from the semantical rules alone, while for the derivation of 38-7 a factual premise is needed.

The foregoing discussion shows that, even if somebody possesses no other information concerning S_1 and S_2 than the semantical rules for these systems formulated in M_e, he is, nevertheless, in a position to know the meanings—that is to say, not only the extensions but also the intensions—which are intended, first, for the primitive descriptive constants and, second, for all designators. All he has to do is to look, first, at the rules of designation themselves and, second, at those statements about designation which follow from the semantical rules alone, leaving aside all those statements in M_e which, although true, can be arrived at only with the help of factual knowledge. In other words, he has to consider only those statements about designation which are L-true in M_e.

It is sometimes said that a metalanguage, in which the semantics of an object language *S* is to be formulated, must contain translations of all expressions or at least of all designators in *S*. If this were right, M_e would not suffice as a semantics language for S_2, because M_e cannot, of course, contain an expression L-equivalent to the intensional sign 'N' in S_2. But the requirement mentioned is only approximately right; strictly speaking, it is too strong. The metalanguage must, indeed, contain for every sentence in *S* an L-equivalent sentence; furthermore, it must be sufficiently equipped with variables and descriptive expressions. It is, however, not necessary that it contain an L-equivalent expression for every logical sign in *S*. Although M_e cannot contain a translation of 'N', it can contain a semantical rule for 'N', for instance, the rule of ranges mentioned above. If $\mathfrak{S}_i$ is a sentence in S_2 containing 'N', then an extensional language like S_1 or M_e cannot, of course, contain a translation of $\mathfrak{S}_i$ in the strong sense of a sentence with the same intensional structure (§ 14). But it can be shown that S_1, and hence M_e, too, always contains a sentence L-equivalent to $\mathfrak{S}_i$. [For full sentences of 'N', this follows simply from the circumstance that they are either L-true or L-false (see 39-2); however, since sentences may contain several occurrences of 'N' and quantifiers in any combination, the general proof is rather complicated.] Further, S_1 and S_2 contain the same variables and descriptive signs. Hence, if M_e is sufficient for the formulation of the semantics of S_1, it is likewise sufficient for that of S_2.

On the basis of these considerations, I am inclined to believe that it is possible to give a complete semantical description even of an intensional language system like S_2 in an extensional metalanguage like M_e. However, this problem requires further investigation.

CHAPTER V

ON THE LOGIC OF MODALITIES

In this chapter we study logical modalities like necessity, possibility, impossibility. We introduce 'N' as a symbol of necessity; the other modal concepts, including necessary implication and necessary equivalence, can be defined with its help. The modal system S_2 is constructed by adding 'N' to our previous system S_1 (§ 39); and the semantical rules for S_2 are stated (§ 41). An analysis of the variables occurring in modal sentences shows that they have to be interpreted as referring to intensions (§ 40); hence a translation in words must be given either in terms of intensions (in the metalanguage M) or in neutral terms (in M') (§ 43). Quine's views on the possibility of combining modalities and variables are discussed (§ 44). Finally, the main results of the discussions in this book are briefly summarized (§ 45).

§ 39. Logical Modalities

We form the modal system S_2 from our earlier system S_1 by the addition of the modal sign 'N' for logical necessity. We regard a proposition as necessary if any sentence expressing it is L-true. Other modalities can be defined in terms of necessity, for example, impossibility, possibility, contingency. With the help of 'N', we define symbols for necessary implication and necessary equivalence; the latter symbol may be regarded as an identity sign for intensions.

In the earlier chapters, modal sentences have sometimes been taken as examples, especially sentences about necessity or possibility, either in words (for instance, in §§ 30 and 31) or in symbols (for instance, § 11, Example II). We use 'N' as a sign for logical necessity; 'N(A)' is the symbolic notation for 'it is (logically) necessary that A'.

Quite a number of different systems of modal logic have been constructed, by C. I. Lewis (see Bibliography) and others.[1] These systems differ from one another in their basic assumptions concerning modalities. There is, for instance, the question of whether all sentences of the form '$Np \supset NNp$' are true, in words: 'if it is necessary that *p*, then it is necessary that it is necessary that *p*'. Some of the systems give an affirmative answer to this question, other systems give a negative answer or leave it undecided. Not only do logicians disagree among themselves on this question, but sometimes also one logician constructs systems which differ in this point, probably because he is doubtful whether he should regard the sentences mentioned as true or false. There are several further points of

[1] For bibliographical references up to 1938, see Church's bibliography in *Journal of Symbolic Logic*, Vols. I and III; the pertinent references are listed in III, 199 ("Modality") and 202 ("Strict Implication").

difference between the systems. All these differences are, I think, due to the fact that the concept of logical necessity is not sufficiently clear; it can, for instance, be conceived in such a way that the sentences mentioned are true, but also in another way such that they, or some of them, are false.

