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UNIFYING QUANTIFIED MODAL LOGIC

1. Introduction

Quantified modal logic (QML) has reputation for complexity. Completeness results for the various systems appear piecemeal. Different tactics are used for different systems, and success of a given method seems sensitive to many factors, including the specific combination of choices made for the quantifiers, terms, identity, and the strength of the underlying propositional modal logic. The lack of a unified framework in which to view QMLs and their completeness properties puts pressure on those who develop, apply, and teach QML to work with the (allegedly) simplest systems, namely those that adopt the Barcan Formulas and predicate logic rules for the quantifiers. In these systems, the quantifier ranges over a fixed domain of possible individuals, so advocates of these logics are sometimes called possibilists. A literature has grown up rationalizing the choice of possibilist logics despite ordinary intuitions that the resulting theorems are too strong (Cresswell, 1991; Linsky and Zalta, 1994). Williamson (1998, p. 262) even takes the view that the complications to be faced within the weaker logics "are a warning sign of philosophical error". It is the purpose of this paper to show that abandonment of the weaker QMLs is excessively fainthearted, since most QMLs can be given relatively simple formulations within one general framework. Given the straightforward nature of the systems and their completeness results, the purported complications evaporate, along with any philosophical warnings one might have associated with them.

1.1. *The Motivation for this Study*

It is not the purpose of this paper to rebut arguments supporting the view that the "right" QML is based on classical predicate logic and the Barcan Formulas. However, some considerations are worth recording here to motivate the unification project. Heavy demands are placed on possibilists to pacify the ontological scruples of those with an aversion to quantification over possibilia. So whatever arguments are presented to support possibilism, some philosophers will remain unconvinced.[1] Given that it is the function of logic to provide reasoning tools that do not beg questions still under dispute, it is of interest to provide technical results exploring the weaker logics and their relationships one to another.

In the battle over possibilism, one may forget that significant applications of logic lie outside of philosophy, notably in artificial reasoning and natural language processing, where disputes over the metaphysics of modal quantification are largely not in play. A serious problem for possibilist logics is that they entail theorems such as $\forall x \Box \exists y y = x$, which are unacceptable when quantifiers are given the reading found in normal conversation: everything necessarily exists. Possibilists will point out that on *their* interpretation $\forall x \Box \exists y y = x$ is acceptable; and if one wants, an actualist quantifier $\forall_a x$ that conforms more closely to ordinary usage can be defined within their logic by $\forall_a x \mathrm{A}x =_{\mathrm{df}} \forall x(\mathrm{E}x \to \mathrm{A}x)$, where E is a predicate letter for 'is actual'. However, this fails to provide a convincing argument for abandoning study of the weaker logics. Kripke semantics shows that modal logics can be embedded in set theory, but that embedding does not disarm the importance of locating consistent and complete systems expressed wholly in the language of modal logic. Similarly, although actualist quantifiers can be embedded in a possibilist logic, it is still important to find out what logical principles adequately govern a language containing only the actualist quantifier.[2]

Even if they win the possibilist–actualist debate, possibilists still face another problem. $\exists x \Box x = t$ is a theorem, but invalid on the possibilist reading of the quantifier when the term t is a non-rigid designator – a term that refers to different objects in different possible worlds.[3] Possibilists will respond by eliminating non-rigid terms from the language in favor of definite descriptions. But again, the possibility of such an embedding into a description theory does not undermine the value of working out the logical principles governing non-rigid terms *in the original language*. So even within the possibilist approach to quantification, weaker logics are needed to give a thorough account of non-rigid designation. Considering the widespread use of non-rigid terms in ordinary language, logics that introduce such terms directly should be explored.

1.2. *Main Ideas Behind the Unification*

The unification of QML presented here begins with a weak but *general* logic G. A collection of axioms and rules for extending G is then introduced. Completeness results for G and (almost) all its extensions are shown in a common framework by demonstrating that the rules and axioms added to G guarantee corresponding conditions on a canonical model. With one notable exception, major systems found in the literature can be brought under this general scheme.

This unification does not depend on deep technical insights. The completeness results are "easy" in that they transparently follow given well known strategies. Instead, progress has resulted from choosing the right generalizations in the proof theory and semantics for G. One useful (but not essential) proof theoretic idea has been to generalize natural deduction formulations for the rules to what I call modal arguments. The reader may already be familiar with the introduction of subproofs indexed by the modal operator $\Box$ in the formulation of natural deduction rules for propositional modal logic (Garson, 2001, p. 298). A variation on this idea is to "promote" the $\Box$ that indexes such a subproof to the head of the subproof, treating the $\Box$ as an "honorary" hypothesis. Within this more general scheme, a *modal argument* L / C is composed of a hypothesis list L consisting of an (ordered) list of sentences and occurrences of $\Box$. The *modal sequent* $\mathrm{L} \vdash \mathrm{C}$ asserts that there is a proof of C within subproofs headed by members of L (in that order). So for example, the sequent $p, \Box, q \vdash r$, expresses that r can be derived in nested subproofs headed by p, $\Box$, and q respectively. Following this idea, the basic propositional modal logic K is formulated with modal sequent rules for propositional logic plus the rules ($\Box$In) and ($\Box$Out):

$$(\Box\mathrm{In}) \quad \frac{\mathrm{L}, \Box \vdash \mathrm{A}}{\mathrm{L} \vdash \Box\mathrm{A}} \qquad (\Box\mathrm{Out}) \quad \frac{\mathrm{L} \vdash \Box\mathrm{A}}{\mathrm{L}, \Box \vdash \mathrm{A}}$$

This greatly simplifies the presentation of the rules for the basic system G, and provides resources for simplifying the soundness and completeness proofs as well.

For example, a major annoyance in formulating universal generalization principles for weaker QMLs has been the need to "prefix" the ordinary rule with arbitrarily long sequences of strict implications as follows.

$$\frac{\mathrm{A}_1 \to (\mathrm{A}_2 \Rightarrow \cdots (\mathrm{A}_i \Rightarrow (\mathrm{E}c \to \mathrm{A}c)) \cdots)}{\mathrm{A}_1 \to (\mathrm{A}_2 \Rightarrow \cdots (\mathrm{A}_i \Rightarrow \forall x\mathrm{A}) \cdots)},$$

where c is a free variable not in the conclusion.

(Here '$\mathrm{A} \Rightarrow \mathrm{B}$' abbreviates $\Box(\mathrm{A} \to \mathrm{B})$.) However, when rules are formulated with modal sequents, the rule takes on a simple form familiar from free logic.

$$\frac{\mathrm{L} \vdash \mathrm{E}c \to \mathrm{A}c}{\mathrm{L} \vdash \forall x\mathrm{A}}, \quad \text{where } c \text{ is a free variable not in the conclusion.}$$

On the semantical side, the various treatments of modal quantification are all brought under a general intensional truth condition which is a generalization of the substantial interpretation of the quantifier (Garson, 2001, p. 283). Quantification in G ranges over *individual concepts*, that is, functions from the set **W** of possible worlds to a set **D** of objects. A global domain of individual concepts **I** is introduced, along with sets **Iw** of individual concepts to serve as local domains for each of the possible worlds **w**. Note that the **Iw** need *not* be subsets of **I**, and the intensions of some terms may be chosen outside any of the **Iw** and even outside **I**. This freedom is important accommodating different treatments of the quantifiers, especially the objectual interpretation. The generalization in the treatment of the domains **Iw** is reflected in the truth conditions for the primitive existence predicate E. The formula Et is true in a world **w** iff the *individual concept* assigned to t is a member of **Iw** (not the object assigned to t at world **w**).[4]

The syntax of G includes symbols $\approx$ for (contingent) identity, and E for existence. Since $\Box$ creates opaque contexts, the normal rule of substitution is restricted to atomic formulas. The existence predicate E creates opaque contexts as well. Therefore the substitution principle $s{\approx}t \rightarrow (\mathrm{E}s \rightarrow \mathrm{E}t)$ is invalid. In its place, the weaker rule (iE) is required, which guarantees substitution of t for s behind E when $s{\approx}t$ is a *theorem*.

$$\text{(iE)} \quad \frac{\vdash \quad s{\approx}t}{\vdash \ \mathrm{E}s \rightarrow \mathrm{E}t}$$

Garson (2001, p. 300) omits this rule, and still manages to prove completeness. But this was possible only because there happened to be no theorems of the form $s{\approx}t$ in systems he considered. If G is to serve as a correct foundation for description theory, mathematics or science, (iE) is crucial.

1.3. *Some Results of the Unification*

One may feel that the "intensional" treatment of quantification just described is excessively general. However, this paper provides evidence that it is fundamental. The substitution interpretation of the quantifier provides a test bed for treating quantification in an ontologically neutral way, since no explicit domain of quantification is required. In Section 6.1, it is shown that G validates exactly the arguments valid for the substitution interpretation of the quantifier.

The standard approach to quantification in QML is the objectual interpretation, where quantifiers range over objects rather than individual concepts. This choice may be captured in G with the condition (Ic) that **I** be the set of all constant functions with values in **D**. Since constant functions may be "identified" with their values, (Ic) has exactly the effect of quantification over objects. (See Section 6.2 for a detailed proof.) By locating the principles that correspond to the condition that **I** contain all and only constant functions in the canonical model, the corresponding logic oG is generated. Constructing oG as an extension of G simplifies the rules for the objectual interpretation, and provides insight into the role of rules previously given in the literature.

The idea that the objectual interpretation can be captured with the stipulation that the domains contain constant functions is nothing new. For example, Garson (2001, Section 1.4) suggests this in passing. However, the situation is more complicated than it may first seem. It is not clear that the result is even forthcoming given the semantics provided there. That semantics contains the local domains **Iw**, but **I** is not introduced, and so not mentioned in the truth condition for the quantifier. The attempt to capture the objectual interpretation within this semantics by stipulating that members of **Iw** are constant functions does not succeed because this validates $\mathrm{E}t\&\mathrm{E}s\&t{\approx}s \rightarrow \Box t{\approx}s$, which is not valid on the objectual interpretation given that terms s and t are non-rigid. When the constant function condition is imposed on **I**, rather than **Iw**, the difficulties are resolved.

The presence of non-rigid terms complicates the formulation of systems for the objectual quantification in another way. The problem has been to define an adequate rule of Universal Instantiation. When the term t is non-rigid, the standard axiom $\forall x\mathrm{A} \rightarrow \mathrm{A}t$ fails, and so does its free logic counterpart $\forall x\mathrm{A} \rightarrow (\mathrm{E}t \rightarrow \mathrm{A}t)$. The failure is due to the fact that the bound variable x in $\forall x\mathrm{A}$ may lie in an opaque context, so that replacement of t for x in A does not preserve truth in world **w**, even when x and t corefer in world **w** (Garson, 2001, p. 303). Hintikka (1970) proposes a very complex alternative instantiation principle but gives no completeness proof. Thomason (1970) shows completeness when the modality is S4 or stronger, where Hintikka's rule can be simplified to $\forall x\mathrm{A} \rightarrow (\exists x\Box x{\approx}t \rightarrow \mathrm{A}t)$. Such complications have caused many to avoid systems that include non-rigid terms.

The solution provided by oG is surprisingly simple, and does not depend on the strength of the underlying propositional modal logic. Here the free variables are treated as rigid designators, thus validating $\forall x\mathrm{A} \rightarrow (\mathrm{E}c \rightarrow \mathrm{A}c)$ – the free logic rule restricted to free variables c. Hintikka and Thomason apparently labor under the misconception that completeness requires some additional (and more complex) rule to govern instantiation for the non-rigid terms, and Garson (2001, p. 304) presumes that the introduction of a primitive existence predicate is required. However, no new instantiation principle is needed to guarantee completeness of the logic, for Hintikka's complex instantiation principles emerge in oG as derived rules.[5]

Furthermore, the existence predicate can be eliminated in oG by defining it in the "normal" way: $\mathrm{E}t =_{\mathrm{df}} \exists x x{\approx}t$.

