If, then, (13) is true, it must say the same thing as

(14) $a\mathrm{I}a$

But (14) is surely valid, and so (13) must surely be valid as well. This is precisely the import of my theorem (8). We would therefore expect, indeed it would be a consequence of the truth of (13), that 'a' is replaceable by 'b' in any context.

Now suppose we come upon a statement like

(15) Scott is the author of *Waverley*.

and we have a decision to make. This decision cannot be made in a formal vacuum, but must depend to a considerable extent on some informal consideration as to what it is we are trying to say in (10) and (15). If we decide that 'the evening star' and 'the morning star' are proper names for the same thing, and that 'Scott' and 'the author of *Waverley*' are proper names for the same thing, then they must be intersubstitutable in every context. In fact it often happens, in a growing, changing language, that a descriptive phrase comes to be used

as a proper name—an identifying tag—and the descriptive meaning is lost or ignored. Sometimes we use certain devices, such as capitalization with or without dropping of the definite article, to indicate the change in use. 'The evening star' becomes 'Evening Star', 'the morning star' becomes 'Morning Star', and they may come to be used as names for the same thing. Singular descriptions such as 'the little corporal', 'the Prince of Denmark', 'the sage of Concord', or 'the great dissenter', are, as we know, often used as alternative proper names of Napoleon, Hamlet, Emerson, and Oliver Wendell Holmes. One might even devise a criterion as to when a descriptive phrase is being used as a proper name. Suppose that, through some astronomical cataclysm, Venus was no longer the first star of the evening. If we continued to call it alternatively 'Evening Star' or 'the evening star' that practice would be a measure of the conversion of the descriptive phrase into a proper name. If, however, we would then regard (10) as false, this would indicate that 'the evening star' was not being used as an alternative proper name of Venus. (We might mention in passing that, although the conversion of descriptions into proper names appears to be asymmetric, we do find proper names used in singular descriptions of something other than the thing named, as in the statement 'Mao Tse-tung is the Stalin of China', where one intends to assert a similarity between the entities named.)

That any language must countenance some entities as things would appear to be a precondition for language. But this is not to say that experience is given to us as a collection of things, for it would appear that there are cultural variations and accompanying linguistic variations as to what sorts of entities are so singled out. It would also appear to be a precondition of language that the singling out of an entity as a thing is accompanied by many—and perhaps an indefinite number—of unique descriptions, for otherwise how would it be singled out? But to assign a thing a proper name is different from giving a unique description. For suppose we took an inventory of all the entities countenanced as things by some particular culture through its own language, with its own set of names and equatable singular descriptions, and suppose that number were finite (this assumption is for the sake of simplifying the exposition). And suppose we randomized as many whole numbers as we needed for a one-to-one correspondence, and thereby tagged each thing. This identifying tag is a proper name of the thing. In taking our inventory we discovered that many of the entities countenanced as things by that language-culture complex already had proper names, although in many cases a singular description may have been used. This tag, a proper name, has no meaning. It simply tags. It is not strongly equatable with any of the singular descriptions of the thing, although singular descriptions may be equatable (in a weaker sense) with each other, where

(16) Desc_1 eq Desc_2

means that Desc_1 and Desc_2 describe the same thing. But here too, what we are asserting would depend on our choice of 'eq'.

Perhaps I should mention that I am not unaware of the awful simplicity of the tagging procedure described above. It makes the assumption of finitude, or, if this is not assumed, then at least the assumption of denumerability of the class of things; also, the assumption that all things countenanced by the language-culture complex are named (or nameable), described (or describable). But my point is only to distinguish tagging from describing, proper names from descriptions. You may describe Venus as the evening star, and I may describe Venus as the morning star, and we may both be surprised that, as an empirical fact, the same thing is being described. But it is not an empirical fact that

(17) Venus I Venus

and if '*a*' is another proper name for Venus that

(18) Venus I *a*

Nor is it extraordinary that we often convert one of the descriptions of a thing into a proper name. Perhaps we ought to be more consistent in our use of upper-case letters, but this is a question of reforming ordinary language. It ought not to be an insurmountable problem for logicians.

What I have been arguing is that to say truly of an identity (in the strongest sense of the word) that it is true, it must be tautologically true or analytically true. The controversial (8) of QS4, the necessity of identity, no more banishes concrete entities from the universe than (12) banishes from the universe red-blooded propositions.

Let us now return to (15) and (10). If they express true identities, then 'Scott' ought to be anywhere intersubstitutable for 'the author of *Waverley*' in modal contexts, and similarly for 'the morning star' and 'the evening star'. If those pairs are not so universally intersubstitutable—that is, if our decision is that they are not simply proper names for the same thing; that (15) and (10) express equivalences that are possibly false, e.g., someone else might have written *Waverley*, the star first seen in the evening might have been different from the

star first seen in the morning—then (15) and (10) do not express identities.

Russell provides one solution; on his analysis in accordance with the theory of descriptions the truth of (15) and (10) does not commit us to their necessary truth, and certainly not to taking 'eq' of (10) or 'is' of (15) as identity, except on the explicit assumption of an extensionalizing axiom. Other and related solutions are in terms of membership in a unit class or applicability of a unit attribute. But, whatever we choose, it will have to permit intersubstitutability or some analogue of intersubstitutability for the members of the pairs 'Scott' and 'the author of *Waverley*', and 'the evening star' and 'the morning star', which is short of being universal. In a language that is implicitly strongly extensional, that is, where all contexts in which such substitutions fail are simply eschewed, of course there is no harm in equating identity with weaker forms of equivalence. But why restrict ourselves in this way when, in a more intensional language, we can still make all the substitutions permissible to this weaker form of equivalence, yet admit contexts in which such substitution is not permitted? To illustrate, I would like to turn to the instances of (4) that are provable[11] in QS4. I will again confine my remarks, for the purpose of exposition, to S4, although it is the generalizations for QS4 that are actually proved. An unrestricted

(19) $x \equiv y \rightarrow z \equiv w$

is not provable, whether '$\rightarrow$' is taken as material implication, strict implication, or a metalinguistic conditional. It would involve us in a contradiction if our interpreted system allowed true statements such as

(20) $(x \equiv y) \cdot \sim \Box(x \equiv y)$

as it must if it is not to reduce itself to the system of material implication. Indeed, the underlying assumption about equivalence that is implicit in the whole "evening star/morning star" controversy is that there are equivalences (misleadingly called "true identities") that are contingently true. Let x and y of (19) be taken as some pair p and q that satisfies (20). Let z be $\Box(p \equiv p)$ and w be $\Box(p \equiv q)$. Then (19) is

(21) $(p \equiv q) \rightarrow (\Box(p \equiv p) \equiv \Box(p \equiv q))$

11. Barcan (Marcus), "A Functional Calculus of First Order Based on Strict Implication." Theorem XIX* corresponds to (23). The restricted (19), given the conditions of the restriction, is clearly provable in the same manner as XIX*.

