III

EXTENSIONALITY

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The continued development of intensional logics, and concern with problems of their interpretation has had a rather curious effect. It has reinforced the notion, unjustifiable in my opinion, that extensionality is an unambiguous concept. This presumed clarity is usually singled out as the virtue of extensional systems, to say nothing of their metaphysical advantages. The assertion that in mathematics and empirical science one does not *need* to traffic in non-extensional notions which are fuzzy and troublesome, has become a virtual platitude. Yet a cursory examination of the literature does not reveal any well-defined theory of extensionality, although it is possible to find a core of agreement. Indeed, there are differences as to (a) what are the principles of extensionality, (b) which objects are or ought to be extensional, and (c) which formal systems are extensional.

My purpose in this paper is to arrive at a characterization of extensionality in terms of these differences which may be helpful in connection with some familiar problems of interpreting intensional systems.

Principles of extensionality. Consider first some unspecified system of material implication L with theory of types. On the propositional level, extensionality takes the form of a substitution principle:

(1) If p is equivalent$_1$ to q then A is equivalent$_2$ to B,

where B is the result of replacing one or more occurrences of p in A by q.

As stated, (1) is of course ambiguous. The ambiguity concerns the meaning of 'equivalence$_1$' and 'equivalence$_2$'. A minimal requirement of an equivalence relation is that it be reflexive, transitive and symmetrical. These conditions are met by a variety of relations ranging from identity to having the same weight, and further interpretation is required. Our concern is with logically definable relations of equivalence.

Using the abbreviations 'eq_1' and 'eq_2', let us first consider principles

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in which eq_1 and eq_2 have the same meaning. If they are taken as identity, (1) becomes

(1.1) If pIq then AIB (where 'I' names the identity relation)

and is merely explicative of the notion of identity. Suppose what is intended is

(1.2) If '$p \equiv q$' is a tautology then '$A \equiv B$' is a tautology.

In what sense is (1.2) an extensionality principle? Only in that it eliminates as possible predicates of propositions, certain intensional predicates such as 'believed by John'. Not all intensional predicates are precluded by (1.2). In particular, modal predicates such as 'logically necessary' would not falsify (1.2). Ordinarily, variables which range over predicates of propositions are dispensable, and consequently (1.1) is often provable as a strong form of the substitution theorem.

Most commonly, eq_1 and eq_2 are interpreted as material equivalence without the modifying condition of (1.2):

(1.3) If $p \equiv q$ then $A \equiv B$.

Here (1) is taken to mean that if p and q have the same truth value, whether contingently or necessarily, then A and B have the same truth value. As contrasted with (1.2), (1.3) is a strongly extensional principle for it disallows all intensional predicates of propositions. Here again, where variables which range over propositional predicates are not introduced, (1.3) is provable as the substitution theorem.

Consider, again on the level of propositions, principles in which eq_1 and eq_2 are not the same. If eq_1 is taken as identity and eq_2 material equivalence, then (1) becomes

(1.4) If pIq then $A \equiv B$.

(1.4) like (1.1) is explicative of the identity relation. [The converse of (1.4) is of course another matter involving as it does the assumption of Leibniz's law, in addition to being strongly extensional.]

If we take eq_1 as material equivalence and eq_2 as identity, we have

(1.5) If $p \equiv q$ then AIB

which in the first instance, where A is p, becomes

(1.51) If $p \equiv q$ then pIq.

In contrast to (1.4), (1.5) is very strongly extensional since it not only eliminates intensional predicates of propositions but assimilates propositions to truth values.