The system oG also throws light on the role of a principle first used by Thomason to develop complete systems for the system Q3–S4.[6] It may be simplified to the following rule of oG.

$$(\exists\mathrm{i}) \quad \frac{\mathrm{L} \vdash \sim t{\approx}c}{\mathrm{L} \vdash \bot}, \quad \text{where } c \text{ does not appear in L, or } t.$$

One wonders whether this rule or its ancestors can be dispensed with, and the answer is that they cannot. In the semantics for G, (∃i) corresponds to the condition that for each world **w** and each term t, there is an individual concept i in **I** such that $i(\mathbf{w}) = \mathbf{a}_{\mathbf{w}}(t)$. The completeness proof supporting this correspondence immediately provides an independence proof for (∃i) within oG, since models of the remainder of oG that lack this condition are easily constructed. (See Section 7.1 for the details.) It also reveals the important role of (∃i) in guaranteeing that the set of constant functions **I** includes a member whose value is d, for each object d in **D**, so that the full force of the objectual interpretation is obtained.

This way of looking at (∃i) has led to the discovery that (∃i) secures a proof of $\mathrm{E}t \leftrightarrow \exists x x{\approx}t$ in oG to support the elimination of E by definition. Given these major simplifications possible in oG, there is no justice in the complaint that actualist logics must be significantly more complex than their possibilist rivals.

The unification also provides interesting results concerning the Barcan Formula. Within G, the Barcan Formula (BF) is too weak to enforce the "Contracting Domains" condition (CD) with which it has been traditionally associated.

(BF) $\forall x\Box\mathrm{A} \rightarrow \Box\forall x\mathrm{A}$
(CD) If **wRv** then the domain for **v** is a subset of the domain for **w**.

It will be shown in Section 7.2 that the axiom (oCD): $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ which captures the idea more directly, is independent in G+(BF), even when the underlying modality is S5. Only in stronger systems such as oG does (BF) suffice. This result resolves a puzzle posed by Fitting and Mendelsohn (1998, p. 185). They presume (BF) is derivable in a logic for "expanding domains" given the symmetry axiom (B), and (CBF), the Converse of the Barcan Formula.

(B) $\mathrm{A} \rightarrow \Box\Diamond\mathrm{A}$
(CBF) $\Box\forall x\mathrm{A} \rightarrow \forall x\Box\mathrm{A}$

However, they report that they cannot find the proof when $\approx$ is absent from the language. In Section 7.2 it is shown that there cannot be such a proof. Note that Corsi (2002) obtains the same result using a very different strategy.

It should be admitted that the results given here do not thoroughly unify QML. First, the completeness proofs given below are not entirely modular, i.e. they do not cover every conceivable combination of principles listed as possible additions to G. Second, there are limitations related to the strength of the underlying propositional modal logic. For example, the method does not work for some conditions on frames (such as convergence) that are expressed in the metalanguage with existential quantification over possible worlds. This is to be expected since some of these QMLs are known to be incomplete. Possibilists decrying the complexities of actualist logic can take little encouragement from these limitations, for completeness fails for possibilist systems as well. For example, systems with Barcan Formulas and convergent frames are not complete (Cresswell, 1995).

Although most of the major approaches to QML can be treated within G, there is one important exception. G is based on rules for free logic, but the standard quantifier rules may be obtained by adopting an additional axiom that entails the classical principle of Universal Instantiation. One consequence of formulating the rules with *modal* sequents is that the standard quantifier rules immediately entail the Barcan Formula (as well as its converse). Therefore the "expanding domain" systems (Hughes and Cresswell, 1996, Ch. 15), which adopt classical quantifier rules without the Barcan Formula, lie outside this unification. (For results unifying such systems see (Corsi, 2002).) Note that the modal sequent version (∀In) of Universal Generalization (given below) is demonstrably stronger than the standard rule of Universal Generalization, for it is well known that (BF) is not provable in the weaker expanding domain systems. On the other hand, it is easy to prove that (∀In) is derivable from the standard rules when (BF) is available, so results given here apply to all systems of this kind.

2. THE BASIC SYSTEM G

2.1. *Syntax*

A language for the system G contains parentheses, ∀, →, ⊥ (for a contradiction), ≈ (for contingent identity), E (for existence), predicate letters $\mathrm{P}', \mathrm{P}'', \ldots$ bound variables $x', x'', \ldots$ and terms, some of which belong to an infinite set C of free variables. (Complex terms such as descriptions and function symbols are omitted in this study but they may be added without difficulty.) To simplify the presentation, it is assumed that the free and bound variables are distinct and that no well-formed formula contains a bound variable x outside the scope of the quantifier $\forall x$. We use 's', 't', and 'u' to range over terms and 'a', 'b', and 'c' to range over the set C of free variables. The logical constants $\sim$, &, $\vee$, $\leftrightarrow$, $\exists$, are defined from this basic vocabulary in the usual way. (For example, $\sim$A is defined by $\mathrm{A} \to \bot$.)

In contexts where a universal sentence $\forall x\mathrm{A}$ has been mentioned, we use the notation 'At' to indicate the result of substituting the term t properly for all occurrences of bound variable x in A.[7] We leave the formation rules for wffs of the language of G to the reader. We use 'l' and 'l'' as metavariables over lists $t_1, \ldots, t_n$ of terms (including null lists), and presume that atomic sentences are formed by concatenating a predicate letter P with a list l whose length is the arity of P.

2.2. *Proof Theory*

The rules of G include principles for propositional logic,[8] axioms for a propositional modal logic meeting the conditions to be described in Section 5.1, plus the following principles governing $\Box$, $\forall$, E, and $\approx$.

($\Box$In) $\dfrac{\mathrm{L}, \Box \vdash \mathrm{A}}{\mathrm{L} \vdash \Box\mathrm{A}}$ ($\Box$Out) $\dfrac{\mathrm{L} \vdash \Box\mathrm{A}}{\mathrm{L}, \Box \vdash \mathrm{A}}$

($\forall$In) $\dfrac{\mathrm{L} \vdash \mathrm{E}c \to \mathrm{A}c}{\mathrm{L} \vdash \forall x\mathrm{A}}$ for c not in L or $\forall x$A. ($\forall$Out) $\dfrac{\mathrm{L} \vdash \forall x\mathrm{A}}{\mathrm{L} \vdash \mathrm{E}c \to \mathrm{A}c}$

($\approx$In) $t{\approx}t$ ($\approx$Out) $\dfrac{\mathrm{L} \vdash t{\approx}s}{\mathrm{L} \vdash \mathrm{P}l, t, l' \to \mathrm{P}l, s, l'}$

(iE) $\dfrac{\vdash s{\approx}t}{\vdash \mathrm{E}s \to \mathrm{E}t}$

2.3. *Semantics*

A *G-model* $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{a}\rangle$ contains a frame $\langle \mathbf{W}, \mathbf{R}\rangle$ appropriate for the underlying propositional modal logic, and a set (non-empty) $\mathbf{D}$ of objects. $\mathbb{I}$ consists of $\mathbf{I}$, a non-empty subset of $\underline{\mathrm{IC}}$, the set of functions from $\mathbf{W}$ into $\mathbf{D}$, and local domains $\mathbf{Iw}$ which are subsets of IC (not necessarily subsets of $\mathbf{I}$), for each possible world $\mathbf{w}$ in $\mathbf{W}$. The assignment function $\mathbf{a}$ assigns an intension $\mathbf{a}(e)$ to each primitive expression e of the language as follows.

(c) $\mathbf{a}(c) \in \mathbf{I}$.
(t) $\mathbf{a}(t) \in \mathrm{IC}$.
(P) $\mathbf{a}(\mathrm{P})$ is a function from $\mathbf{W}$ into the powerset of the set of all lists of objects in $\mathbf{D}$ of length i, where i is the arity of P.

Note that (c) is required to guarantee the validity of ($\forall$Out). When $\mathbf{a}(e)$ is the intension of an expression e, we abbreviate the extension $\mathbf{a}(e)(\mathbf{w})$ of $\mathbf{a}(e)$ in world $\mathbf{w}$ by '$\mathbf{a}_\mathbf{w}(e)$'. When l is the list $t_1, \ldots, t_n$, the following abbreviation is in force.

(l) $\mathbf{a}_\mathbf{w}(l) = \mathbf{a}_\mathbf{w}(t_1, \ldots, t_n) = \mathbf{a}_\mathbf{w}(t_1), \ldots, \mathbf{a}_\mathbf{w}(t_n)$.

The extension given by the assignment function for any sentence is defined as either T (true) or F (false) by the following.

Basic Truth Conditions.

($\bot$) $\mathbf{a}_\mathbf{w}(\bot) = \mathrm{F}$.
($\approx$) $\mathbf{a}_\mathbf{w}(s{\approx}t) = \mathrm{T}$ iff $\mathbf{a}_\mathbf{w}(s) = \mathbf{a}_\mathbf{w}(t)$.
(Pl) $\mathbf{a}_\mathbf{w}(\mathrm{P}l) = \mathrm{T}$ iff $\mathbf{a}_\mathbf{w}(l) \in \mathbf{a}_\mathbf{w}(\mathrm{P})$.
(iE) $\mathbf{a}_\mathbf{w}(\mathrm{E}t) = \mathrm{T}$ iff $\mathbf{a}(t) \in \mathbf{Iw}$.
($\to$) $\mathbf{a}_\mathbf{w}(\mathrm{A} \to \mathrm{B}) = \mathrm{T}$ iff $\mathbf{a}_\mathbf{w}(\mathrm{A}) = \mathrm{F}$ or $\mathbf{a}_\mathbf{w}(\mathrm{B}) = \mathrm{T}$.
($\Box$) $\mathbf{a}_\mathbf{w}(\Box\mathrm{A}) = \mathrm{T}$ iff for each $\mathbf{v}$ such that $\mathbf{wRv}$, $\mathbf{a}_\mathbf{v}(\mathrm{A}) = \mathrm{T}$.
($\forall$) $\mathbf{a}_\mathbf{w}(\forall x\mathrm{A}) = \mathrm{T}$ iff all $i \in \mathbf{I}$, if $i \in \mathbf{Iw}$, then $\mathbf{a}_\mathbf{w}(\mathrm{A}i) = \mathrm{T}$.

The notation '$\mathbf{a}_\mathbf{w}(\mathrm{A}i) = \mathrm{T}$' in condition ($\forall$) abbreviates '$\mathbf{a}^i/_{c\mathbf{w}}(\mathrm{A}c) = \mathrm{T}$', where c is the alphabetically first free variable chosen so that it does not appear in $\forall x$A, and $\mathbf{a}^i/_c$ is the assignment function exactly like $\mathbf{a}$ save that $\mathbf{a}^i/_c(c) = i$. Truth conditions ($\sim$), ($\Diamond$), and ($\exists$) for defined notation are fixed by the basic truth conditions, and are left to the reader. This completes the definition of a G-model.

Validity for *modal* arguments L/C is defined as follows. The notation '$\mathbf{a}_\mathbf{w}(\mathrm{L}) = \mathrm{T}$' (for 'L is satisfied by $\mathbf{a}$ at $\mathbf{w}$'), is defined recursively on the length of L as follows.

When L is empty, $\mathbf{a}_\mathbf{w}(\mathrm{L}) = \mathrm{T}$.
$\mathbf{a}_\mathbf{w}(\mathrm{L}, \mathrm{A}) = \mathrm{T}$ iff $\mathbf{a}_\mathbf{w}(\mathrm{L}) = \mathrm{T}$ and $\mathbf{a}_\mathbf{w}(\mathrm{A}) = \mathrm{T}$.
$\mathbf{a}_\mathbf{w}(\mathrm{L}, \Box) = \mathrm{T}$ iff for some $\mathbf{v}$ in $\mathbf{W}$, $\mathbf{vRw}$ and $\mathbf{a}_\mathbf{v}(\mathrm{L}) = \mathrm{T}$.

So for example, $\mathbf{a}_\mathbf{w}(p, \Box, q) = \mathrm{T}$ provided that there is a world $\mathbf{v}$ in $\mathbf{W}$ such that $\mathbf{a}_\mathbf{v}(p) = \mathrm{T}$ and $\mathbf{vRw}$ and $\mathbf{a}_\mathbf{w}(q) = \mathrm{T}$. When S is some condition on G-models, we say *L is S-satisfiable* iff L is satisfied by $\mathbf{a}$ at $\mathbf{w}$ in some G-model which obeys S. Argument L/C has an *S-counterexample* iff the list L, $\sim$C is S-satisfiable, and L/C is *S-valid* ($\mathrm{L} \models_\mathrm{S} \mathrm{C}$) iff L/C has no S-counterexample. A rule *preserves S-validity* just in case whenever the premise(s) of the rule is (are) S-valid so is the conclusion. Officially, axioms are treated as zero premise rules, and should be written as modal sequents with the form: L ⊢ C. However, we omit 'L ⊢' in the statement of axioms. An *axiom preserves S-validity* iff it is S-valid.