From (20), simplification, *modus ponens,* and $\Box(p \equiv p)$, which is a theorem of S4, we can deduce $\Box(p \equiv q)$. $\Box(p \equiv q)$, simplification of (20), and conjunction lead to the contradiction

(22) $\Box(p \equiv q) \cdot \sim\Box(p \equiv q)$

A restricted form of (19) is provable: (19) is provable where z does not contain any modal operators. And these are the contexts that support substitution in Sm (the system of material implication), without at the same time banishing modal contexts. Indeed, a slightly stronger (19) is provable. (19) is provable where x does not fall within the scope of a modal operator in z.

Where in (4), eq_1 and eq_2 are both taken as strict equivalence, the substitution theorem

(23) $(x \equiv y) \rightarrow (z \equiv w)$

is provable without restriction, and also where eq_1 is taken as strict equivalence and eq_2 is taken as material equivalence, as in

(24) $(x \equiv y) \rightarrow (z \equiv w)$

But (23) is also an extensionalizing principle, for it fails in epistemic contexts such as contexts involving 'knows that' or 'believes that', for consider the statement

(25) When Professor Quine reviewed the paper on identity in QS4, he knew that $\vdash a\mathrm{I}_m b \equiv a\mathrm{I}_m b$.

and

(26) When Professor Quine reviewed the paper on identity in QS4 he knew that $\vdash a\mathrm{I}b \equiv a\mathrm{I}_m b$.

Although (25) is true, (26) is false, even though (7) holds in QS4. But rather than repeat the old mistakes by abandoning epistemic contexts to the shelf labeled "referential opacity" after having rescued modal contexts as the most intensional permissible contexts to which such a language is appropriate, we need only conclude that (23) confines us to limits of applicability of such modal systems. *If* it should turn out that statements involving 'knows that' and 'believes that' permit of formal analysis, then such an analysis would have to be embedded in a language with a stronger equivalence relation for sentences than strict equivalence. Carnap's intensional isomorphism, Lewis's analytical comparability, and perhaps Anderson and Belnap's mutual entailment are relations that point in that direction. But they too would be short of identity, for there surely are contexts in which substitutions

allowed by such stronger equivalences would convert a truth into a falsehood.

It is my opinion[12] that the identity relation need not be introduced for anything other than the entities we countenance as things, such as individuals. Increasingly strong substitution theorems give the force of universal substitutivity without explicit axioms of extensionality. We can talk of equivalence between propositions, classes, attributes, without thereby conferring on them thinghood by equating such equivalences with the identity relation. QS4 has no explicit extensionality axiom. Instead we have (23), the restricted (19), and their analogues for attributes or classes.

The discussion of identity and substitution in QS4 would be incomplete without touching on the other familiar example:

(27) 9 eq the number of planets

is said to be a true identity for which substitution fails in

(28) $\Box(9 > 7)$

for it leads to the falsehood

(29) $\Box$(the number of planets > 7)

Since the argument holds (27) to be contingent [$\sim\Box$(9 eq the number of planets)], 'eq' of (27) must be unpacked into an appropriate analogue of material equivalence, and consequently the step from (28) to (29) is not valid, since the substitution would have to be made in the scope of the square. It was shown that (19) is not an unrestricted theorem in QS4.

On the other hand, since in QS4

(30) $(5+4) =_s 9$

where '$=_s$' is the appropriate analogue for attributes (classes) of strict equivalence, '5+4' may replace '9' in (28) in accordance with (23). If, however, the square were dropped from (28), as it validly can be since

(30a) $\Box p \mathrel{-\!\!\!3} p$

is provable, then, by the restricted (19), the very same substitution available to Sm is available here.

12. See my "Extensionality," *Mind,* n.s., LXIX (1960): 55–62, which overlaps to some extent the present paper.

The Interpretation of Quantification: A Substitutional View

The second prominent area of criticism of quantified modal logic involves interpretation of the operations of quantification when combined with modalities. It appears to me that at least some of the problems stem from an absence of an adequate, unequivocal, colloquial translation of the operations of quantification. It is often not quantification but our choice of reading and the implicit interpretive consequences of such a reading that lead to difficulties. Such difficulties are not confined to modal systems. The most common reading of existential quantification is

(31) There is (exists) at least one (some) thing (person) which (who) . . .

Strawson,[13] for example, does not even admit the possibility of significant alternatives, for he says of (31): "We might think it strange that the whole of modern formal logic after it leaves the propositional logic and before it crosses the boundary into the analysis of mathematical concepts, should be confined to the elaboration of sets of rules giving the logical interrelations of formulae which, however complex, begin with these few rather strained and awkward phrases." Indeed, taking (31) at face value, Strawson gets into a muddle about tense [(30) is in the present tense], and the ambiguities of the word 'exist'. What we would like to have and do not have, is a direct, unequivocal colloquial reading of

(32) $(\exists x)\varphi x$

which gives us the force of the following:

(33) Some substitution instance of 'φx' is true.

I am suggesting not that (33) provides translations of (32), but only that what is wanted is a translation with the force of (32).

As seen from (33), quantification has to do primarily with truth and falsity and with open sentences. Readings in accordance with (31) may entangle us unnecessarily in ontological perplexities, for if quantification has to do with things and if variables for attributes or classes can be quantified upon, then in accordance with (31) attributes and classes are things. If we still want to distinguish the identifying from the classifying function of language, then we are involved in a classification of different kinds of things and the accompanying platonic

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involvements. The solution is not to banish quantification binding variables other than individual variables but only not to be taken in by (31). We do in fact have some colloquial counterparts of (33): the nontemporal 'sometimes' or 'in some cases' or 'in at least one case', which have greater ontological neutrality than (31).