What I am trying to make apparent by this necessarily crude and informal analysis, is that even on the level of propositions, we cannot talk of *the* thesis of extensionality but only of stronger and weaker extensionality principles. I will call a principle *extensional* if it either (a) *directly or indirectly imposes restrictions on the possible values of the functional variables such that some intensional functions are prohibited or* (b) *it has the consequence of equating identity with a weaker form of equivalence.* Obviously (a) and (b) are interdependent. On the basis of this characterization, (1.2), (1.3), and (1.5) are all principles of extensionality, in order of increasing strength. It should now be clear why there is often disagreement as to whether a given formal system is or is not extensional. There is, for example, a literature of tiresome arguments as to whether the formal system of *Principia* is extensional. It is all a matter of deciding how extensional a system must be to be properly so-called. There are by contrast, logicians such as Alonzo Church who does talk in terms of degrees of extensionality. A more reasonable approach would be to assert, in connection with *Principia*, that the formal system as interpreted in the first edition is less extensional than the interpretation proposed by the second edition, since the latter assumes an analogue of (1.5) which is stronger than (1.2).

Consider next another set of principles which are more frequently associated with the theory of extensionality. Principles which relate the equivalence of classes (or attributes) to the equivalence of their defining functions.

(2) If $(x)(F(x) \text{ eq}_1 G(x))$ then $F \text{ eq}_2 G$.

In addition to the interpretation of eq_1 and eq_2, one must specify whether F and G are predicate variables, class variables, or non-committal functional variables. I cannot give an exhaustive account of the many possible variations of (2). In a weakly extensional system, eq_1 might be taken as tautological equivalence, eq_2 as identity, F and G functional variables, as follows:

(2.1) If $(x)(F(x) \equiv G(x))$ is tautological, then FIG.

By the criterion of extensionality stated above, (2.1) is weakly extensional since it precludes some intensional contexts. On the other hand (2.1) permits us to state an identity between the terms '9' and '3^2' but not '9' and (on the assumption that it can be construed as an expression of proper type level) 'the number of planets'. A stronger alternative (referred to most often as *the* extensionality principle) asserts identity of functions as a consequence of formal equivalence.

(2.2) If $(x)(F(x) \equiv G(x))$ then FIG.

In languages which distinguish classes from attributes, the distinction is sometimes maintained by postulating (2.2) for classes and perhaps (2.1) for attributes. This has the effect of eliminating intensional contexts involving class names but allowing such contexts for attribute names. Such is the interpreted procedure of *Principia.* The concept of identity in *Principia* is systematically ambiguous not only as prescribed by the theory of types, but on the same type level. In the second order predicate calculus, 'identity' means something different for classes than for attributes, and has still another import for individuals. My preference is for the alternative procedure of giving uniform meaning to 'identity' and to talk of attributes and classes as being equal, but not identical. Functional *equality* would be defined as

(3) $(F = G) \rightarrow_{\text{df}} (x)(F(x) \equiv G(x))$

where $F = G$ is *not* equated to FIG. On the basis of known substitution theorems, the substitution of F for G in strongly extensional contexts is still permissible and in such contexts $F = G$ is *like* FIG. This permits us to say that the class of mermaids and the class of Greek gods are equal but not identical and that in strongly extensional contexts (arithmetic ones for example) we are concerned only with their equality, so that the name of one may be substituted for the other.

It seems to me that much of the discussion these past few years concerning apparent breakdowns of substitutivity principles in intensional contexts and its presumably devastating results for logic and mathematics are largely terminological. I am not (as Quine[1] insists in his review of two of my papers on quantified modal logic) proposing that there be more than one kind of identity, but only that the distinctions between stronger and weaker equivalences be made explicit before, for one avowed purpose or another, they are obliterated.

The usual reason given for reducing identity to equality [(3) (2.2)] is that it provides a simpler base for mathematics, mathematics being concerned with aggregates discussed in truth functional contexts, not with predicates in intensional contexts. Under such restrictive conditions, the substitution theorem can generally be proved for equal (formally equivalent) classes, with the result that equality functions *as* identity.

Establishing the foundations of mathematics is not the only purpose of logic, particularly if the assumptions deemed convenient for mathematics do violence to both ordinary and philosophical usage. I am not

[1] W. V. O. Quine, *Journal of Symbolic Logic*, 12 (1947), 95–6.

disturbed by the possibility of equal, non-identical classes or attributes, *e.g.* man and featherless biped. To me it seems reasonable that there are many empty classes of the same type, *e.g.* mermaids and Greek gods, equal but not identical. And why should the non-identity of the numbers n and $n+1$ depend on the enumeration of different things in the world? To subsume mathematics under logic is not to equate them. A much broader base is indicated in the direction of intensional systems such as the modal logics. I will try to show that the apparent difficulties of interpreting such systems are not genuine, but analogous to a rejection of a non-Euclidean geometry because it allows parallel lines to meet.