3. Extensions of G

Here rules and axioms (hereafter called rules) will be listed which can be used to extend G to a stronger logic S. For each rule or rules, we list to its left a corresponding condition on models. With certain restrictions to be noted in Section 5.1, any system S formed by adding these rules to G is adequate (i.e. both sound and complete) with respect to S-validity where S is the conjunction of the corresponding conditions for those rules. For convenience, rules, systems and their corresponding conditions are given the same names. For condition (eE) below, the (*e*xtensional) domain **Dw** for each world **w** is defined by (Dw).

(Dw) $\mathbf{Dw} = \{f(\mathbf{w}) : f \in \mathbf{Iw}\}$.

Semantic Condition	*Rule(s)*
(∃E) **Iw** is not empty	$\exists x Ex$
(Ec) $\mathbf{a}(c) \in \mathbf{Iw}$	Ec
(Et) $\mathbf{a}(t) \in \mathbf{Iw}$	Et
(EI) If **wRv**, then $\mathbf{Iw} \subseteq \mathbf{Iv}$	$Et \rightarrow \Box Et$
(CI) If **wRv**, then $\mathbf{Iv} \subseteq \mathbf{Iw}$	$\Diamond Et \rightarrow Et$
(∃i) for some $i \in \mathbf{I}$, $i(\mathbf{w}) = \mathbf{a}_{\mathrm{w}}(t)$	$L \vdash \sim t \approx c / L \vdash \bot$, c not in L, t
(rc) $\mathbf{a}(c)$ is a constant function	$(b \approx c \rightarrow \Box b \approx c)$& $(\sim b \approx c \rightarrow \Box \sim b \approx c)$
(Ic) **I** is the set of all constant functions with values in **D**.	(Ic) = (∃i) + (rc)
(eE) $\mathbf{a}_{\mathrm{w}}(Et) = T$ iff $\mathbf{a}_{\mathrm{w}}(t) \in \mathbf{Dw}$	$s \approx t \rightarrow (Es \rightarrow Et)$

System oS for the objectual interpretation is defined as S + (Ic) + (eE). All the correspondences above hold for oS plus the following.

Semantic Condition	*Rule*
(oEc) $\mathbf{Dw} = \mathbf{D}$	Ec
(oED) If **wRv**, then $\mathbf{Dw} \subseteq \mathbf{Dv}$	$Ec \rightarrow \Box Ec$ or (CBF): $\Box \forall x A \rightarrow \forall x \Box A$
(oCD) If **wRv**, then $\mathbf{Dv} \subseteq \mathbf{Dw}$	$\Diamond Ec \rightarrow Ec$ or (BF): $\forall x \Box A \rightarrow \Box \forall x A$

4. Consistency

4.1. *Consistency of G*

The adequacy proofs for the QMLs described in this paper depend on the following two theorems, which can be proven by induction on the structure of A.

INSTANCE THEOREM. *If* $\mathbf{a}(t) = i$ *then* $\mathbf{a}_{\mathrm{w}}(At) = \mathbf{a}_{\mathrm{w}}(Ai)$.

NO c THEOREM. *If two* S *models are identical save that their respective assignment functions* **a** *and* **b** *agree on all predicate letters and terms except for a single term c, and c does not appear in* $\forall x A$, *then* $\mathbf{a}_{\mathrm{w}}(Ai) = \mathbf{b}_{\mathrm{w}}(Ai)$, *and if c does not appear in* A, *then* $\mathbf{a}_{\mathrm{w}}(A) = \mathbf{b}_{\mathrm{w}}(A)$, *hence when c is not in* L, $\mathbf{a}_{\mathrm{w}}(L) = \mathbf{b}_{\mathrm{w}}(L)$.

To show that G is consistent, one shows that the rules of G preserve G-validity. Details related to rules (□In) and (∀In) are given here, and the rest is left to the reader. (Throughout this paper, subscripts on '⊢' and '⊨' are dropped where they can be recovered from the context.) The proofs that (□In) and (□Out) preserve S-validity depend on the following feature of the definition of $\mathbf{a}_{\mathrm{w}}(L)$.

(L□) If $\mathbf{a}_{\mathrm{w}}(L, \Box) = T$ iff for some $\mathbf{v} \in \mathbf{W}$, $\mathbf{a}_{\mathrm{v}}(L) = T$ and **vRw**.

(□In) To show that if L, □ ⊨ A then L ⊨ □A, assume that L, □ ⊨ A, and suppose there is a G-model with a world **v** in **W** where $\mathbf{a}_{\mathrm{v}}(L) = T$, and show $\mathbf{a}_{\mathrm{v}}(\Box A) = T$ as follows. Suppose that **w** is any world such that **vRw**. By (L□), $\mathbf{a}_{\mathrm{w}}(L, \Box) = T$, and so by L, □ ⊨ A, $\mathbf{a}_{\mathrm{w}}(A) = T$. So whenever **vRw**, $\mathbf{a}_{\mathrm{w}}(A) = T$ and $\mathbf{a}_{\mathrm{v}}(\Box A) = T$ follows by the truth condition (□).

The proof for (□Out) is similar.

(∀In) Assuming L ⊨ Ec → Ac and c is not in L nor in $\forall x A$, we must show L ⊨ $\forall x A$. To do that, suppose ⟨**W**, **R**, **D**, 𝕀, **a**⟩ is any G-model where $\mathbf{a}_{\mathrm{w}}(L) = T$, and i is any member of both **I** and **Iw**, and demonstrate $\mathbf{a}_{\mathrm{w}}(Ai) = T$ as follows. Construct a new G-model ⟨**W**, **R**, **D**, 𝕀, **b**⟩ with **b** identical to $\mathbf{a}^{i}/_{c}$, hence $\mathbf{b}(c) = i$. We have $\mathbf{b}(c) \in \mathbf{Iw}$, so by (iE), $\mathbf{b}_{\mathrm{w}}(Ec) = T$. Since c does not appear in L, the No c Theorem entails $\mathbf{b}_{\mathrm{w}}(L) = T$, and so by L ⊨ E$c$ → Ac, $\mathbf{b}_{\mathrm{w}}(Ec \rightarrow Ac) = T$. Since $\mathbf{b}_{\mathrm{w}}(Ec) = T$, (→) guarantees $\mathbf{b}_{\mathrm{w}}(Ac) = T$. By the Instance Theorem, $\mathbf{b}_{\mathrm{w}}(Ai) = T$. Since c is not in $\forall x A$, $\mathbf{a}_{\mathrm{w}}(Ai) = T$ follows from the No c Theorem.

4.2. *Consistency of Extensions of G*

It is not difficult to show that each rule mentioned in Section 3 preserves S-validity when its corresponding condition is in S. Details for (∃i) and (rc) are given here, and the rest left to the reader.

($\exists$i) For some $i \in \mathbf{I}$, $i(\mathbf{w}) = \mathbf{a_w}(t)$ $\quad$ $L \vdash \sim t{\approx}c / L \vdash \bot$, c not in L, or t

Suppose c is not in L or t, that $L \models \sim t{\approx}c$, and that $L \nvDash \bot$ for *reductio*. Then there is a S-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{a}\rangle$ with world $\mathbf{w}$ in $\mathbf{W}$ where $\mathbf{a_w}(L) =$ T. By the condition ($\exists$i), there is a function $i \in \mathbf{I}$ such that $i(\mathbf{w}) = \mathbf{a_w}(t)$. Now construct the model $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{b}\rangle$ where $\mathbf{b}$ is $\mathbf{a}^{i}/_{c}$. Hence $\mathbf{b_w}(c) = i(\mathbf{w}) = \mathbf{a_w}(t)$. Since c does not appear in L nor in t, it follows by the No c Theorem that $\mathbf{b_w}(L) = \mathbf{a_w}(L) =$ T and $\mathbf{b_w}(t) = \mathbf{a_w}(t) = \mathbf{b_w}(c)$. So by ($\approx$), $\mathbf{b_w}(t{\approx}c) =$ T. Since $L \models \sim t{\approx}c$, it follows by $\mathbf{b_w}(L) =$ T that $\mathbf{b_w}(t{\approx}c) =$ F, which yields the desired contradiction.

(rc) $\mathbf{a_w}(c) = \mathbf{a_v}(c)$ for $c \in$ C $\quad$ $(b{\approx}c \rightarrow \Box b{\approx}c)$ & $(\sim b{\approx}c \rightarrow \Box{\sim}b{\approx}c)$

To show the left conjunct, assume $\mathbf{a_w}(b{\approx}c) =$ T and $\mathbf{wRv}$ for any world $\mathbf{v}$. We have $\mathbf{a_w}(b) = \mathbf{a_w}(c)$ by ($\approx$), and by condition (rc), it follows that $\mathbf{a_v}(b) = \mathbf{a_v}(c)$, hence $\mathbf{a_v}(b{\approx}c) =$ T. Therefore, for any world such that $\mathbf{wRv}$, $\mathbf{a_v}(b{\approx}c) =$ T, and $\mathbf{a_w}(\Box b{\approx}c) =$ T by ($\Box$). The proof for the right conjunct is similar.

5. Completeness

5.1. *Limits on the Completeness Results*

Here the restrictions on the completeness results are explained. Once (rc) is in S, it is necessary to include the other two rules that make up oG. Call this *the (rc) restriction.*

(rc) Restriction. If (rc) is in S, then so are ($\exists$i) and (eE).

There are also limitations on the choice of the underlying modality. The method only works for a propositional modal logic whose frame conditions *shrink*, which means they hold in the frame of its canonical model, and in all its subframes. ($\langle \mathbf{W}', \mathbf{R}'\rangle$ is a *subframe* of frame $\langle \mathbf{W}, \mathbf{R}\rangle$ provided $\mathbf{W}' \subseteq \mathbf{W}$ and $\mathbf{R}'$ is $\mathbf{R}$ restricted to $\mathbf{W}$.) Frame conditions that shrink include Reflexivity, Transitivity, Symmetry and others that are expressed in the metalanguage only with universal quantification over worlds. Conditions involving *existential* quantification over $\mathbf{W}$ such as Seriality, Density and Convergence do not shrink since the claim that there exists a member $\mathbf{u}$ in $\mathbf{W}$ with a certain property might fail in a subset of $\mathbf{W}$ that omits $\mathbf{u}$. Even though Seriality does not shrink, the seriality axiom guarantees that the frame is serial in the canonical models to be given below, so the completeness method works in this case. The method does not work for Density and Convergence. The failure for Convergence is expected, since it has been shown that possibilist logics are incomplete for this condition (Cresswell, 1995). Notice that the condition that $\mathbf{R}$ be universal (i.e. that $\mathbf{wRv}$ for all worlds $\mathbf{w}$ and $\mathbf{v}$), does not shrink because this condition does not hold for the frame of the canonical model of S5. Completeness can be shown for such systems, but only when the additional axiom (E$\Box$) is present.[9]

(E$\Box$) $\Box s{\approx}t \rightarrow (\text{E}s \rightarrow \text{E}t)$

5.2. *Saturated Sets*

Completeness is proven for any QML meeting the above restrictions following essentially strategy 4 of Garson (2001, p. 297), which generalizes the definition of saturation used for predicate logic. When L/C is a modal argument, the result of applying ($\Box$In) and Conditional Proof repeatedly to it produces its *corresponding sentence* $L \Rightarrow C$. For example, when L/C is the modal argument A, $\Box$, B, $\Box$/C, the corresponding sentence $L \Rightarrow C$ is $A \rightarrow \Box(B \rightarrow \Box C)$. The following lemma should be obvious; it serves as a modal generalization of the Deduction Theorem.

$\Rightarrow$ LEMMA. *When* L′ *is finite,* L, $L' \vdash C$ *iff* $L \vdash L' \Rightarrow C$.

When we write '$M \vdash A$' in the case of a *set* M, we mean that $H \vdash_S A$ for some finite list H whose members are all in M. M is a *r-set* (for a given language) if and only if it obeys the following two conditions.

($\forall$-set) If $M \vdash L \Rightarrow (\text{E}c \rightarrow \text{A}c)$ for every free variable c, then $M \vdash L \Rightarrow \forall x\text{A}$.

($\approx$-set) If ($\exists$i) is in S, and $M \vdash L \Rightarrow \sim t{\approx}c$ for every free variable c, then $M \vdash L \Rightarrow \sim t{\approx}t$.