Some of the arguments involving modalities and quantification are closely connected with questions of substitution and identity. At the risk of boring my readers I will go through one of these again. In QS4 the following definitions are introduced:[14]

(34) $(\varphi =_m \psi) =_{df} (x)(\varphi x \equiv \psi x)$

(35) $(\varphi =_s \psi) =_{df} \Box(\varphi =_m \psi)$

Note that '$=_s$' and '$=_m$' represent equivalence relations weaker than identity and describable as strict and material equality. Individual descriptions can be interpreted as higher-order terms, or, if the theory of descriptions is strictly applied, the solution to some of the substitution puzzles also falls under the theorems that, as argued above, constrain substitution of material equivalents in modal contexts. Since the equality in (10) is contingent, it is the case that there is a *descriptive* reading such that

(36) the evening star $=_m$ the morning star

It is also the case that

(37) $\Diamond \sim$(the evening star $=_m$ the morning star)

One way of writing (11) is as

(38) $\Box$(the evening star $=_m$ the evening star)

By existential generalization in (38), it follows that

(39) $(\exists\varphi)\Box(\varphi =_m$ the evening star)

In the words of (31), (39) becomes

(40) There is a thing such that it is necessary that it is equal to the evening star.

14. See my "A Functional Calculus of First Order Based on Strict Implication" and "The Identity of Individuals in a Strict Functional Calculus." Abstracts are introduced and attributes (or classes) may be equated with abstracts. Among the obvious features of such a calculus of attributes (classes), is the presence of equivalent, nonidentical, empty attributes (classes). If the *null* attribute (class) is defined in terms of identity, it will be intersubstitutible with any abstract on a contradictory function.

The stubborn unlaid ghost rises again. Which thing, the evening star that, by (36), is equal to the morning star? But such a substitution would lead to the falsehood

(41) $\Box$(the evening star $=_m$ the morning star)

The argument may be repeated for (27) through (29).

In QS4 the solution is clear, for, since (37) holds and since in (39) 'φ' occurs within the scope of a square, we cannot go from (39) to (41). On the other hand, the step from (38) to (39) (existential instantiation) is entirely valid, for surely there is a substituend of 'φ' for which

$\Box(\varphi =_m$ the evening star)

is true. In particular, the case where 'φ' is replaced by 'the evening star'.

There is also the specific problem of interpreting quantification in (6) $[\Diamond(\exists x)\varphi x \prec (\exists x)\Diamond\varphi x]$, which is a postulate of QS4. Read in accordance with (31) as

(42) If it is logically possible that there is something that φs, then there is something such that it is logically possible that it φs.

it is admittedly odd. The antecedent seems to be about what is logically possible and the consequent about what there is. How can one go from possibility to existence? Read in accordance with (33) we have the clumsy but not so paradoxical

(43) If it is logically possible that φx for some substituend of 'x', then there is some substituend of 'x' such that it is logically possible that φx.

Although the emphasis has now been shifted from things to statements, and the ontological consequences of (42) are absent, (43) is still indirect and awkward. It would appear that questions such as the acceptability or nonacceptability of (6) are best solved in terms of some semantical construction. We will return to this, but first some minor matters.

Modalities Misunderstood

A defense of modal logic would be incomplete without touching on criticisms of modalities that stem from confusion about what is or is

not provable in such systems. One criticism is that of Paul Rosenbloom,[15] who seized on my proof[16] that a strong deduction theorem is not available in QS4 as a reason for discarding strict implication as relevant in any way to the deducibility relation. Rosenbloom failed to note that a weaker and perhaps more appropriate deduction theorem is available. Indeed, Anderson and Belnap,[17] in their attempt to formalize entailment without modalities, reject the strong form of the deduction theorem as "counter-intuitive for entailment."

Another example occurs in *Word and Object*;[18] it can be summarized as follows:

(44) Modalities yield talk of a difference between necessary and contingent attributes.

(45) Mathematicians may be said to be necessarily rational and not necessarily two-legged.

(46) Cyclists are necessarily two-legged and not necessarily rational.

(47) *a* is a mathematician and a cyclist.

(48) Is this concrete individual necessarily rational and contingently two-legged or vice versa?

(49) "Talking referentially of the object, with no special bias toward a background grouping of mathematicians as against cyclists . . . there is no semblance of sense in rating some of his attributes as necessary and others as contingent."

Quine says that (44) to (47) are supposed to "evoke the appropriate sense of bewilderment," and they surely do, for I know of no interpreted modal system that countenances necessary attributes in the manner suggested. Translating (45) to (47), we have one of the equivalent

(50) $(x)(Mx \prec Rx) \equiv (x)\Box(Mx \supset Rx) \equiv (x) \sim \Diamond(Mx \cdot \sim Rx)$

conjoined in (45) with one of the equivalent

(51) $(x) \sim \Box(Mx \supset Tx)$

15. *The Elements of Mathematical Logic* (New York: Dover, 1950), p. 60.
16. "Strict Implication, Deducibility, and the Deduction Theorem," *Journal of Symbolic Logic*, XVIII (1953): 234–236.
17. A. R. Anderson and N. D. Belnap, *The Pure Calculus of Entailment* (pre-print).
18. *Word and Object*, pp. 199–200.

$$\equiv (x) \Diamond \sim (Mx \supset Tx)$$
$$\equiv (x) \Diamond (Mx \cdot \sim Tx)$$

Also one of the equivalent

(52) $(x)(Cx ⥽ Tx) \equiv (x) \Box (Cx \supset Tx) \equiv (x) \sim \Diamond (Cx \cdot \sim Tx)$

conjoined in (46) with one of the equivalent

(53) $(x) \sim \Box (Cx \supset Rx) \equiv (x) \Diamond \sim (Cx \supset Rx)$
$\equiv (x) \Diamond (Cx \cdot \sim Rx)$

And in (47)

(54) $Ma \cdot Ca$

Among the conclusions we can draw from (49) to (54) are $\Box(Ma \supset Ra)$, $\sim \Diamond (Ma \cdot \sim Ra)$, $\Diamond (Ma \cdot \sim Ta)$, $\sim \Box (Ma \supset Ta)$, $\Box (Ca \supset Ta)$, $\sim \Diamond (Ca \cdot \sim Ta)$, $\Diamond (Ca \cdot \sim Ra)$, $\sim \Box (Ca \cdot \sim Ra)$, Ta, Ra, $Ta \cdot Ra$; but nothing to answer question (48), or to make any sense of (49). Quine would appear to be assuming that

(55) $(p ⥽ q) ⥽ (p ⥽ \Box q)$

is provable in QS4, but it is not so provable, except where $p \equiv \Box r$ for some r. Keeping in mind that, if we are dealing with logical modalities, we can see that none of the attributes (M, R, T, C) in (50) to (54), taken separately or conjoined, are logically necessary. These are not the sort of attributes that modal logic, even derivatively, countenances as being logically necessary. But a word is appropriate here about the derivative sense in which we *can* speak of necessary and contingent attributes.