To complete this analysis, I must consider another set of modifications of identity involving Leibniz's law.

The identity of indiscernibles. Sometimes identity is introduced as

(4) If $(\phi)(\phi(x) \text{ eq } \phi(y))$ then xIy.

The usual interpretation of eq is as material equivalence:

(4.1) If $(\phi)(\phi(x) \equiv \phi(y))$ then xIy.

Another possibility is in terms of tautological equivalence.

(4.2) If $(\phi)(\phi(x) \equiv \phi(x))$ is a tautology, then xIy.

The converse of (4) is of course explicative of identity. The status of (4) itself is not quite so clear. It has also been accepted as a truism and merely explicative. However, I do not regard Ramsey's[2] reservations about (4) as entirely spurious. He objected to taking (4.1) as definitive of identity on the ground that it is logically possible for two things to have all their properties in common and still be two. Such a possibility is excluded by (4.1). To argue that if they are two then they are distinguishable as having two different names will not do for Ramsey, since they may be unknown, unnamed, and still two.

According to our characterization of extensionality, instances of (4) may therefore be interpreted as extensionality principles in that they equate identity with the slightly weaker relation of indiscernibility which requires that to be distinct means to be *discernibly* distinct.[3]

[2] F. P. Ramsey, *The Foundations of Mathematics* (London: Routledge & Kegan Paul; and New York: Harcourt Brace, 1931), pp. 30–2.

[3] I am aware that interpreting (4) in this way is somewhat paradoxical since (4) has the effect of establishing a logical priority of the concept of 'property' over that of 'class' whereas in an extensional system the emphasis is held to be on the class concept. If such usage is intolerable the characterization of extensionality can be appropriately modified.

Interpreting intensional systems. Quine[4] states: 'When modal logic is extended (as by Miss Barcan) to include quantification theory, . . . serious obstacles to interpretation are encountered.' These difficulties revolve about the substitution of equivalences in contexts involving 'knows that', 'is aware that', and in particular 'is necessary that', and 'is possible that'. Quine describes such contexts as being referentially opaque. It is the point of this paper to show that the opacity lies with Quine's use of such terms as 'identity', 'true identity', 'equality'. The above analysis leads to the dissolution of at least some of the problems of interpretation associated with intensional contexts.

Among the equivalence relations which can be introduced into L are identity, indiscernibility, tautological equivalence, material equivalence. On the functional level, these are listed in order of decreasing strength, for there is some model, some permissible range of values, which prevents our equating them except by explicit postulate. It should be noted that for variables of lowest type, there are only identity and indiscernibility. Indeed a recent paper of Bergmann[5] on *Individuals* may be understood as an attempt to explicate the notion of individuals as those entities for which only the strongest equivalence relation holds.

Consider now modal functional calculi such as my[6] extension of the Lewis systems. In such languages (2.1) can be stated directly as

(5) If $N((x)(F(x) \equiv G(x)))$ then FIG (where N is interpreted as logical necessity)

(4.2) becomes

(6) If $N((\phi)(\phi(x) \equiv \phi(y)))$ then xIy

and (1.1) is

(7) If $N(P \equiv Q)$ then PIQ.

[4] W. Quine, 'The Problem of Interpreting Modal Logic', *Journal of Symbolic Logic*, 12 (1947), 43–8.

[5] G. Bergmann, 'Individuals', *Philosophical Studies*, IX (1958), 78–85. (My interpretation of this paper rests on the assumption that the statement of (Ext) involves a typographical error.)