A set M *is consistent* iff $M \nvdash \bot$. M *is maximal* iff for every sentence A, either A or $\sim$A is in M. M is *saturated* for a language if and only if M is maximal, consistent, r-set. M is *ready* when it is a r-set or there are infinitely many free variables of the language not appearing in sentences of M. The next three lemmas are basic to the completeness result. Their proofs (and unremarkable proofs for other lemmas) are given in the notes.

ADDITION LEMMA. *If* M′ *is finite, and* M *is ready, then so is* $M \cup M'$.[10]

READY LEMMA. *If* M *is ready and consistent for a given language, then* M *satisfies both of the following*:

(a) *If* $\sim(L \Rightarrow \forall x\text{A}) \in M$, *then* M, $\sim(L \Rightarrow (\text{E}c \rightarrow \text{A}c)) \nvdash \bot$, *for some free variable c of the language.*
(b) *If* ($\exists$i) *is in S, and* $\sim(L \Rightarrow \sim t{\approx}t) \in M$, *then* M, $\sim(L \Rightarrow \sim t{\approx}c) \nvdash \bot$, *for some free variable c of the language.*[11]

SATURATED SET LEMMA. *If* M *is consistent and ready, then* M *has a saturated extension.*[12]

5.3. *The Canonical Model*

To show completeness for S, a familiar strategy is used: to show that any consistent set H is S-satisfiable. Suppose $H \nvdash \perp$. Now consider a language with infinitely many free variables not in the syntax of S. H is ready in this language, and so, according to the Saturated Set Lemma, can be extended to a saturated set **h**. We then define the canonical model for S. If it can be established that **h** belongs to the **W** of that model and $\mathbf{a}_{\mathbf{h}}(\mathrm{H}) = \mathrm{T}$, H will be S-satisfiable.

So the main task will be to define the canonical model and show it satisfies any consistent set **h**. The set U of terms is defined so that when (rc) is in S, U is C, otherwise U is the set of all terms. In the canonical model, we take the extension $\mathbf{a}_{\mathbf{w}}(t)$ of a term t to be an equivalence class of members of U such that $\mathbf{w} \vdash t{\approx}u$.

The canonical model $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{a}\rangle$ for S, is defined as follows.

(R) **wRv** iff for all sentences B, if $\mathbf{w} \vdash \Box\mathrm{B}$ then $\mathbf{v} \vdash \mathrm{B}$.
(W) When S includes (rc), **W** is the set of all saturated sets in the ancestry of **h**; otherwise **W** is the set of all saturated sets.

(*The ancestry of* **h** is defined as the smallest set such that **h** is in the ancestry of **h**, and if **w** is in the ancestry of **h** and **wRv** then so is **v**.)

(t) $u \in \mathbf{a}_{\mathbf{w}}(t)$ iff $\mathbf{w} \vdash t{\approx}u$ and $u \in \mathrm{U}$.
(D) $\mathbf{D} = \{\mathbf{a}_{\mathbf{w}}(t) : t$ is a term and **w** is some member of $\mathbf{W}\}$.
(I) $\mathbf{I} = \{\mathbf{a}(c) : \text{for some } c \in \mathrm{C}\}$.
(Iw) $\mathbf{Iw} = \{\mathbf{a}(s) : \mathbf{w} \vdash \mathrm{E}s, \text{ for some term } s\}$.
(P) $\mathbf{a}_{\mathbf{w}}(\mathrm{P}) = \{\mathbf{a}_{\mathbf{w}}(l) : \mathbf{w} \vdash \mathrm{P}l, \text{ for some list of terms } l\}$.

The values of the sentences on the canonical model are determined by the Basic Truth Clauses.

We are now ready to prove the main result concerning the canonical model.

TRUTH LEMMA. $\mathbf{a}_{\mathbf{w}}(\mathrm{A}) = \mathrm{T}$ *iff* $\mathbf{w} \vdash \mathrm{A}$.

Proof of the Truth Lemma. The proof is by induction on the structure of A. Details are given here for sentences with shapes $s{\approx}t$, Et, $\Box$A, and $\forall x$A; the rest is left to the reader.

The case for $s{\approx}t$. The proof relies on the next two lemmas and ($\approx\vdash$), which follows from the rules for $\approx$.

($\approx\vdash$) If $\mathbf{w} \vdash s{\approx}t$, then $\mathbf{w} \vdash s{\approx}u$ iff $\mathbf{w} \vdash t{\approx}u$.
If $\mathbf{w} \vdash t{\approx}u$ and $\mathbf{w} \vdash s{\approx}u$, then $\mathbf{w} \vdash s{\approx}t$.

c LEMMA. *When* ($\exists$i) *is in* S, $\mathbf{w} \vdash t{\approx}c$ *for some free variable* c.[13]

$\approx$LEMMA. *For all* $u \in \mathrm{U}$, $\mathbf{w} \vdash s{\approx}t$ *iff*, $\mathbf{w} \vdash s{\approx}u$ *iff* $\mathbf{w} \vdash t{\approx}u$.

Proof. The proof from left to right follows from ($\approx\vdash$). For the proof from right to left, assume for any term $u \in U$, $\mathbf{w} \vdash s{\approx}u$ iff $\mathbf{w} \vdash t{\approx}u$. If U is the set of all terms, this assumption yields $\mathbf{w} \vdash s{\approx}t$ iff $\mathbf{w} \vdash t{\approx}t$, from which $\mathbf{w} \vdash s{\approx}t$ follows from ($\approx$In). In case U is C, (rc) is in S. By the uc Restriction, ($\exists$i) is in S. But then the c Lemma yields that there is a free variable c such that, $\mathbf{w} \vdash t{\approx}c$. The original assumption now guarantees that $\mathbf{w} \vdash s{\approx}c$. By ($\vdash\approx$), $\mathbf{w} \vdash s{\approx}t$.

The case for sentences with the shape $s{\approx}t$ follows from ($\approx$), (t) and the $\approx$Lemma.

The case for Et. The case rests on the next two lemmas.

Ev LEMMA. *If* $\mathbf{w} \vdash \mathrm{E}s$ *and* $\mathbf{w} \vdash {\sim}\mathrm{E}t$, *then for some* **v** *in* **W**, $\mathbf{v} \vdash {\sim}t{\approx}s$.

Proof of the Ev Lemma. Suppose $\mathbf{w} \vdash \mathrm{E}s$ and $\mathbf{w} \vdash {\sim}\mathrm{E}t$. The proof varies depending on whether S contains (rc) or not.

S contains (rc). The (rc) Restriction requires that S contains (eE), and we have $\mathbf{w} \vdash t{\approx}s \rightarrow (\mathrm{E}s \rightarrow \mathrm{E}t)$. Hence given $\mathbf{w} \vdash \mathrm{E}s$ and $\mathbf{w} \vdash {\sim}\mathrm{E}t$, $\mathbf{w} \nvdash t{\approx}s$, since otherwise **w** is not consistent. Since **w** is maximal, $\mathbf{w} \vdash {\sim}t{\approx}s$. World **w** is in the ancestry of **h**, so **w** is the **v** in **W**, such that $\mathbf{v} \vdash {\sim}t{\approx}s$.

S does not contain (rc). S contains rule (iE), so we have that if $\vdash t{\approx}s$, then $\vdash \mathrm{E}s \rightarrow \mathrm{E}t$, and hence $\mathbf{w} \vdash \mathrm{E}s \rightarrow \mathrm{E}t$. So given $\mathbf{w} \vdash \mathrm{E}s$ and $\mathbf{w} \vdash {\sim}\mathrm{E}t$, it cannot be that $\vdash t{\approx}s$ for this would make **w** inconsistent. So $\nvdash t{\approx}s$, hence ${\sim}t{\approx}s \nvdash \perp$ by propositional logic. So the set containing ${\sim}t{\approx}s$ as its only member is consistent. Since it is finite, there are infinitely many free variables not in it, and so it is ready. Therefore it can be extended to a saturated set **v** containing ${\sim}t{\approx}s$. When S does not satisfy (rc), **W** is the set of all saturated sets, so $\mathbf{v} \in \mathbf{W}$, and hence $\mathbf{v} \vdash {\sim}t{\approx}s$.

E LEMMA. $\mathbf{w} \vdash \mathrm{E}t$ *iff for some term* s, $\mathbf{a}(t) = \mathbf{a}(s)$ *and* $\mathbf{w} \vdash \mathrm{E}s$.

Proof of the E Lemma. For the proof from left to right, assume $\mathbf{w} \vdash \mathrm{E}t$ and note that $\mathbf{a}(t) = \mathbf{a}(t)$, so there is a term s (namely t) such that $\mathbf{a}(t) = \mathbf{a}(s)$ and $\mathbf{w} \vdash \mathrm{E}s$. For the proof from right to left, assume for some term s, $\mathbf{a}(t) = \mathbf{a}(s)$ and $\mathbf{w} \vdash \mathrm{E}s$. Assume $\mathbf{w} \nvdash \mathrm{E}t$ for *reductio*. Since **w** is maximal, $\mathbf{w} \vdash \sim Et$. By the Ev Lemma, for some $\mathbf{v}$ in $\mathbf{W}$, $\mathbf{v} \vdash \sim t{\approx}s$. Since $\mathbf{a}(t) = \mathbf{a}(s)$, we have $\mathbf{a_v}(t) = \mathbf{a_v}(s)$, from which $\mathbf{a_v}(t{\approx}s) = T$ by ($\approx$). From the base case for sentences with the shape $s{\approx}t$, we obtain $\mathbf{v} \vdash t{\approx}s$, and this contradicts the consistency of $\mathbf{v}$.

The case for sentences with the shape Et follows from (iE), (Iw) and the E Lemma.

The case for $\Box A$. Two definitions and some lemmas concerning $\Box$ are needed. The first three are standard results left to the reader and proofs for others are found in the notes.

(V) $V = \{A : \mathbf{w} \vdash \Box A\}$.
($\Box$V) $\Box V = \{\Box A : A \in V\}$

V-LEMMA. $\mathbf{w}$ *is an extension of* $\Box V$.

In the next three lemmas, $V \cup \{\sim A\}$ is abbreviated by: V, $\sim$A.

CONSISTENCY LEMMA. *If* $\mathbf{w} \nvdash \Box A$, *then* V, $\sim$A *is consistent.*

R LEMMA. *If* $\mathbf{v}$ *is an extension of* V, $\sim$A, *then* $\mathbf{wRv}$.

r-Set LEMMA. V, $\sim$A *is an r-set.*[14]

($\vdash\Box$) $\mathbf{w} \vdash \Box A$, iff for all $\mathbf{v} \in \mathbf{W}$, if $\mathbf{wRv}$, then $\mathbf{v} \vdash A$.[15]

The case for sentences with the shape $\Box A$ follows from ($\Box$), the inductive hypothesis and ($\vdash\Box$).

The case for $\forall xA$. To show the case for sentences with the shape $\forall xA$, ($\vdash\forall$) is proven.

($\vdash\forall$) $\mathbf{w} \vdash \forall xA$ iff for all $c \in C$, $\mathbf{w} \vdash Ec \to Ac$.

Proof. The proof from left to right, is guaranteed by ($\forall$Out). For the proof from right to left, assume for all $c \in C$, $\mathbf{w} \vdash Ec \to Ac$. Since $\mathbf{w}$ is saturated, $\mathbf{w}$ is a $\forall$-set, and so in the special case where L is empty, it follows that $\mathbf{w} \vdash \forall xAx$.

In light of ($\vdash\forall$), ($\to$) and the inductive hypothesis, the case for $\forall xA$ follows since (s$\forall$) can be shown. (Note that (s$\forall$) is the truth clause for the substitution interpretation.)

(s$\forall$) $\mathbf{a_w}(\forall xA) = T$ iff for all $c \in C$, if $\mathbf{a_w}(Ec) = T$ then $\mathbf{a_w}(Ac) = T$.[16]

The Truth Lemma has now been established. All that remains for the completeness proof is to show that the corresponding conditions for the principles of S hold of its canonical model. When this is established, we will have constructed an S-model that satisfies H, because the saturated set $\mathbf{h}$ constructed from H entails all members of H, so by the Truth Lemma, $\mathbf{a_h}(H) = T$. By definition, $\mathbf{h}$ is in the ancestry of $\mathbf{h}$, so $\mathbf{h} \in \mathbf{W}$ even when $\mathbf{W}$ is restricted to saturated sets in the ancestry of $\mathbf{h}$. So $\mathbf{h}$ qualifies as a world in $\mathbf{W}$ where $\mathbf{a_h}(H) = T$ as desired.