In QS4 abstracts are introduced such that to every function there corresponds an abstract, e.g.,

(56) $x \varepsilon \hat{y} A =_{df} B$

where B is the result of substituting every free occurrence of y in A by x, given familiar constraints.

If r is some abstract, we can define

(57) $x \varepsilon \boxdot r =_{df} \Box (x \varepsilon r)$ $\qquad \vdash \boxdot r =_{df} (x)(x \varepsilon \boxdot r)$

and

(58) $x \varepsilon ⟐ r =_{df} \Diamond (x \varepsilon r)$ $\qquad \vdash ⟐ r =_{df} (x)(x \in ⟐ r)$

It is clear that among the abstracts to which $\vdash\boxdot$ may validly be affixed will be those corresponding to valid functions, e.g., $\hat{y}(yIy)$, $\hat{y}(\varphi x \vee \sim \phi x)$, etc. It would be appropriate to call these *necessary* attributes, and the symbol '⊡' is a derivative way of applying modalities to attributes.

Similarly, all the constituent attributes of (50) to (54) could in the sense of (58) be called possible, where '⟐' is the derivative modality for possibility of attributes. A contingent attribute would be possible but not necessary. However, *if* (50) is true, then the attribute of being either not a mathematician or rational could appropriately be called necessary, for it would follow from the premises that

(59) $(x) \Box (x \varepsilon \hat{y} (\sim My \vee Ry))$

but some ground for importing the necessity of the premise would be required.

Semantic Constructions

I would like in conclusion to suggest that the polemics of modal logic are perhaps best carried out in terms of some explicit semantical construction. As we know from disputes about interpretation in connection with (6) $[\Diamond (\exists x) A ⥽ (\exists x) \Diamond A]$, to argue without some construction is awkward at best and at worst has the character of a quibble.

Let us reappraise (6) in terms of such a construction.[19] Consider, for example, a language (L), with truth-functional connectives, a modal operator ($\Diamond$), a finite number of individual constants, an infinite number of individual variables, one two-place predicate (R), quantification, and the usual criteria for being well-formed. A domain (D) of individuals is then considered, named by the constants of L. A model of L is defined as a class of ordered couples (possibly empty) of D. The members of a model of L are exactly those pairs between which R holds. To say, therefore, that the atomic sentence $R(a_1 a_2)$ of L holds or is true in M is to say that the ordered couple (b_1, b_2) is a member of M, where a_1 and a_2 are the names in L of b_1 and b_2. If a sentence A of L is of the form $\sim B$, A is true in M if and only if B is not true in M. If A is of the form $(B_1 \cdot B_2)$, then A is true in M if and only if both B_1 and B_2 are true in M. If A is of the form $(\exists x) B$, then A is true in M if and only if at least one substitution in-

19. The construction here outlined is close to that of R. Carnap, *Meaning and Necessity* (Chicago: University of Chicago Press, 1947). The statement of the construction is in accordance with a method of J. C. C. McKinsey. See also McKinsey, "On the Syntactical Construction of Systems of Modal Logic," *Journal of Symbolic Logic*, X (1946): 88–94; "A New Definition of Truth," *Synthese*, VII (1948/49): 428–433.

stance of B is true (holds) in M. If A is $\Diamond$ B then A is true in M if and only if B is true in some model M_1.

We see on the construction that a true sentence of L is defined relative to a model and a domain of individuals. A logically true sentence is a sentence that would be true in every model. We are now in a position to give a rough proof of (6). Suppose (6) is false in some M. Then

$$\sim\Diamond(\Diamond(\exists x)\varphi x\cdot\sim(\exists x)\Diamond\varphi x)$$

is false in M. Therefore

$$\Diamond(\Diamond(\exists x)\varphi x\cdot\sim(\exists x)\Diamond\varphi x)$$

is true in M. So

$$\Diamond(\exists x)\phi x\cdot\sim(\exists x)\Diamond\phi x$$

is true in some M_1. Therefore

(60) $\Diamond(\exists x)\varphi x$

and

(61) $\sim(\exists x)\Diamond\varphi x$

are true in M_1. Consequently, from (60),

(62) $(\exists x)\varphi x$

is true in some model M_2. Therefore there is a member of D (b) such that

(63) φb

is true in M_2. But from (61)

$$(\exists x)\Diamond\varphi x$$

is not true in M_1. Consequently, there is no member b of D such that

(64) $\Diamond\varphi b$

is true in M_1. So there is no model M_2 such that

$$\varphi b$$

is true in M_2. But this result contradicts (63). Consequently, in such a construction, (6) must be true in every model.

If this is the sort of construction one has in mind, then we are persuaded of the plausibility of (6). Indeed, going back to (43), it can be seen that this was the sort of construction that was being assumed.

If (6) is to be regarded as offensive, it must be in terms of some other semantic construction that ought to be made explicit.[20]

We see, that, though the rough outline above corresponds to the Leibnizian distinction between true in a possible world and true in all possible worlds, it is also to be noted that there are no specifically intensional objects. No new entity is spawned in a possible world that isn't already in the domain in terms of which the class of models is defined. In such a model modal operators have to do with truth relative to the model. On this interpretation,[21] Quine's "flight from intension" may have been exhilarating, but unnecessary.

20. A criticism of the construction here outlined is the assumption of the countability of members of D. McKinsey points this out in the one chapter I have seen (chap. 1, vol. 2) of a projected (unpublished) two-volume study of modal logic, and indicates that his construction will not assume the countability of members of D. Whereas Carnap's construction leads to a system at least as strong as S5, McKinsey's (he claims) will be at least as strong as S4. I have not seen or been able to locate any parts of this study, in which the details were to have been worked out along with completeness proofs for some of the Lewis systems. See also J. Myhill, in *Logique et analyse* (1958), pp. 74–83; and S. Kripke, *Journal of Symbolic Logic,* XXIV (1959): 323–324 (abstract).

21. If one wishes to talk about possible things, then of course such a construction is inadequate.

APPENDIX 1A: DISCUSSION

This appendix appeared in *Synthese,* XIV (September 1962). It includes comments of some of the participants in the Boston Colloquium for the Philosophy of Science on the occasion of the presentation of "Modalities and Intensional Languages" in February 1962. The transcript of the tape was circulated to the participants for final editing. Not all comments of participants were included in the published version.

The present printing includes some editorial corrections and a footnote to a passage that needs clarification and correction.