Our task will be to find clear and exact concepts to replace the vague concepts of the modalities as used in common language and in traditional logic. In other words, we are looking for explicata for the modalities. It seems to me that a simple and convenient way of explication consists in basing the modalities on the semantical L-concepts. The concept of logical necessity, as explicandum, seems to be commonly understood in such a way that it applies to a proposition p if and only if the truth of p is based on purely logical reasons and is not dependent upon the contingency of facts; in other words, if the assumption of not-p would lead to a logical contradiction, independent of facts. Thus we see a close similarity between two explicanda, the logical necessity of a proposition and the logical truth of a sentence. Now for the latter concept we possess an exact explicatum in the semantical concept of L-truth, defined on the basis of the concepts of state-description and range (2-2). Therefore, the most natural way seems to me to take as the explicatum for logical necessity that property of propositions which corresponds to the L-truth of sentences. Accordingly, we lay down the following convention for 'N':

39-1. For any sentence '. . .', 'N(. . .)' is true if and only if '. . .' is L-true.

We shall construct the system S_2 by adding to the system S_1 the sign 'N' with suitable rules such that the convention just stated is fulfilled (§ 41). This convention may be regarded as a rule of truth for the full sentences of 'N'. S_2 thus contains all the signs and the sentences of S_1.

On the basis of our interpretation of 'N', as given by the convention 39-1, the old controversies can be solved. Suppose that 'L-true in S_2' is defined in such a way that our earlier convention 2-1, which says that a sentence is L-true if and only if it is true in virtue of the semantical rules alone, independently of any extra-linguistic facts, is fulfilled. Let 'A' be an abbreviation for an L-true sentence in S_2 (for example, 'Hs $\vee \sim$ Hs'). Then 'N(A)' is true, according to 39-1. And, moreover, it is L-true, because its truth is established by the semantical rules which determine the truth and thereby the L-truth of 'A', together with the semantical rule for 'N', say 39-1. Thus, generally, if 'N(. . .)' is true, then 'NN(. . .)' is true; hence any sentence of the form '$\mathrm{N}p \supset \mathrm{NN}p$' is true. This constitutes an affirmative answer to the controversial question mentioned in the beginning. It can be shown in a similar way that every sentence of the

form '$\sim \mathrm{N}p \supset \mathrm{N} \sim \mathrm{N}p$' is true. This settles another one of the controversial questions.[2]

This analysis leads to the result that, if 'N(. . .)' is true, it is L-true; and if it is false, it is L-false; hence:

39-2. Every sentence of the form 'N(. . .)' is L-determinate.

Therefore, the convention 39-1 may be replaced by the following more specific one:

39-3. For any sentence '. . .' in S_2, 'N(. . .)' is L-true if '. . .' is L-true; and otherwise 'N(. . .)' is L-false.

On the basis of the concept of logical necessity, the other logical modalities can easily be defined, as is well known. For example, 'p is impossible' means 'non-p is necessary'; 'p is contingent' means 'p is neither necessary nor impossible'; 'p is possible' means 'p is not impossible' (we adopt this interpretation in agreement with the majority of contemporary logicians, in distinction to other philosophers who use 'possible' in the sense of our 'contingent'). Let us use the diamond, '$\Diamond$', as a sign of possibility; we define it on the basis of 'N':

39-4. *Abbreviation.* '$\Diamond$(. . .)' for '$\sim \mathrm{N} \sim$(. . .)'.

It would also be possible to take '$\Diamond$' as primitive, as Lewis does, and then to define 'N(. . .)' by '$\sim \Diamond \sim$(. . .)'.