In the case of G, all that remains is to show that the frame for the canonical model of G meets the frame conditions for G. We have assumed that the frame conditions for S shrink, which means that they hold in the frame $\langle \mathbf{Wp}, \mathbf{Rp} \rangle$ of canonical model for the propositional modal logic from which S is formed, and in all its subframes. Consider the frame $\langle \mathbf{W}+, \mathbf{R}+ \rangle$, where $\mathbf{W}+$ is the set of all maximally consistent sets for S, and $\mathbf{R}+$ is given by (R), the definition for $\mathbf{R}$ used in the canonical model. By exactly the reasoning that established that $\langle \mathbf{Wp}, \mathbf{Rp} \rangle$ meets the frame conditions for S, $\langle \mathbf{W}+, \mathbf{R}+ \rangle$ meets them also.[17] Since the frame conditions shrink, and $\langle \mathbf{W}, \mathbf{R} \rangle$ is a subframe of $\langle \mathbf{W}+, \mathbf{R}+ \rangle$, they hold for $\langle \mathbf{W}, \mathbf{R} \rangle$ as well. It is also possible to show that Seriality holds for the canonical model if the axiom $A \to \Diamond A$ is in S.[18]

5.4. *Completeness for Extensions of G*

The completeness of extensions of G is shown by demonstrating that the canonical model for G obeys the corresponding condition for each rule added to G. Proofs for some of the cases are given below.

(EI) If $\mathbf{wRv}$, then $\mathbf{Iw} \subseteq \mathbf{Iv}$ $\quad Et \to \Box Et$

The proof is similar to the one for the next case.

(CI) If $\mathbf{wRv}$, then $\mathbf{Iv} \subseteq \mathbf{Iw}$ $\quad \Diamond Et \to Et$

To show (CI), assume $\mathbf{wRv}$ and $f \in \mathbf{Iv}$, and then prove $f \in \mathbf{Iw}$. We know by (Iw) and $f \in \mathbf{Iv}$ that for some term t, $f = \mathbf{a}(t)$ and $\mathbf{v} \vdash Et$. By ($\Box\vdash$), the definition of $\Diamond$, and the fact that $\mathbf{w}$ is maximal, $\mathbf{w} \vdash \Diamond Et$, and so by the axiom, $\mathbf{w} \vdash Et$. By (Iw), $\mathbf{a}(t) = f \in \mathbf{Iw}$.

($\exists$i) For some $i \in \mathbf{I}$, $i(\mathbf{w}) = \mathbf{a_w}(t)$ $\quad L \vdash \sim t{\approx}c / L \vdash \bot$, c not in L, t.

To show ($\exists$i), let t be any term and $\mathbf{w}$ any member of $\mathbf{W}$. Since rule ($\exists$i) is in S, the c Lemma yields that $\mathbf{w} \vdash t{\approx}c$ for some free variable c. By the Truth Lemma and ($\approx$), $\mathbf{a_w}(c) = \mathbf{a_w}(t)$. By (c), $\mathbf{a}(c) \in \mathbf{I}$, so there is a member i of $\mathbf{I}$ (namely $\mathbf{a}(c)$) such that $i(\mathbf{w}) = \mathbf{a_w}(t)$.

To prepare for the case of (rc), we will prove a lemma.

rc LEMMA. *When* (rc) *is in* S, *then for all* $\mathbf{w}$, $\mathbf{v}$ *in* $\mathbf{W}$, $\mathbf{w} \vdash b{\approx}c$ *iff* $\mathbf{v} \vdash b{\approx}c$.

Proof of the rc Lemma. We first establish two facts for any world $\mathbf{w}$ in the ancestry of $\mathbf{h}$.

(1) If $\mathbf{h} \vdash b{\approx}c$, then $\mathbf{w} \vdash b{\approx}c$.
(2) If $\mathbf{h} \vdash {\sim}b{\approx}c$, then $\mathbf{w} \vdash {\sim}b{\approx}c$.

The proof of these facts is by induction on the definition of the ancestry of $\mathbf{h}$. In the base case we must show that (1) and (2) when $\mathbf{w}$ is $\mathbf{h}$, which is trivial. Next we must show that assuming that (1) and (2) are true and $\mathbf{wRv}$ then (1) and (2) are also true when $\mathbf{v}$ is replaced for $\mathbf{w}$. So assume (1) and (2), and $\mathbf{wRv}$. Assume $\mathbf{h} \vdash b{\approx}c$. By the hypothesis of the induction, $\mathbf{w} \vdash b{\approx}c$. Then by (rc), $\mathbf{w} \vdash b{\approx}c \rightarrow \Box(b{\approx}c)$ and so $\mathbf{w} \vdash \Box b{\approx}c$. Since $\mathbf{wRv}$, it follows by the definition of $\mathbf{R}$ in the canonical model that $\mathbf{v} \vdash b{\approx}c$. So if $\mathbf{h} \vdash b{\approx}c$, then $\mathbf{v} \vdash b{\approx}c$ as desired. Similar reasoning can be used to show that when $\mathbf{h} \vdash {\sim}b{\approx}c$, $\mathbf{v} \vdash {\sim}b{\approx}c$. Given facts (1) and (2), we are ready to prove the lemma. Suppose (rc) is in S and let $\mathbf{w}$ and $\mathbf{v}$ be any worlds in $\mathbf{W}$. Since (rc) is in S, $\mathbf{W}$ is the set of all worlds in the ancestry of $\mathbf{h}$. So $\mathbf{w}$ and $\mathbf{v}$ are in the ancestry of $\mathbf{h}$. To show $\mathbf{w} \vdash b{\approx}c$ iff $\mathbf{v} \vdash b{\approx}c$ from left to right, suppose $\mathbf{w} \vdash b{\approx}c$. Then $\mathbf{h} \vdash b{\approx}c$ for otherwise $\mathbf{h} \nvdash b{\approx}c$, and by maximality we would get $\mathbf{h} \vdash {\sim}b{\approx}c$, which yields $\mathbf{w} \vdash {\sim}b{\approx}c$ by fact (2), which conflicts with the consistency of $\mathbf{w}$. But if $\mathbf{h} \vdash b{\approx}c$, it follows by fact (1) that $\mathbf{v} \vdash b{\approx}c$. The proof from right to left is similar.

(rc) $\mathbf{a}(c)$ is a constant function $\quad (b{\approx}c \rightarrow \Box b{\approx}c) \,\&\, ({\sim}b{\approx}c \rightarrow \Box{\sim}b{\approx}c)$

When (rc) is in S, $\mathbf{W}$ is defined as the set of all saturated sets in the ancestry of $\mathbf{h}$. So we know that $\mathbf{w}$ and $\mathbf{v}$ are in the ancestry of $\mathbf{h}$, from which it follows from the rc Lemma that $\mathbf{w} \vdash c{\approx}b$ iff $\mathbf{v} \vdash c{\approx}b$. By (t), for all $b \in C$, $b \in \mathbf{a_w}(c)$ iff $b \in \mathbf{a_v}(c)$. Since (rc) is in S, U = C and so for all $u \in U$, $u \in \mathbf{a_w}(c)$ iff $u \in \mathbf{a_v}(c)$, which means that the two sets $\mathbf{a_w}(c)$ and $\mathbf{a_v}(c)$ are identical. So we have shown that for arbitrary worlds $\mathbf{w}$, and $\mathbf{v}$, $\mathbf{a_w}(c) = \mathbf{a_v}(c)$ and so $\mathbf{a}(c)$ is a constant function.

(Ic) I is the set of all constant functions with values in $\mathbf{D}$. $\quad$ (Ic) = (∃i) + (rc)

We just showed that when (rc) is in S, it holds that $\mathbf{a}(c)$ is a constant function. By (I), $\mathbf{I} = \{\mathbf{a}(c) \;:\; c \text{ is a constant}\}$, so $\mathbf{I}$ contains nothing but constant functions. What remains is to show that for each $d \in \mathbf{D}$, there is a member i of $\mathbf{I}$ such that, $i(\mathbf{w}) = d$. So let d be any member of $\mathbf{D}$. By (D), there is a term t and world $\mathbf{v}$ such that $\mathbf{a_v}(t) = d$. By (∃i), there is a constant b such that $\mathbf{a_v}(b) = \mathbf{a_v}(t) = d$. Since $\mathbf{a}(b)$ is a constant function, we have $\mathbf{a_w}(b) = d$. So $\mathbf{a}(b)$ is a member i of $\mathbf{I}$ such that $i(\mathbf{w}) = d$.

(eE) $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$ iff $\mathbf{a_w}(t) \in \mathbf{Dw} \quad s{\approx}t \rightarrow (\mathrm{E}s \rightarrow \mathrm{E}t)$

For the proof from left to right, suppose $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$. By (iE), $\mathbf{a}(t) \in \mathbf{Iw}$. So there is a f (namely $\mathbf{a}(t)$) such that $f \in \mathbf{Iw}$ and $f(\mathbf{w}) = \mathbf{a_w}(t)$. So by (Dw), $\mathbf{a_w}(t) \in \mathbf{Dw}$. For the proof from right to left, suppose $\mathbf{a_w}(t) \in \mathbf{Dw}$. Then by (Dw) there is an $f \in \mathbf{Iw}$ such that $f(\mathbf{w}) = \mathbf{a_w}(t)$. By (Iw), $f = \mathbf{a}(s)$ and $\mathbf{w} \vdash \mathrm{E}s$ for some term s. Since $f = \mathbf{a}(s)$, $f(\mathbf{w}) = \mathbf{a_w}(s)$, and so $\mathbf{a_w}(s) = \mathbf{a_w}(t)$. So by (≈), $\mathbf{a_w}(s{\approx}t) = \mathrm{T}$, and by the Truth Lemma, $\mathbf{w} \vdash s{\approx}t$, from which it follows by the axiom that $\mathbf{w} \vdash \mathrm{E}t$. By the Truth Lemma, $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$.

Next it is shown the rules enforce their corresponding conditions on canonical models for oG. Note that conditions (Ic) and (eE) hold in this model by the reasoning already given.

(oEc) $\mathbf{Dw} = \mathbf{D} \quad \mathrm{E}c$

To show condition (oEc), prove $d \in \mathbf{Dw}$ iff $d \in \mathbf{D}$ as follows. To prove this from left to right, assume $d \in \mathbf{Dw}$. By (Dw), there is a function $f \in \mathbf{Iw}$ and $f(\mathbf{w}) = d$. Since $f \in \mathbf{Iw}$, $f = \mathbf{a}(t)$ for some term t by (Iw). So $\mathbf{a_w}(t) = d$. By (D), $d \in \mathbf{D}$. For the other direction, suppose that $d \in \mathbf{D}$. Then by (Ic), there is a free variable c such that $\mathbf{a_w}(c) = d$, and by the axiom, we have $\mathbf{w} \vdash \mathrm{E}c$, and so by the Truth Lemma, $\mathbf{a_w}(\mathrm{E}c) = \mathrm{T}$, and hence $\mathbf{a}(c) \in \mathbf{Iw}$ by (iE). So there is a function f (namely $\mathbf{a}(c)$) such that $f(\mathbf{w}) = d$ and $f \in \mathbf{Iw}$, and so $d \in \mathbf{Dw}$, by (Dw).

(oED) If $\mathbf{wRv}$, then $\mathbf{Dw} \subseteq \mathbf{Dv} \quad \mathrm{E}c \rightarrow \Box\mathrm{E}c$

The proof is similar to the next case.

(oCD) If $\mathbf{wRv}$, then $\mathbf{Dv} \subseteq \mathbf{Dw} \quad \Diamond\mathrm{E}c \rightarrow \mathrm{E}c$

To show (oCD), assume $\mathbf{wRv}$ and $d \in \mathbf{Dv}$, and then prove $d \in \mathbf{Dw}$ as follows. By (Ic), $\mathbf{a_v}(c) = d$ for some free variable c, so $\mathbf{a_v}(c) \in \mathbf{Dv}$. By (eE), $\mathbf{a_v}(\mathrm{E}c) = \mathrm{T}$, from which $\mathbf{v} \vdash \mathrm{E}c$ follows by the Truth Lemma. By (□⊢), the definition of ◊, and the fact that $\mathbf{w}$ is maximal, $\mathbf{w} \vdash \Diamond\mathrm{E}c$, and so by the axiom, $\mathbf{w} \vdash \mathrm{E}c$. By the Truth Lemma and (eE), $\mathbf{a_w}(c) \in \mathbf{Dw}$. By (rc), $\mathbf{a_v}(c) = \mathbf{a_w}(c)$, so $\mathbf{a_v}(c) = d \in \mathbf{Dw}$, as desired.