I debated as to whether the discussion should be included in the present volume but decided that it had sufficient interest, including historical interest, to be worth inclusion. ■

PROF. MARCUS: We seem still at the impasse I thought to resolve at this time. The argument concerning (12) was informal, and parallels as I suggested, questions raised in connection with the 'paradox' of analysis. One would expect that if a statement were analytic, and it bore a strong equivalence relation to a second statement, the latter would be analytic as well. Since (12) cannot be represented in Sm without restriction, the argument reveals material equivalence to be insufficient and weak. An adequate representation of (12) requires a modal framework.

The question I have about essentialism is this: Suppose these modal systems are extended in the manner of *Principia* to higher order.

Then

$\Box((5+4)=9)$

will hold. Here '$=$' may be taken as either '$=_s$' or '$=_m$' of the present paper, (since the reiterated squares telescope), whereas

$\Box((5+4) = \text{the number of planets})$

does not hold. Our interpretation of these results commits us only to the conclusion that the equivalence relation that holds between 5+4

and 9 is stronger than the one that holds between 5+4 and the number of planets. More specifically, the stronger one is the class or attribute analogue of $\equiv$. No mysterious property is being conferred on either 9 or the number of planets that they do not already have in the extensional $((5+4) =_m \text{the number of planets})$.

PROF. QUINE: May I ask if Kripke has an answer to this? . . . Or I'll answer, or try to.

MR. KRIPKE: As I understand Professor Quine's essentialism, it isn't what's involved in either of these two things you wrote on the board that causes trouble. It is in inferring that there exists an x, which necessarily $= 5+4$ (from the first of the two). (To Quine:) Isn't that what's at issue?

PROF. QUINE: Yes.

MR. KRIPKE: So this attributes necessarily equalling 5+4 to an object.

PROF. MARCUS: But that depends on the suggested interpretation of quantification. We prefer a reading that is not in accordance with things, unless, as in the first-order language, there are other reasons for reading in accordance with things.

PROF. QUINE: That's true.

PROF. MARCUS: So the question of essentialism arises only on your reading of quantification. For you, the notion of reference is univocal, absolute, and bound up with the expressions, of whatever level, on which quantification is allowed. What I am suggesting is a point of view that is not new to the history of philosophy and logic: That all terms may "refer" to objects, but that not all objects are things, where a thing is at least that about which it is appropriate to assert the identity relation. We note a certain historical consistency here, as, for example, the reluctance to allow identity as a relation proper to propositions. If one wishes, one could say that object-reference (in terms of quantification) is a wider notion than thing-reference, the latter being also bound up with identity and perhaps with other restrictions as well, such as spatiotemporal location. If one wishes to use the word 'refer' exclusively for thing-reference, then we would distinguish those names that refer, from those that name other sorts of objects. Considered in terms of the semantical construction proposed at the end of the paper, identity is a relation that holds between individuals; and their names have thing-reference. To say of a thing a that it necessarily has a property φ, $(\Box(\varphi a))$, is to say that φa is true in every model. Self-identity would be such a property.

PROF. QUINE: Speaking of the objects or the referential end of things in terms of identity, rather than quantification, is agreeable to me in the sense that for me these are interdefinable anyway. But what's appropriately regarded as the identity matrix, or open sentence, in the theory is for me determined certainly by consideration of quantification. Quantification is a little bit broader, a little bit more generally applicable to the theory because you don't always have anything that would fulfill this identity requirement. As to where essentialism comes in: what I have in mind is an interpretation of this quantification where you have an x here (in $\Box((5+4)=x)$). Now, I appreciate that from the point of view of modal logic, and of things that have been done in modal logic in Professor Marcus's pioneer system, this would be regarded as true rather than false:

$$\Box((5+4)=9)$$

This is my point, in spite of the fact that if you think of this ($\Box((5+4)=$ the number of planets)) as what it is generalized from, it ought to be false.

PROF. MARCUS: $\Box((5+4)=$ the number of planets) *would* be false. But this does not preclude the truth of

$$(\exists x)\Box((5+4)=x)$$

any more than the falsehood

$$13 = \text{the number of Christ's disciples}$$

precludes the truth of

$$(\exists x)(13=x)$$

(We would, of course, take '$=$' as '$=_m$' here.)

PROF. QUINE: That's if we use quantification in the ordinary ontological way and that's why I say we put a premium on the *nine* as over against the *number of planets;* we say this term is what is going to be *maßgebend* for the truth value of this sentence in spite of the fact that we get the opposite whenever we consider the other term. This is the sort of specification of the number that counts:

$$5+4=9$$

This is not:

$$5+4=\text{number of planets}$$

I grant further that essentialism does not come in if we interpret quantification in your new way. By quantification I mean quantification in the ordinary sense rather than a new interpretation that might fit most if not all of the formal laws that the old quantification fits. I say 'if not all', because I think of the example of real numbers again. If on the other hand we do not have quantification in the old sense, then I have nothing to suggest at this point about the ontological implications or difficulties of modal logic. The question of ontology wouldn't arise if there were no quantification of the ordinary sort. Furthermore, essentialism certainly wouldn't be to the point, for the essentialism I'm talking about is essentialism in the sense that talks about objects, certain objects; that an object has certain of these attributes essentially, certain others only accidentally. And no such question of essentialism arises if we are only talking of the terms and not the objects that they allegedly refer to. Now, Professor Marcus also suggested that possibly the interpretation could be made something of a hybrid between the two—between quantification thought of as a formal matter, and just talking in a manner whose truth conditions are set up in terms of the expression substituted rather than in terms of the objects talked about; and that there are other cases where we can still give quantification the same old force. Now, that may well be: we might find that in the ordinary sense of quantification I've been talking about there is quantification into nonmodal contexts and no quantification but only this sort of quasi-quantification into the modal ones. And this conceivably might be as good a way of handling such modal matters as any.

PROF. MARCUS: It is not merely a way of coping with perplexities associated with intensional contexts. I think of it as a better way of handling quantification.

You've raised a problem that has to do with the real numbers. Perhaps the Cantorian assumption is one we can abandon. We need not be particularly concerned with it here.

PROF. QUINE: It's one thing I would certainly be glad to avoid, if we can get all of the classical mathematics that we do want.