There are six modalities, that is, purely modal properties of propositions (as distinguished from mixed modal properties, for instance, contingent truth, see 30-1). The accompanying table shows how they can be expressed in terms of 'N' and in terms of '$\Diamond$'. The last column gives the corresponding semantical concepts; a proposition has one of the modal

THE SIX MODALITIES

Modal Property of a Proposition	With 'N'	With '$\Diamond$'	Semantical Property of a Sentence
Necessary	$\mathrm{N}p$	$\sim \Diamond \sim p$	L-true
Impossible	$\mathrm{N} \sim p$	$\sim \Diamond p$	L-false
Contingent	$\sim \mathrm{N}p \cdot \sim \mathrm{N} \sim p$	$\Diamond \sim p \cdot \Diamond p$	Factual
Non-necessary	$\sim \mathrm{N}p$	$\Diamond \sim p$	Non-L-true
Possible	$\sim \mathrm{N} \sim p$	$\Diamond p$	Non-L-false
Noncontingent	$\mathrm{N}p \vee \mathrm{N} \sim p$	$\sim \Diamond \sim p \vee \sim \Diamond p$	L-determinate

[2] The two questions and the reasons for our affirmative answers are discussed in more detail in [Modalities], § 1.

properties if and only if any sentence expressing the proposition has the corresponding semantical property.

Every proposition with respect to a given system *S* is either necessary or impossible or contingent. This classification is, according to our interpretation of the modalities, analogous to the classification of the sentences of *S* into the three classes of L-true, L-false, and factual sentences. There is, however, one important difference between the two classifications. The number of L-true sentences may be infinite, and it is, indeed, infinite for each of the systems discussed in this book. On the other hand, there is only one necessary proposition, because all L-true sentences are L-equivalent with one another and hence have the same intension. [This result holds only for that use of the term 'proposition' which is based on L-equivalence as the condition of identity. It is, of course, possible to choose a stronger requirement for identity, for instance, intensional isomorphism. In this case the intensional structures are called 'propositions'. And their number is infinite.] Likewise, there is only one impossible proposition, because all L-false sentences are L-equivalent. But the number of contingent propositions (with respect to a system with an infinite number of individuals) is infinite, like that of factual sentences.

It should be noted that the two sentences 'N(A)' and 'the sentence 'A' is L-true in S_2' correspond to each other merely in the sense that, if one of them is true, the other must also be true; in other words, they are L-equivalent (assuming that L-terms are defined in a suitable way so as to apply also to the metalanguage). This correspondence cannot be used as a *definition* for 'N', because the second sentence belongs, not to the object language S_2 as the first one does, but to the metalanguage M. The second sentence is not even a *translation* of the first in the strict sense which requires not only L-equivalence but intensional isomorphism (§ 14). If M contains the modal term 'necessary', then 'N(A)' can be translated into M by a sentence of the form 'it is necessary that . . .' (where '. . .' is the translation of 'A'). If M contains no modal terms, then there is no strict translation for 'N(A)'. But the correspondence stated makes it possible in any case to give an *interpretation* for 'N(A)' in M with the help of the concept of L-truth, for example, by laying down the truth-rule, 39-1.

On the basis of 'N', we introduce two further modal signs for modal relations between propositions:

39-5. *Abbreviation.* Let '. . .' and '- - -' be sentences in S_2. '. . . ⥽ - - -' for 'N(. . . ⊃ - - -)'.

39-6. *Abbreviation.* Let '. . .' and '- - -' be any designators in S_2 (sentences or otherwise). '. . . ≡ - - -' for 'N(. . . ≡ - - -)'.

Thus '⥽' is a sign for necessary implication between propositions (Lewis' strict implication). The symbol '≡' is a sign for necessary equivalence. The sign '≡' in S_2 is the analogue to the term 'L-equivalent' in its non-semantical use in M (5-4) or M′ (§ 34), where it designates a relation between intensions, not between designators. When standing between sentences, it corresponds to Lewis' sign '=' for strict equivalence. We have seen earlier that '≡', standing between designators of any type, is a sign for the identity of extensions (see remark on 5-3). Here in S_2, '≡' is, similarly, a sign for the identity of intensions. For example, 'H ≡ RA' is short for 'N(H ≡ RA)'. Hence, according to the rule 39-1, 'H ≡ RA' is true if and only if 'H ≡ RA' is L-true, hence if and only if 'H' and 'RA' are L-equivalent, in other words, have the same intension.

We have earlier formulated the two principles of interchangeability (12-1 and 12-2). For the first principle we have given, in addition to the chief formulation in semantical terms (12-1a), alternative formulations with the help of sentences of the object language containing '≡' (12-1b and c). Now, with the help of '≡', we can provide analogous formulations for the second principle. The following theorems 39-7b and c, which may be added to 12-2a as 12-2b and c, follow from 12-2a because $\mathfrak{A}_j$ and $\mathfrak{A}_k$ are L-equivalent if and only if $\mathfrak{A}_j$ ≡ $\mathfrak{A}_k$ is true.