6. Substitution and Objectual Interpretations of the Quantifier

In Section 1.3, it was claimed the substitution and objectual interpretations of the quantifier may be accommodated within G as special cases. The proof of those claims follows.

6.1. *The Substitution Interpretation*

For the substitution interpretation, a *sG-model* $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{a}\rangle$ is like a G-model save that $\mathbb{I}$ is missing, and the truth condition for the quantifier reads as follows.

(s∀) $\mathbf{a}_\mathbf{w}(\forall x\mathrm{A}) = \mathrm{T}$ iff for all $c \in \mathrm{C}$, if $\mathbf{a}_\mathbf{w}(\mathrm{E}c) = \mathrm{T}$ then $\mathbf{a}_\mathbf{w}(\mathrm{A}c) = \mathrm{T}$.

The structure $\mathbb{I}$ can be defined in sG-models by (I) and (sIw).

(I) $\mathbf{I} = \{\mathbf{a}(c) : \text{for } c \in \mathrm{C}\}$.
(sIw) $\mathbf{Iw} = \{\mathbf{a}(t) : \mathbf{a}_\mathbf{w}(\mathrm{E}t) = \mathrm{T}\}$.

This means that the conditions introduced in Section 3 for extending G-models may also be added to sG-models to define corresponding concepts of sS-validity.

It is easy to see that notions of validity using the substitution interpretation match up perfectly with those using the intensional truth clause (∀) instead. It was proven in Section 5.3 that the canonical model for S obeys (s∀). Since that proof appeals to no special property of the canonical model other than (I), we have the following.

s LEMMA. *Any S-model that obeys* (I) *obeys* (s∀).

It follows from this that an sS-counterexample can be constructed for every argument H/C not provable in S; simply remove $\mathbb{I}$ from the canonical model for S to obtain a sS-model that is a counterexample to H/C. This means that S must be complete for sS-validity, and the reader can verify that it is also consistent for sS-validity. It follows by the consistency and completeness of S for S-validity proven in Sections 4 and 5 that the classes of S-valid and sS-valid arguments are identical.

6.2. *The Objectual Interpretation*

On the objectual interpretation of the quantifier, an dS-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{D}, \mathbf{a}\rangle$ is defined to be like a S-model except that $\mathbb{I}$ is replaced with $\mathbb{D}$, which consists of subsets $\mathbf{Dw}$ of $\mathbf{D}$ for each world $\mathbf{w}$. In dS-models the assignment function $\mathbf{a}$ obeys the following clauses for E and the quantifier.

(eE) $\mathbf{a}_\mathbf{w}(\mathrm{E}t) = \mathrm{T}$ iff $\mathbf{a}_\mathbf{w}(t) \in \mathbf{Dw}$, for any term t.
(o∀) $\mathbf{a}_\mathbf{w}(\forall x\mathrm{A}) = \mathrm{T}$ iff for all $d \in \mathbf{Dw}$, $\mathbf{a}_\mathbf{w}(\mathrm{A}d) = \mathrm{T}$.

(The notation '$\mathbf{a}_\mathbf{w}(\mathrm{A}d) = \mathrm{T}$' in condition (d∀) abbreviates '$\mathbf{a}^{\mathbf{i}}\!/_{c\mathbf{w}}(\mathrm{A}c) = \mathrm{T}$', where c is chosen foreign to $\forall x\mathrm{A}$ and $\mathbf{a}^{\mathbf{i}}\!/_c$ is the assignment function exactly like $\mathbf{a}$ save that $\mathbf{a}^{\mathbf{i}}\!/_{c\mathbf{w}}(c) = d$.) Here we will show that the objectual interpretation is exactly captured within oS models by proving the following lemma. (Remember, oS models are S-models that obey (Ic) and (eE).)

o LEMMA. *Any* oS-*model that obeys* (I) *obeys* (o∀).

Proof. To show (o∀), we will appeal to the following lemma for oS models, which is shown by induction on the form of A.

id LEMMA. *If i is a constant function with value d,* $\mathbf{a}_\mathbf{w}(\mathrm{A}i) = \mathbf{a}_\mathbf{w}(\mathrm{A}d)$.

The proof then proceeds by showing that (or) and (ir) are equivalent in any oS-model that satisfies (I).[19]

(or) For all $d \in \mathbf{Dw}$, $\mathbf{a}_\mathbf{w}(\mathrm{A}d) = \mathrm{T}$.
(ir) For all $i \in \mathbf{I}$, if $i \in \mathbf{Iw}$, then $\mathbf{a}_\mathbf{w}(\mathrm{A}i) = \mathrm{T}$.

The o Lemma immediately establishes completeness for oS with respect to the corresponding dS semantics. Simply define a canonical dS-model by replacing $\mathbb{I}$ in the canonical S-model with a structure $\mathbb{D}$ defined by (Dw), note (eE) holds, and use the o Lemma to establish that (d∀) holds. As in the case for the substitution interpretation, this result guarantees that dS-validity and oS-validity are equivalent.

7. Independence Results

7.1. (∃i) *is independent in oG*

It is natural to wonder whether (∃i) and its more complex ancestors in the literature can be dispensed with in systems that use the objectual interpretation.

$$(\exists\mathrm{i})\quad \frac{\mathrm{L} \vdash \sim t{\approx}c}{\mathrm{L} \vdash \bot}, \quad \text{where } c \text{ does not appear in L, or } t.$$

When the language contains only free variables, (∃i) is admissible, but it is shown here that (∃i) is essential in general, for (∃i) is independent within oG. We remarked in Section 1.3 that (∃i) plays a role in the proof of $\mathrm{E}t \leftrightarrow \exists x x{\approx}t$ in oG.[20] So all it will take to show that (∃i) is independent in oG is to show that the system oG- that results from removing (∃i) from oG cannot prove $\mathrm{E}t \rightarrow \exists x x{\approx}t$. To show this, it suffices to show that $\mathrm{E}t/\exists x x{\approx}t$ is G + (eE) + (rc)-invalid, by locating a G-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{a}\rangle$ satisfying conditions (eE) and (rc), where $\mathbf{a}_\mathbf{w}(\mathrm{E}s) = \mathrm{T}$ and $\mathbf{a}_\mathbf{w}(\exists x x{\approx}s) = \mathrm{F}$, for some term s. To do this, let $\langle \mathbf{W}, \mathbf{R}\rangle$ be any frame with at least one world $\mathbf{w}$, and let $\mathbf{D} = \{1, 2\}$. Let $\mathbf{I} = \{i\}$ where i is the constant function with value 1. Let $\mathbf{a}(c) = i$ for each free variable c, and let $\mathbf{a}(t) = j$, the constant function with value 2 for terms t that are not free variables. Finally let $\mathbf{Iw} = \{i, j\}$. This model clearly satisfies (rc), and it satisfies (eE) by the following reasoning. By (iE), $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$ iff $\mathbf{a}(t) \in \mathbf{Iw}$. So since $\mathbf{Iw} = \{i, j\}$, $\mathbf{a}(t) \in \mathbf{Iw}$ for any term t. By (Dw), $\mathbf{Dw} = \{\mathbf{a_w}(t) : \mathbf{a}(t) \in \mathbf{Iw}\} = \{1, 2\}$, so $\mathbf{a_w}(t) \in \mathbf{Dw}$ for every term t. Hence $\mathbf{a}(t) \in \mathbf{Iw}$ iff $\mathbf{a_w}(t) \in \mathbf{Dw}$, and (eE) holds: $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$ iff $\mathbf{a_w}(t) \in \mathbf{Dw}$. Let s be any term that is not a free variable. Clearly $\mathbf{a_w}(\mathrm{E}s) = \mathrm{T}$; but by the (intensional) truth condition ($\exists$), $\mathbf{a_w}(\exists x x{\approx}s) = \mathrm{F}$, because there is no object in $\mathbf{I}$ whose value in $\mathbf{w}$ is identical to that of $\mathbf{a}(s) = 2$.

7.2. *Results for the Barcan Formulas*

In this section, we show that the Barcan Formula (BF) is too weak to capture the intensional contracting domains condition (CI) in G. There are G-models of (BF) where (CI) fails and so $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is invalid.

(CI) If $\mathbf{wRv}$, then $\mathbf{Iv} \subseteq \mathbf{Iw}$.

As a result, $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is not provable in G + (BF). The result holds even when the underlying propositional modal logic is as strong as S5. It will follow as a corollary of this result that the converse Barcan Formula (CBF) is not provable in the system given by Fitting and Mendelsohn (1998, p. 135).

We begin with a definition. For i and j in IC, '$j =_{\mathbf{w}} i$' means that $j(\mathbf{v}) = i(\mathbf{v})$ for all $\mathbf{v}$ in the ancestry of $\mathbf{w}$. Let *the system GBF* be G plus (BF), ($\exists$E) and the axioms of S5. The following lemma establishes that $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is not provable in GBF, nor in any weaker logic.

BF LEMMA. $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ *is not provable in* GBF.

Proof. Let a GBF-model be a G-model that satisfies (GBF), ($\exists$E) and (S5).

(GBF) If $i \in \mathbf{Iv}$ and $\mathbf{wRv}$ then for some $j \in \mathrm{I}$, $j =_{\mathbf{w}} i$ and $j \in \mathbf{Iw}$, and $j \in \mathbf{Iv}$.
($\exists$E) $\mathbf{Iw}$ is not empty.
(S5) The frame $\langle\mathbf{W}, \mathbf{R}\rangle$ is reflexive, transitive, and symmetric.

It will be shown that GBF is consistent for GBF-validity, but that there is a GBF-model where $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ fails, because (CI) does not hold. It is straightforward to show that ($\exists$E), and the rules of G and S5 preserve GBF-validity, so the only hard part is to show that axiom (BF) is GBF-valid. The proof of that depends on the following lemma, which is shown by induction on the form of A.

$=_{\mathbf{w}}$ LEMMA. *If* $j =_{\mathbf{w}} i$, $j \in \mathbf{Iw}$, *and* $i \in \mathbf{Iw}$, *then* $\mathbf{a_w}(\mathrm{A}j) = \mathbf{a_w}(\mathrm{A}i)$.

(The condition that both $j \in \mathbf{Iw}$ and $i \in \mathbf{Iw}$ is needed to guarantee the base case where $\mathrm{A}j$ is $\mathrm{E}j$.)

To show that (BF) is GBF-valid, assume that $\mathbf{a_w}(\forall x\Box\mathrm{A}) = \mathrm{T}$ in any GBF-model. To show that $\mathbf{a_w}(\Box\forall x\mathrm{A}) = \mathrm{T}$, assume that $\mathbf{wRv}$ and that i is any member of $\mathbf{I}$ such that $i \in \mathbf{Iv}$, and show $\mathbf{a_v}(\mathrm{A}i) = \mathrm{T}$ as follows. By condition (GBF), there is a $j \in \mathbf{I}$ such that $j =_{\mathbf{w}} i$ and $j \in \mathbf{Iw}$, and $j \in \mathbf{Iv}$. By $\mathbf{a_w}(\forall x\Box\mathrm{A}) = \mathrm{T}$, and $j \in \mathbf{Iw}$, it follows by ($\forall$) that $\mathbf{a_w}(\Box\mathrm{A}j) = \mathrm{T}$. So $\mathbf{a_v}(\mathrm{A}j) = \mathrm{T}$ by ($\Box$). Since $j =_{\mathbf{w}} i$, and $\mathbf{wRv}$, $j =_{\mathbf{v}} i$. Both j and i are members of $\mathbf{Iv}$, so the $=_{\mathbf{w}}$ Lemma guarantees $\mathbf{a_v}(\mathrm{A}i) = \mathrm{T}$ as desired.