MR. KRIPKE: This is what I thought the issue conceivably might be, and hence I'll raise it explicitly in this form: Suppose this system contains names, and suppose the variables are supposed to range over numbers, and using '9' as the name of the number of planets, and the usual stock of numerals, '0', '1', '2' . . . and in addition various other primitive terms for numbers, one of which would be 'NP' for the 'number of planets', and suppose '$\Box(9>7)$' is true, according to our system. But say we also have '$\sim\Box(NP>7)$'. Now suppose 'NP' is taken to be as legitimate a name for the number of planets as

'9' (i.e., for this *number*) as the numeral itself. Then we get the odd-seeming conclusion (anyway in your [Marcus's] quantification) that

$$(\exists x, y)(x=y \cdot \Box(x>7) \cdot \sim\Box(y>7))$$

On the other hand, if 'NP' is not taken to be as legitimate a name for the number of planets as '9', then, in that case, I presume that Quine would reply that this sort of distinction amounts to the distinction of essentialism itself. (To Quine:) Would this be a good way of stating your position?

PROF. QUINE: Yes. And I think this formula is one that Professor Marcus would accept under a new version of quantification. Is that right?

PROF. MARCUS: No . . . this wouldn't be true under my interpretation, if the '=' (of Kripke's expression) is taken as identity. If it were taken as identity, it would be not only odd-seeming but contradictory. If it is taken as '$=_m$', then it is not odd-seeming but true. What we must be clear about is that in the extended modal systems with which we are dealing here, we are working within the framework of the theory of types. On the level of individuals, we have only identity as an equivalence relation between individuals. On the level of predicates, or attributes, or classes, or propositions, there are other equivalence relations that are weaker. Now, the misleading aspect of your [Kripke's] formulation is that when you say, "Let the variables range over the numbers," we seem to be talking about individual variables, '=' must then name the identity relation and we are in a quandary. But within a type framework, if *x* and *y* can be replaced by names of numbers, then they are higher-type variables and the weaker equivalence relations are appropriate in such contexts.

MR. KRIPKE: Well, you're presupposing something like the Frege-Russell definition of number, then?

PROF. MARCUS: All right. Suppose numbers are generated as in *Principia* and suppose 'the number of planets' may be properly equated with '9'. The precise nature of this equivalence will of course depend on whether 'the number of planets' is interpreted as a description or a predicate, but in any case, it will be a much weaker equivalence.

MR. KRIPKE: Nine and the number of planets do not in fact turn out to be identically the same?

PROF. MARCUS: No, they're not. That's just the point.

MR. KRIPKE: Now, do you admit the notion of "identically the same" at all?

PROF. MARCUS: That's a different question. I admit identity on the level of individuals certainly. Nor do I foresee any difficulty in allowing the identity relation to hold for objects named by higher-type expressions (except perhaps propositional expressions), other than the ontological consequences discussed in the paper. What I am *not* admitting is that "identically the same" is indistinguishable from weaker forms of equivalence. It is explicit or implicit extensionalizing principles that obliterate the distinction. On this analysis, we could assert that

9 is identically the same as 9

but not

9 is identically the same as $(5+4)$

without some very weak extensionalizing principle that reduces identity to logical equivalence.

MR. KRIPKE: Supposing you have any identity, and you have something varying over individuals.

PROF. MARCUS: In the theory of types, numbers are values for predicate variables of a kind to which several equivalence relations are proper.

MR. KRIPKE: Then, in your opinion the use of *numbers* (rather than individuals) in my example is very important.

PROF. MARCUS: It's crucial.

PROF. QUINE: That's what I used to think before I discovered the error in Church's criticism. And if I understand you, you're suggesting now what I used to think was necessary; namely, in order to set these things up, we're going to have, as the values of variables, not numbers, but assorted number properties, that are equal, but different numbers—the number of planets on the one hand, 9 on the other. What I say now is that this proliferation of entities isn't going to work. For example, take *x* as just as narrow and intensional an object as you like . . .

PROF. MARCUS: Yes, but not on the level of individuals, where only one equivalence relation is present. (We are omitting here consideration of such relations as congruence.)

PROF. QUINE: No, my *x* isn't an individual. The values of 'x' may be properties, or attributes, or propositions, that is, as intensional as you like. I argue that if $\varphi(x)$ determines *x* uniquely, and if *p* is not implied by $\varphi(x)$, still the conjunction $p \cdot \varphi(x)$ will determine that same highly abstract attribute, or whatever it was, uniquely, and yet these two conditions will not be equivalent, and therefore this kind of argument can be repeated for it. My point is, we can't get out of the difficulty by splitting up the entities; we're going to have to get out of it by essentialism. I think essentialism, from the point of view of the modal logician, is something that ought to be welcome. I don't take this as being a *reductio ad absurdum.*

PROF. MCCARTHY: (MIT): It seems to me you can't get out of the difficulty by making 9 come out to be a class. Even if you admit your individuals to be much more inclusive than numbers. For example, if you let them be truth values. Suppose you take the truth value of the 'number of planets is nine', then this is something which is true, which has the value truth. But you would be in exactly the same situation here. If you carry out the same problem, you will still get something that will be 'there exists x, y such that $x=y$ and it is necessary that x is true, but it is not necessary that y is true'.

PROF. MARCUS: In the type framework, the individuals are neither numbers, nor truth values, nor any object named by higher-type expressions. Nor are the values of sentential variables truth values. Sentential or propositional variables take as substituends sentences (statements, names of propositions if you will). As for your example, there is no paradox, since your '=' would be a material equivalence, and by virtue of the substitution theorem, we could not replace 'x' by 'y' in '$\Box x$' (y being contingently true).

PROF. MCCARTHY: Then you don't have to split up numbers, regarding them as predicates either, unless you also regard truth functions as predicates.

PROF. MARCUS: About "splitting up". If we must talk about objects, then we could say that the objects in the domain of individuals are extensions, and the objects named by higher-order expressions are intensions. If one is going to classify objects in terms of the intension—extension dualism, then this is the better way of doing it. It appears to me that a failing of the Carnap approach to such questions, and one that generated some of these difficulties, is the passion for symmetry. Every term (or name) must, according to Carnap, have a dual role. To me it seems unnecessary and does proliferate entities unnecessarily. The kind of evidence relevant here is informal. We do, for example, have a certain hesitation about talking of identity of propositions, and we do acknowledge a certain difference between talking of identity of attributes as against identity in connection with individuals. And to speak of the intension named by a proper name strikes one immediately as a distortion for the sake of symmetry.

FØLLESDAL: The main question I have to ask relates to your argument against Quine's examples about mathematicians and cyclists.

You say that (55) is not provable in QS4. Is your answer to Quine that it is not provable?

PROF. MARCUS: No. My answer to Quine is that I know of no modal system, extended of course, to include the truth of:

It is necessary that mathematicians are rational.

and

It is necessary that cyclists are two-legged.

by virtue of meaning postulates or some such, where his argument applies. Surely if the argument was intended as a criticism of modal logic, as it seems to be, he must have had *some* formalization in mind, in which such paradoxes might arise.