[6] R. C. Barcan (Marcus): 'A Functional Calculus of First Order Based on Strict Implication', *Journal of Symbolic Logic*, 11 (1946), 1–16; 'The Deduction Theorem in a Functional Calculus of First Order Based on Strict Implication', ibid. 115–18; 'The Identity of Individuals in a Strict Functional Calculus of Second Order', ibid. 12, 12–15.

Within these extended systems, I have been able to prove theorems which relate different kinds of equivalence. It is possible to show

(8) Given $P \equiv Q$, P is not everywhere interchangeable with Q, but only in restricted non-modal contexts. Given $N(P \equiv Q)$, then the substitution theorem is unrestricted.

This theorem has the effect of prohibiting the substitution of 'Socrates is a featherless biped' for 'Socrates is a man' in 'It is necessary that if Socrates is a man then Socrates is a man'. That the substitution theorem for strict equivalence differs from the theorem for material equivalence, is not paradoxical, but a more adequate formalization of a known distinction.

It is when Quine[7] refers to

(9) The number of planets equals nine

as a 'true identity', without hint of ambiguity that we become aware that his fundamental criticism is directed not toward presumed paradoxes but toward the intensional point of view. As indicated above (9) is *not* unambiguous except in a strongly extensional language.

Let us assume for the moment that '9' and 'the number of planets' are expressions of the same type level and can meaningfully be equated. Quite apart from interpretation in terms of the theory of descriptions, within the modal language the problem revolves about substituting 'the number of planets' for '9' in

(10) $N(9 > 7)$.

But such a substitution is prohibited by (8), for (9) does not assert tautological equivalence, and the substitution would have to be made within the scope of a modal operator. The paradox evaporates. By the same token, since

(11) $N(9 = (5+4))$, '5+4' can replace '9' in (10).

The problem of the Morning Star and the Evening Star is resolved in an analogous way.[8] For, like (9),

[7] W. Quine, *From a Logical Point of View* (Cambridge, Mass.: Harvard Univ. Press, 1953), p. 144. [See above, p. 21].

[8] The paragraph which follows restates a point made by F. B. Fitch in 'The Problem of the Morning Star and the Evening Star', *Philosophy of Science*, XVI (1949), 137–40.

(12) The Evening Star equals the Morning Star

is not unambiguous. If (12) involves proper names of individuals then 'the Evening Star' may replace 'the Morning Star' without paradox in

(13) It is necessary that the Evening Star is the Evening Star

for the only equivalence relation between individuals are identity and indiscernibility. Indeed, although it appears as if (4.1) and (4.2) express two kinds of indiscernibility, they can be *proved* strictly equivalent within a modal system. Quine's[9] failure to note the latter in his review of my paper had the unfortunate result of perpetuating a non-existent paradox.

If, on the other hand, (12) is about classes or properties, then it states a non-tautological equality, not an identity, and consequently, the conditions of the substitution theorem (8) prevent the substitution of 'the Morning Star' for one of the occurrences of 'the Evening Star' in (13). At the risk of too much repetition, we are not asserting that the substitution *ought* not to be made on the basis of some pre-formal analysis, but that they are prohibited by the theorems provable[10] in such extended systems.

I have tried in this brief paper, to characterize the theory of extensionality, and to show that logical systems are more or less extensional. Their extensionality depends on the kinds of contexts and predicates which are prohibited, and the degree to which the relation of identity is equated to weaker forms of equivalence. I also tried to show that a more broadly based logic in the direction of modalities need not do violence to the foundations of mathematics, and the supposed paradoxes involved in interpreting such intensional systems are not genuine.

[9] See n. 4 above. Quine's failure to notice that (4.1) and (4.2) are materially equivalent in $s2^2$ and strictly equivalent in $s4^2$ and $s5^2$ leads him to conclude that modal logic must deal with individual concepts rather than individuals. In a recent letter Quine forwarded copies of a note to the editor of *The Journal of Symbolic Logic*, and his publisher correcting the error.

[10] The substitution principle (8) is a rough restatement of substitution theorems for some of the extended modal calculi. The theorems are proved at the end of the first paper listed in n. 6 p. 49 above.