To show that $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is GBF-invalid, a GBF-model, $\langle\mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbb{I}, \mathbf{a}\rangle$, is defined where $\mathbf{a_w}(\Diamond\mathrm{E}c) = \mathrm{T}$ and $\mathbf{a_w}(\mathrm{E}c) = \mathrm{F}$. Let $\mathbf{W} = \{\mathbf{w1}, \mathbf{w2}, \mathbf{w3}\}$, $\mathbf{R} = \{\langle\mathbf{w1}, \mathbf{w1}\rangle, \langle\mathbf{w2}, \mathbf{w2}\rangle, \langle\mathbf{w3}, \mathbf{w3}\rangle, \langle\mathbf{w1}, \mathbf{w2}\rangle, \langle\mathbf{w2}, \mathbf{w1}\rangle\}$, and $\mathbf{D} = \{1, 2\}$. Let f be the constant function with value 1 and g be the function with value 1 in worlds $\mathbf{w1}$ and $\mathbf{w2}$ but with value 2 in $\mathbf{w3}$. $\mathbb{I}$ is defined as follows. $\mathbf{I} = \{f, g\}$, $\mathbf{Iw1} = \{g\}$, $\mathbf{Iw2} = \{f, g\}$, $\mathbf{Iw3} = \{g\}$. Finally let $\mathbf{a}(c) = f$. Note (CI) fails, because $\mathbf{w1Rw2}$, $f \in \mathbf{Iw2}$, and $f \notin \mathbf{Iw1}$. Since $\mathbf{a}(c) \notin \mathbf{Iw1}$, $\mathbf{a_{w1}}(\mathrm{E}c) = \mathrm{F}$. But $\mathbf{a_{w1}}(\Diamond\mathrm{E}c) = \mathrm{T}$, because in world $\mathbf{w2}$, $\mathbf{w1Rw2}$ and $\mathbf{a}(c) \in \mathbf{Iw2}$ so that $\mathbf{a_{w2}}(\mathrm{E}c) = \mathrm{T}$. Note that the frame of the model is an S5 frame, and each $\mathbf{Iw}$ is not empty, so all that remains is to show that this model satisfies condition (GBF).

Since g is a member of $\mathbf{Iw}$ for each $\mathbf{w}$ in $\mathbf{W}$, it follows that for some j (namely g) $j =_{\mathbf{v}} g$ and $j \in \mathbf{Iw}$ and $j \in \mathbf{Iv}$, for any $\mathbf{w}$, $\mathbf{v}$ in $\mathbf{W}$. So (GBF) holds for $i = g$. Since f is only a member of $\mathbf{Iw2}$, all that remains is to show that there is a j such that $j =_{\mathbf{w}} f$ and $j \in \mathbf{Iw}$ and $j \in \mathbf{Iw2}$, for $\mathbf{w}$ in $\{\mathbf{w1}, \mathbf{w2}\}$. But this holds for $j = g$.

Now consider the system GBF-, the system like GBF save that the language lacks $\approx$, and the rules ($\approx$In) ($\approx$Out) and (IE). It is a simple exercise to show that $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is provable in GBF- plus (CBF).[21] It follows immediately that (CBF) is independent in GBF-, for were (CBF) provable there, so would $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ be, and $\Diamond\mathrm{E}c \rightarrow \mathrm{E}c$ is not even provable in the stronger system GBF by the BF Lemma. So the following lemma holds.

CBF LEMMA. (CBF) *is not provable in* GBF-.

Fitting and Mendelsohn's system BF results from adding the Barcan Formula (BF) and the "symmetry" axiom (B) to the "Kripke-style" quantifier system defined by Definition 6.2.1 (1998, p. 135) and discussed in Hughes and Cresswell (1996, p. 304). The Kripke-style system consists of tautologies of propositional logic, Necessitation, Modus Ponens, and the following principles, for a language that allows variables to be both bound and free.

Vacuous Quantification	$\forall x\mathrm{A} \leftrightarrow \mathrm{A}$ when x does not occur free in A
Universal Distributivity	$\forall x(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow (\forall x\mathrm{A} \rightarrow \forall x\mathrm{B})$

Permutation	$\forall x\forall y\mathrm{A} \leftrightarrow \forall y\forall x\mathrm{A}$
Universal Instantiation	$\forall y(\forall x\mathrm{A} \rightarrow \mathrm{A}y)$
Universal Generalization	$\mathrm{A}/\forall x\mathrm{A}$

By closing Universal Instantiation one obtains a "classical" axiom, which is still valid in systems that employ an actualist semantics. Free logic rules are thereby avoided. Fitting and Mendelsohn (1998, p. 138) point out that (B) corresponds to the symmetry of the accessibility relation R, and given symmetry, if one of the following holds, so must the other as well.

(Expanding Domains) If $w\mathrm{R}v$ then $\mathrm{D}w \subseteq \mathrm{D}v$

(Contracting Domains) If $w\mathrm{R}v$ then $\mathrm{D}v \subseteq \mathrm{D}w$

They accept the standard assumption that (CBF) corresponds to (Expanding Domains) and (BF) to (Contracting Domains), and so expect to be able to find a proof of (CBF) in system BF. (The reverse deduction from (B) and (CBF) to (BF) is easily shown.) They report not being able to find a proof and request the readers' help.

However, no such proof is forthcoming. It is a simple matter to verify that the axioms and rules of BF are all derivable in GBF-, and so everything provable in BF is provable in GBF-. It follows immediately from the CBF Lemma that (CBF) is not provable in BF. This reveals a problem with identity-free Kripke-style systems. The attempt to avoid free logic rules by adopting closed classical principles results in logics that are too weak.

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NOTES

[1] Even if possibilitists manage to win the day in the case of logics for necessity, there is no guarantee that their reasoning will transfer to temporal, deontic or alethic modalities.

[2] See Fitting (2004) who makes the same point, and who agrees with us in recommending a more liberal attitude in the construction of quantified modal logic.

[3] The reason is that there may be no *one* possible object that satisfies $\Box x = t$ when t picks out different possible objects in two or more possible worlds of a model. Essentially, this point is made by Garson (2001, p. 278).

[4] The reader may wonder at the use of an intensional existence predicate here. Note first that a more standard extensional treatment follows as a special case when (eE) is adopted. (See below.) When (eE) is available, the completeness proof can be simplified

somewhat. There are two reasons that (eE) was not adopted in the basic system. First, I do not know how to obtain the independence results of Section 7 within the stronger logics. Second, there are intuitions that support the need for an intensional existence predicate. For example, consider a temporal logic where members of D are time slices of objects. The domain I of individual concepts then contains functions that correspond to temporally enduring objects. Whether an object exists at t is intuitively not to be determined by whether a time slice for that object occurs at t, for that occurrence is (logically) compatible with no occurrence of any other times slices for that object in prior or later times. Under these circumstances, we do not want to be forced to grant the existence of an object.

[5] This is reminiscent of the situation in free logic, where correct instantiation principles for terms referring in an outer domain are derivable from an instantiation rule restricted to free variables only (Lambert and van Fraassen, 1972, p. 137).

[6] See Rules R6 and R7 of Thomason (1970, p. 63) and (G=) of Garson (2001, p. 304).

[7] In case t is a complex term containing free variables, it is assumed that bound variables in $\mathrm{A}x$ are adjusted to leave them free.

[8] For example, the following modal sequent rules are adequate.

(Hyp) $\mathrm{L}, \mathrm{A} \vdash \mathrm{A}$
(MP) $\mathrm{L} \vdash \mathrm{A}, \mathrm{L} \vdash \mathrm{A} \rightarrow \mathrm{B} \,/\, \mathrm{L} \vdash \mathrm{B}$
(CP) $\mathrm{L}, \mathrm{A} \vdash \mathrm{B} \,/\, \mathrm{L} \vdash \mathrm{A} \rightarrow \mathrm{B}$
(Reit) $\mathrm{L} \vdash \mathrm{A} \,/\, \mathrm{L}, \mathrm{B} \vdash \mathrm{A}$
(DN) $\mathrm{L} \vdash {\sim}{\sim}\mathrm{A} \,/\, \mathrm{L} \vdash \mathrm{A}$

[9] To obtain the completeness result, restrict **W** in the canonical model to saturated sets in the ancestry of **h**. This will guarantee that the frame is Universal. The proof of the Ev Lemma must be modified as follows. We have $\mathbf{w} \vdash \Box s{\approx}t \rightarrow (\mathrm{E}s \rightarrow \mathrm{E}t)$. So given $\mathbf{w} \vdash \mathrm{E}s$ and $\mathbf{w} \vdash {\sim}\mathrm{E}t$, it follows that $\mathbf{w} \nvdash \Box t{\approx}s$, since otherwise **w** is not consistent. By $(\Box\vdash)$, there must be some world **v** in **W** such that **wRv** and $\mathbf{v} \nvdash t{\approx}s$. Since **v** is maximal, $\mathbf{v} \vdash {\sim}t{\approx}s$. **W** is restricted to saturated sets in the ancestry of **h**. So **w** is in the ancestry of **h**. But **wRv**, so **v** is also in the ancestry of **h**, and we have the desired result.

[10] ADDITION LEMMA. *If* M' *is finite, and* M *is ready, then so is* $\mathrm{M} \cup \mathrm{M}'$.

Proof of the Addition Lemma. Suppose M' is finite and M is ready. If the reason M is ready is that there are infinitely many free variables of the language not in M, then since M' is finite there will still be infinitely many free variables not in $\mathrm{M} \cup \mathrm{M}'$ and so $\mathrm{M} \cup \mathrm{M}'$ is ready. If the reason M is ready is that it is a r-set, then $\mathrm{M} \cup \mathrm{M}'$ is also a r-set by the following reasoning. Suppose that $\mathrm{M} \cup \mathrm{M}' \vdash \mathrm{L} \Rightarrow (\mathrm{E}c \rightarrow \mathrm{A}c)$ for every free variable c of the language. It follows by the $\Rightarrow$ Lemma that $\mathrm{M} \vdash \mathrm{H}, \mathrm{L} \Rightarrow (\mathrm{E}c \rightarrow \mathrm{A}c)$, for every free variable c, where H is a list of the members of M'. Since M is a $\forall$-set, it follows then that $\mathrm{M} \vdash \mathrm{H}, \mathrm{L} \Rightarrow \forall x\mathrm{A}$, from which it follows by the $\Rightarrow$ Lemma that $\mathrm{M} \cup \mathrm{M}' \vdash \forall x\mathrm{A}$. Therefore $\mathrm{M} \cup \mathrm{M}'$ is a $\forall$-set. That $\mathrm{M} \cup \mathrm{M}'$ is also a $\approx$-set is shown using the same strategy.

[11] READY LEMMA. *If* M *is ready and consistent for a given language, then* M *satisfies both of the following*:

(a) *If* ${\sim}(\mathrm{L} \Rightarrow \forall x\mathrm{A}) \in \mathrm{M}$, *then* $\mathrm{M}, {\sim}(\mathrm{L} \Rightarrow (\mathrm{E}c \rightarrow \mathrm{A}c)) \nvdash \bot$, *for some free variable* c *of the language.*

(b) *If* $(\exists\mathrm{i})$ *is in* S, *and* ${\sim}(\mathrm{L} \Rightarrow {\sim}t{\approx}t) \in \mathrm{M}$, *then* $\mathrm{M}, {\sim}(\mathrm{L} \Rightarrow {\sim}t{\approx}c) \nvdash \bot$, *for some free variable* c *of the language.*

Proof of the Ready Lemma. Suppose M is ready and consistent. To show (a), suppose ${\sim}(\mathrm{L} \Rightarrow \forall x\mathrm{A}) \in \mathrm{M}$. There must be a free variable c of the language such that M,

$\sim(L \Rightarrow (Ec \rightarrow Ac)) \nvdash \perp$, because otherwise M, $\sim(L \Rightarrow (Ec \rightarrow Ac)) \vdash \perp$, for every free variable c, which leads to a contradiction as follows. By propositional logic, it follows that $M \vdash L \Rightarrow (Ec \rightarrow Ac)$, for every free variable c. If the reason that M is ready is that M is a r-set, then it follows immediately that $M \vdash L \Rightarrow \forall xA$, which conflicts with $\sim(L \Rightarrow \forall xA) \in M$ and M's consistency. If M is ready because there are infinitely many free variables of the language not in M, then $M \vdash L \Rightarrow \forall xA$ also holds for the following reason. L is finite, so there are infinitely many free variables not in M, L or $\forall xAx$. Let b be one of these free variables. Since $M \vdash L \Rightarrow (Ec \rightarrow Ac)$, for any c, we have $M \vdash L \Rightarrow (Eb \rightarrow Ab)$, and so $H \vdash L \Rightarrow (Eb \rightarrow Ab)$ for some list H of members of M. So by the $\Rightarrow$ Lemma, H, $L \vdash Eb \rightarrow Ab$. Apply ($\forall$In) to H, $L \vdash Eb \rightarrow Ab$ to obtain H, $L \vdash \forall xAx$. By the $\Rightarrow$ Lemma, it follows that $H \vdash L \Rightarrow \forall xAx$, and hence $M \vdash L \Rightarrow \forall xAx$. So whichever reason M is ready, it follows that $M \vdash L \Rightarrow \forall xAx$. But we also have $\sim(L \Rightarrow \forall xAx)$ is in M, and this conflicts with the consistency of M. Therefore it follows that M, $\sim(L \Rightarrow (Ec \rightarrow Ac)) \nvdash \perp$ for some constant c. The proof of (b) is similar.