FØLLESDAL: It seems to me that the question is not whether the formula is provable, but whether it's a well-formed formula, and whether it's meaningful.

PROF. MARCUS: The formula in question is entirely meaningful, well formed if you like, given appropriate meaning postulates (defining statements) that entail the necessity of

All mathematicians are rational.

and

All cyclists are two-legged.

I merely indicated that there would be no way of *deriving* from these meaning postulates (or defining statements) as embedded in a modal logic, anything like

It is necessary that John is rational.

given the truth:

John is a mathematician.

although both statements are well formed and the relation between 'mathematician' and 'rational' is analytic. The paradox simply does not arise. What I *did* say is that there is a derivative sense in which one can talk about necessary attributes, in the way that abstraction is derivative. For example, since it is true that

$(x)\Box(x\mathrm{I}x)$

which with abstraction gives us

$$(x)\Box(x \in \hat{y}(yIy))$$

which as we said before, would give us

$$\vdash \boxdot \hat{y}(yIy))$$

The property of self-identity may be said to be necessary, for it corresponds to a tautological function. Returning now to Professor Quine's example, if we introduced constants like 'cyclist', 'mathematician', etc., and appropriate meaning postulates, then the attributes of being either a nonmathematician or rational would also be necessary. Necessary attributes would correspond to analytic functions in the broader sense of analytic. These may be thought of as a kind of essential attribute, although necessary attribute is better here. For these are attributes that belong necessarily to every object in the domain, whereas the usual meaning of essentialism is more restricted. Attributes like mathematician and cylist do not correspond to analytic functions.

PROF. QUINE: I've never said or, I'm sure, written that essentialism could be proved in any system of modal logic whatever. I've never even meant to suggest that any modal logician even was aware of the essentialism he was committing himself to, even implicitly in the sense of putting it into his axioms. I'm talking about quite another thing—I'm not talking about theorems, I'm talking about truth, I'm talking about true interpretation. And what I have been arguing is that if one is to quantify into modal contexts and one is to interpret these modal contexts in the ordinary modal way and one is to interpret quantification as quantification, not in some quasi-quantificatory way that puts the truth conditions in terms of substitutions of expressions, then in order to get a coherent interpretation one has to adopt essentialism and I already explained a while ago just how that comes about. But I did not say that it could ever be deduced in any of the S-systems or any system I've ever seen.

PROF. MARCUS: I was not suggesting that you contended that essentialism could be proved in any system of modal logic. But only that I know of no interpreted modal system, even where extended to include predicate constants such as those of your examples, where properties like being a mathematician would necessarily belong to an object. The kind of uses to which *logical* modalities are put have nothing to do with essential properties in the old ontological sense. The introduction of physical modalities would bring us closer to this sort of essentialism.

FØLLESDAL: That's what creates the trouble when one thinks about properties of this kind, like being a cyclist.

PROF. QUINE: But then you can't use quantifiers as quantifiers.

PROF. MARCUS: The interpretation of quantification has advantages other than those in connection with modalities. For example, many of the perplexities in connection with quantification raised by Strawson in *Introduction to Logical Theory* are clarified by the proposed reading of quantification. Nor is it my conception. One has only to turn to the introduction of *Principia Mathematica* where universal and existential quantification is discussed in terms of 'always true' and 'sometimes true'. It is a way of looking at quantification that has been neglected. Its neglect is a consequence of the absence of a uniform, colloquial way of translating, although we can always find some adequate locution in different classes of cases. It is *easier* to say, "There is a thing which . . ." and since it is adequate some of the time it has come to be used universally with unfortunate consequences.

PROF. QUINE: Well, Frege, who started quantification theory, had the regular ontological interpretation. Whitehead and Russell fouled it up because they confused use and mention.

FØLLESDAL:: It seems from the semantical considerations that you have at the end of the paper, that you need your special axiom.

PROF. MARCUS: Yes, for that construction. I have no strong preferences. It would depend on the uses to which some particular modal system is to be put.

FØLLESDAL:: You think you might have other constructions?

PROF. MARCUS: Indeed. Kripke, for example, has suggested other constructions. My use of this particular construction is to suggest that in discussions of the kind we are having here today, and in connection with the type of criticism raised by Professor Quine in *Word and Object* and elsewhere, it is perhaps best carried out with respect to some construction.

MR. KRIPKE: Forgetting the example of numbers, and using your interpretation of quantification—there's nothing seriously wrong with it at all—does it not require that for any two names, '*A*' and '*B*', of individuals, '*A*=*B*' should be *necessary,* if true at all? And if '*A*' and '*B*' are names of the same individual, that any necessary statement containing 'A' should remain necessary if '*A*' is replaced by '*B*'?

PROF. MARCUS: We might want to say that for the sake of clarity and ease of communication, it would be convenient if to each object there were attached a single name. But we can and we do attach more than one name to a single object. We are here talking of proper names in the ideal sense, as tags and not descriptions. Presumably, if a single object had more than one tag, there would be a way of finding out, such as having recourse to a dictionary[1] or some analogous inquiry, which would resolve the question as to whether the two tags denote the same thing. If 'Evening Star' and 'Morning Star' are considered to be two proper names for Venus, then finding out that they name the same thing that 'Venus' names is different from finding out what is Venus's mass, or its orbit. It is perhaps admirably flexible, but also very confusing, to obliterate the distinction between such linguistic and properly empirical procedures.

MR. KRIPKE: That seems to me like a perfectly valid point of view. It seems to me the only thing Professor Quine would be able to say and therefore what he must say, I hope, is that the assumption of a distinction between tags and empirical descriptions, such that the truth-values of identity statements between tags (but not between descriptions) are ascertainable merely by recourse to a dictionary, amounts to essentialism itself. The tags are the "essential" denoting phrases for individuals, but empirical descriptions are not, and thus we look to statements containing "tags", not descriptions, to ascertain the essential properties of individuals. Thus the assumption of a distinction between "names" and "descriptions" is equivalent to essentialism.

PROF. QUINE: My answer is that this kind of consideration is not revelant to the problem of essentialism because one doesn't ever need descriptions or proper names. If you have notations consisting of simply propositional functions (that is to say, predicates) and quantifiable variables and truth functions, the whole problem remains. The distinction between proper names and descriptions is a red herring. So are the tags. (Marcus: Oh, no.)

All it is is a question of open sentences that uniquely determine. We can get this trouble every time, as I proved with my completely general argument of p in conjunction with φx where x can be as finely discriminated an intension as one pleases—and in this there's no singular term at all except the quantifiable variables or pronouns themselves. This was my answer to Smullyan years ago, and it seems to me the answer now.

MR. KRIPKE: Yes, but you have to allow the writer what she herself says, you see, rather than arguing from the point of view of your own interpretation of the quantifiers.

1. Since such entries are usually described as "nonlexical", the dictionary here functions as an encyclopedia. But this whole passage needs clarification and emendation. As indicated in the earlier text, discovering that Hesperus and Phosphorus have the same path is an empirical discovery that entails that they are identical and hence that 'Hesperus' and 'Phosphorus' name the same thing. But the identity, once given, is necessary [added 1991].

PROF. QUINE: But that changes the subject, doesn't it? I think there are many ways you can interpret modal logic. I think it's been done. Prior has tried it in terms of time and one thing and another. I think any consistent system can be found an intelligible interpretation. What I've been talking about is quantifying, in the quantificational sense of quantification, into modal contexts in a modal sense of modality.

MR. KRIPKE: Suppose the assumption in question is right—that every object is associated with a tag, which is either unique or unique up to the fact that substituting one for the other does not change necessities—is that correct? Now, then granted this, why not read "there exists an x such that necessarily p of x" as (put in an ontological way if you like) "there exists an object x with a name a such that p of a is analytic." Once we have this notion of name, it seems unexceptionable.

PROF. QUINE: It's not very far from the thing I was urging about certain ways of specifying these objects being by essential attributes and that's the role that you're making your attributes play.

MR. KRIPKE: So, as I was saying, such an assumption of names is equivalent to essentialism.

PROF. COHEN: I think this is a good friendly note on which to stop.

APPENDIX 1B: SMULLYAN ON MODALITY AND DESCRIPTION

The following commentary is taken from a review in the *Journal of Symbolic Logic*, XIII (1948): 149–150. ■

In his paper "Modality and Description" (*Journal of Symbolic Logic*, XIII (1948): 31–37) Arthur F. Smullyan is concerned with an antinomy that allegedly arises from substituting equals for equals in modal contexts. Quine for example has argued that the true premises

A. It is logically necessary that 9 is less than 10.

B. 9 = the number of planets.

lead to the false conclusion

C. It is logically necessary that the number of planets is less than 10.

Smullyan attempts to resolve this dilemma within a logical system such as that of *Principia Mathematica* that has been appropriately extended to include modal operators. He presents two analogous solutions. In the first, the expression

D. the number of planets

abbreviates a descriptive phrase, and descriptions are treated in accordance with *14 of *Principia Mathematica*. In the second, D is interpreted as an abstract, and abstracts are introduced in a manner similar to that of Russell.

Where D is construed as a description, A, B, C can be regarded as an instance of

A′. $N(Fy)$

B′. $y = (\iota x)(\phi x)$

C′. $N[F(\iota x)(\phi x)]$

Smullyan shows that at most, A′ and B′ yield

$$(\exists x)((\phi z) \equiv_z z = x \cdot N(Fx))$$

which is equivalent to

$$[(\imath x)(\phi x)] \cdot N(F(\imath x)(\phi x))$$

If D is interpreted as an abstract, the method of solution is similar, provided that the definition employed is unambiguous with respect to the scope of the abstract. Smullyan chooses the definition

E. $[\hat{x}(Gx)] \cdot F\hat{x}(G\hat{x}) =_{df} (\exists \alpha)(Gx \equiv_x x \in \alpha \cdot Fa)$
where α is a class variable

A, B, C may then be regarded as illustrating

A″. $N[f\hat{x}(Ax)]$

B″. $\hat{x}(Ax) = \hat{x}(Bx)$

C″. $N[f\hat{x}(Bx)]$

If A″, B″, and C″ are expanded in accordance with E, it is apparent that C″ cannot be inferred from the premises A″ and B″.

In the reviewer's opinion, Smullyan is justified in his contention that the solution of Quine's dilemma does not require any radical departure from a system such as that of *Principia Mathematica*. Indeed, since such a solution is available, it would seem to be an argument in favor of Russell's method of introducing abstracts and descriptions.

If modal systems are not to be entirely rejected, a solution of the dilemma A, B, C requires that the equality relation that holds between expressions such as '9' and 'the number of planets' must be distinguished from the equality relation that holds, for example, between the expressions '9' and '7+2'. This distinction must be such that '7+2' may replace '9' in modal contexts but 'the number of planets' may not. In a system of the kind with which Smullyan is presumably concerned, an unrestricted substitution theorem can be proved only for expressions that are necessarily equivalent. (See the reviewer's paper, *Journal of Symbolic Logic*, XI [1946]). Thus where E is construed in terms of abstracts (and this is the more natural interpretation), the deduction of C″ from A″ and B″ is precluded by the aforementioned restriction on substitution. A″ is an abbreviation for

$$N(\exists \alpha)(Ax \equiv_x x \in \alpha \cdot f\alpha)$$

and B″ is equivalent to

$$Ax \equiv_x Bx$$

Since 'Ax' comes within the scope of a modal operator in A″, 'Ax' cannot be replaced by 'Bx'.

Although Smullyan constructs his modal system informally, his assumption

S2. $N[x \in \alpha \equiv_x x \in \beta \supset \alpha = \beta]$

(p. 36) seems to lead to the result that if two classes have the same members, then it is necessary that they have the same members. Let '$A \prec B$' and '$A \equiv B$' abbreviate '$N(A \supset B)$' and '$N(A \equiv B)$' respectively. The major steps in the proof of this result are as follows: $N(\alpha = \alpha)$, $(\alpha = \beta) \prec (N(\alpha = \alpha) \supset N(\alpha = \beta))$, $(\alpha = \beta) \prec N(\alpha = \beta)$, $(x \in \alpha \equiv_x x \in \beta) \equiv (\alpha = \beta)$, $(x \in \alpha \equiv_x x \in \beta) \prec N(x \in \alpha \equiv_x x \in \beta)$.

S2 could be replaced by

S2′. $N(x \in \alpha \equiv_x x \in \beta) \supset (\alpha = \beta)$

without altering Smullyan's general analysis. If S2′ were assumed in place of S2, it would not be necessary to reject

S3. $(\exists \alpha) N(\phi x \equiv_x x \in \alpha)$

as an assumption that leads to seemingly paradoxical results.

2. *Iterated Deontic Modalities*

The source of this paper appeared in *Mind,* LXXV, n.s. 300 (1966): 580–582. Although deontic logic has had a considerable evolution and refinement since that time, many of the problems of interpretation remain. ■