[12] SATURATED SET LEMMA. *If* M *is consistent and ready, then* M *has a saturated extension.*

Proof of the Saturated Set Lemma. Order the sentences $A_1, \ldots, A_i, \ldots$ and create a series of sets: $M_1, M_2, \ldots, M_i, \ldots$ following the main outlines of the recipe used for the Lindenbaum Lemma, starting with $M_1 = M$. (Here again 'M, A' abbreviates '$M \cup \{A\}$'.)

$$M_{i+1} = M_i, A_i \quad \text{if } M_i, A_i \nvdash \perp,$$
$$M_{i+1} = M_i, \sim A_i \quad \text{if } M_i, A_i \vdash \perp.$$

However, there will be two changes in the definition of the M_{i+1}.

When A_i has the form $\sim(L \Rightarrow \forall xA)$ and the addition of this sentence would be consistent (i.e. if $M_i, \sim(L \Rightarrow \forall xA) \nvdash \perp$) then add both $\sim(L \Rightarrow \forall xA)$ and $\sim(L \Rightarrow (Eb \rightarrow Ab))$ to M_i, to form M_{i+1}, where b is a free variable chosen so that M_{i+1} is consistent. That there is such a b is proven as follows. M is ready and only finitely many sentences were added to M to form M_i, $\sim(L \Rightarrow \forall xA)$, so by the Addition Lemma, $M_i, \sim(L \Rightarrow \forall xA)$ is ready. Since $M_i, \sim(L \Rightarrow \forall xA) \nvdash \perp$, and $\sim(L \Rightarrow \forall xA)$ is in $M_i, \sim(L \Rightarrow \forall xA)$, it follows by the Ready Lemma that $M_i, \sim(L \Rightarrow \forall xA), \sim(L \Rightarrow (Ec \rightarrow Ac)) \nvdash \perp$ for some free variable c.

If ($\exists$i) is in S and a sentence of the form $\sim(L \Rightarrow \sim t\approx t)$ is A_i, and $M_i, \sim(L \Rightarrow \sim t\approx t) \nvdash \perp$, then add both $\sim(L \Rightarrow \sim t\approx t)$, and $\sim(L \Rightarrow \sim t\approx b)$ to form M_{i+1}, where b is chosen so that M_{i+1} is consistent. That there is such a b is also easily proven from the Addition and Ready Lemmas.

Let m be the union of M with the set of all sentences added in forming any of the M_i. It is a straightforward matter to show that m is the desired saturated extension of M.

[13] c LEMMA. *When* ($\exists$i) *is in* S, $\mathbf{w} \vdash t\approx c$ *for some free variable* c.

Proof of the c Lemma. Assume ($\exists$i) is in S. Because **w** is a saturated set, it is an r-set, and so a $\approx$-set. In the special case where L is empty, this yields (1).

(1) If $\mathbf{w} \vdash \sim t\approx c$ for every free variable c, then $\mathbf{w} \vdash \sim t\approx t$.

But **w** is consistent, so by ($\approx$In), it follows that $\mathbf{w} \nvdash \sim t\approx t$. Therefore the antecedent of (1) must be false, that is, there is a free variable c such that, $\mathbf{w} \nvdash \sim t\approx c$. By the maximality of **w**, it follows that $\mathbf{w} \vdash t\approx c$ for this c.

[14] r-Set LEMMA. V, $\sim$A *is an r-set.*

Proof of the r-Set Lemma. To show V, $\sim$A is a $\forall$-set, assume V, $\sim A \vdash L \Rightarrow (Ec \rightarrow Bc)$ for all c, and show that V, $\sim A \vdash L \Rightarrow \forall xB$ as follows. From the assumption it follows by propositional logic that $V \vdash \sim A, L \Rightarrow (Ec \rightarrow Bc)$ for all c. By principles of the modal logic K, $\Box V \vdash \Box, \sim A, L \Rightarrow (Ec \rightarrow Bc)$ for all c. By the V-Lemma, **w** is an extension of $\Box$V, so it follows that $\mathbf{w} \vdash \Box, \sim A, L \Rightarrow (Ec \rightarrow Bc)$ for all c. But **w** is a $\forall$-set, and so it follows that $\mathbf{w} \vdash \Box, \sim A, L \Rightarrow \forall xB$. By (V), $\sim A, L \Rightarrow \forall xB$ is in V. So $V \vdash \sim A, L \Rightarrow \forall xB$, and V, $\sim A \vdash L \Rightarrow \forall xB$ by propositional logic. The proof that V, $\sim$A is also a $\approx$-set is similar.

[15] ($\vdash\Box$) $\mathbf{w} \vdash \Box A$, iff for all $\mathbf{v} \in \mathbf{W}$, if $\mathbf{wRv}$, then $\mathbf{v} \vdash A$.

Proof of ($\vdash\Box$). For the proof from left to right, suppose $\mathbf{w} \vdash \Box A$, and let **v** be any member of **W** such that **wRv**. By (R), for any sentence B, if $\mathbf{w} \vdash \Box B$, then $\mathbf{v} \vdash B$. So $\mathbf{v} \vdash A$. For the proof from right to left, assume

(1) For all $\mathbf{v} \in \mathbf{W}$, if **wRv**, then $\mathbf{v} \vdash A$.

Suppose $\mathbf{w} \nvdash \Box A$ for *reductio*. Consider the set V defined by (V). The Consistency Lemma guarantees that V, $\sim$A is consistent. The r-Set Lemma guarantees that V, $\sim$A is ready. So by the Saturated Set Lemma, we can extend V, $\sim$A to a saturated set **v** in **W**. Since **v** is an extension of V, $\sim$A it follows by the R-Lemma that **wRv**. Since **v** is an extension of V, $\sim$A, it also follows that $\sim A \in \mathbf{v}$, and hence by the consistency of **v**, that $\mathbf{v} \nvdash A$. In case **W** is the set of saturated sets, we know immediately that $\mathbf{v} \in \mathbf{W}$. If instead **W** is the set of saturated sets in the ancestry of **h**, we know **w** is in the ancestry of **h**. Since **wRv**, **v** is also in the ancestry of **h**, and so $\mathbf{v} \in \mathbf{W}$. Either way, **v** is a saturated set in **W** such that **wRv** and $\mathbf{v} \nvdash A$. This conflicts with (1), so we conclude that $\mathbf{w} \vdash \Box A$.

[16] (s$\forall$) $\mathbf{a_w}(\forall xA) = T$ iff for all $c \in C$, if $\mathbf{a_w}(Ec) = T$ then $\mathbf{a_w}(Ac) = T$.

Proof of (s$\forall$). In light of the truth clause ($\forall$), all that is necessary is to show that the right-hand sides (sr) and (ir) of (s$\forall$) and ($\forall$) (respectively) are equivalent.

(sr) for all $c \in C$, if $\mathbf{a_w}(Ec) = T$ then $\mathbf{a_w}(Ac) = T$.
(ir) for all $i \in \mathbf{I}$, if $i \in \mathbf{Iw}$, then $\mathbf{a_w}(Ai) = T$.

For the proof from (sr) to (ir), assume (sr) and that i is any member of **I** such that $i \in \mathbf{Iw}$. By (I), $\mathbf{a}(b) = i$ for some choice of $b \in C$. Since $\mathbf{a}(b) \in \mathbf{Iw}$, it follows by (iE) that $\mathbf{a_w}(Eb) = T$. By (sr), it follows that $\mathbf{a_w}(Ab) = T$, and so $\mathbf{a_w}(Ai) = T$ follows by the Instance Theorem.

For the proof from (ir) to (sr), assume (ir) and that c is any member of C such that $\mathbf{a_w}(Ec) = T$. By (iE), $\mathbf{a}(c) \in \mathbf{Iw}$. But $\mathbf{a}(c) \in \mathbf{I}$ and there must be some $j \in \mathbf{I}$ such that $\mathbf{a}(c) = j$. By (ir), it follows that $\mathbf{a_w}(Aj) = T$, and so $\mathbf{a_w}(Ac) = T$ follows by the Instance Theorem.

[17] Strictly speaking, the result depends on the fact that frame conditions in the canonical model are not affected by changing from the language of propositional logic to that of G. That this is so should be clear from the fact that neither the definition of the frame of the canonical model nor proofs for the frame conditions depend on the nature of the language.

[18] Let **w** be any member of **W**. Since **w** is maximal, it must contain any theorem of S including $\sim\Box\perp$, which is derivable from $A \rightarrow \Diamond A$. So $\mathbf{w} \vdash \sim\Box\perp$ and by (Def **b**) the canonical model is such that $\mathbf{b_w}(\sim\Box\perp) = \mathbf{T}$ and $\mathbf{b_w}(\Box\perp) = F$. By ($\Box$), there must be a world **v** in **W** such that **wRv** and $\mathbf{b}(\Box\perp) = F$. So for each world **w** in **W**, there is a word **v** in **W** such that **wRv**.

[19] (or) for all $d \in \mathbf{Dw}$, $\mathbf{a_w}(Ad) = T$.

(ir) for all $i \in \mathbf{I}$, if $i \in \mathbf{Iw}$, then $\mathbf{a_w}(Ai) = T$.

For the proof from (or) to (ir), assume (or), and that i is any member of $\mathbf{I}$ such that $i \in \mathbf{Iw}$. Then by (Uc), i is the constant function with value e, for some e in $\mathbf{D}$. By (Dw) and $i \in \mathbf{Iw}$, $e \in \mathbf{Dw}$. By (or), it follows that $\mathbf{a_w}(Ae) = T$, and so $\mathbf{a_w}(Ai) = T$ follows by the id Lemma.

For the proof from (ir) to (or), assume (ir) and that d is any member of $\mathbf{Dw}$. By (Ic), there is a constant function j in $\mathbf{I}$ such that $j(w) = d$. By (I), there is a member b of C such that $\mathbf{a}(b) = j$. Since $\mathbf{a_w}(b) = d$, $\mathbf{a_w}(b) \in \mathbf{Dw}$, and so it follows by (eE) that $\mathbf{a_w}(Eb) = T$. By (iE), it follows that $\mathbf{a}(b) \in \mathbf{Iw}$, hence $j \in \mathbf{Iw}$. By (ir), it follows that $\mathbf{a_w}(Aj) = T$, and so $\mathbf{a_w}(Ad) = T$ follows by the id Lemma.

[20] In the hard direction, the proof proceeds as follows.

$Et, \sim\exists x x{\approx}t, c{\approx}t \vdash \forall x \sim x{\approx}t$	(Definition of $\exists$ and $\sim\sim A \vdash A$)
$Et, \sim\exists x x{\approx}t, c{\approx}t \vdash Ec \to \sim c{\approx}t$	($\forall$Out)
$Et, \sim\exists x x{\approx}t, c{\approx}t \vdash Ec$	(eE)
$Et, \sim\exists x x{\approx}t, c{\approx}t \vdash \sim c{\approx}t,$	Modus Ponens
$Et, \sim\exists x x{\approx}t \vdash \sim c{\approx}t$	Conditional Proof and $A \to \sim A \vdash \sim A$
$Et, \sim\exists x x{\approx}t \vdash \bot$	($\exists$i)
$Et \vdash \exists x x{\approx}t$	Indirect Proof

[21] The proof:

$Ec \to Ec$	propositional logic
$\forall x Ex$	($\forall$In)
$\Box\forall x Ex$	Necessitation
$\forall x \Box Ex$	(CBF)
$Ec \to \Box Ec$	($\forall$Out)
$\Diamond Ec \to \Diamond\Box Ec$	principles of modal logic K
$\Diamond Ec \to Ec$	(B)

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