Second Principle of Interchangeability (alternative formulations):

39-7. Under the conditions of 12-2, the following holds:

b. (12-2b). ($\mathfrak{A}_j$ ≡ $\mathfrak{A}_k$) ⊃ (. . $\mathfrak{A}_j$ . . ≡ . . $\mathfrak{A}_k$ . .) is true (in *S*).

c. (12-2c). Suppose the system *S* contains variables for which $\mathfrak{A}_j$ and $\mathfrak{A}_k$ are substitutable, say '*u*' and '*v*'; then '(*u*)(*v*)[(*u* ≡ *v*) ⊃ (. . *u* . . ≡ . . *v* . .)]' is true (in *S*).

§ 40. Modalities and Variables

Problems concerning the interpretation of variables in modal sentences are discussed, in preparation for the semantical rules given in the next section. It is found that a universal quantifier preceding 'N' is to be interpreted as if it followed the 'N'. It is generally shown that variables in modal sentences are to be understood as referring to intensions rather than to extensions. Thus an individual variable in S_2 is interpreted as referring to individual concepts rather than to individuals. We decide to take as values of these variables not only those individual concepts which are expressible by descriptions in S_2 but the wider class of all individual concepts with respect to S_2. A concept of this kind is represented by any assignment of exactly one individual constant to each state-description in S_2.

So far we have given an interpretation for 'N' only in the case in which the argument-expression of 'N' is a sentence. But in a system which contains variables we also have to solve the problem of interpreting occurrences of 'N' followed by a matrix with free variables, e.g., 'N(P*x*)'. Let us investigate this problem in a general way for a system *S* containing a variable '*u*' of any type. How should we interpret the sentence '(*u*)[N(. . *u* . .)]', where '. . *u* . .' is a matrix containing '*u*' as the only free variable? Let us first consider the case in which '*u*' has only a finite number of values, say *n*, and all these values are expressible in *S*, say by the designators 'U_1', 'U_2', . . . 'U_n'. (As we shall see later, the interpretation of a variable in a modal sentence has to be given in terms of value-intensions, not value-extensions. Therefore, the statement just made is to be understood as saying that there are *n* value-intensions for '*u*' and that they are the intensions of the designators 'U_1', etc.) Now any universal sentence, whether in an extensional or in a modal language, always means that all values of the variable possess the property expressed by the matrix. Therefore, if the number of values is *n*, the universal sentence means the same as the conjunction of the *n* substitution instances of the matrix. In our example, '(*u*)[N(. . *u* . .)]' means the same as 'N(. . U_1 . .) • N(. . U_2 . .) • . . . • N(. . U_n . .)'.

A conjunction of *n* components ($n \geq 2$) is L-true if and only if every one of the components is L-true. Therefore, the following holds, in virtue of the correspondence between necessity and L-truth (39-1):

40-1. If 'A_1', . . . 'A_n' are any sentences, 'N(A_1 • A_2 • . . . • A_n)' is L-equivalent to 'N(A_1) • N(A_2) • . . . • N(A_n)'.

If we apply this to the above result, we find that '(*u*)[N(. . *u* . .)]' means the same as 'N[(. . U_1 . .) • (. . U_2 . .) • . . . • (. . U_n . .)]' and hence the same as 'N[(*u*)(. . *u* . .)]'. Thus the result is that '(*u*)' and 'N' may exchange their places.

Next, let us consider the case in which the variable '*u*' has an infinite, but denumerable, number of values, all of which are expressible in *S*, say by the designators 'U_1', 'U_2', etc. Here we cannot form a conjunction of the substitution instances, but we can still consider their class. If we interpret a class of sentences as a joint assertion of its sentences, in accord with the usual procedure, then we can apply semantical concepts to it in the following way: We define the range of a class of sentences as the product of the ranges of the sentences. This leads to the following two results:

(i) A class of sentences is true if and only if all its sentences are true.

(ii) A class of sentences is L-true if and only if all its sentences are L-true.

Now the sentence '(*u*)[N(. . *u* . .)]' is true if and only if the class of the instances 'N(. . U_n . .)' for $n = 1, 2$, etc., is true; hence, according to (i), if and only if every sentence of the form 'N(. . U_n . .)' is true; hence, according to 39-1, if and only if every sentence of the form '. . U_n . .' is L-true; hence, according to (ii), if and only if the class of these sentences is L-true; hence, if and only if '(*u*)(. . *u* . .)' is L-true; hence, according to 39-1, if and only if 'N[(*u*)(. . *u* . .)]' is true. Thus the result is that, in the case of infinitely many values also, the quantifier '(*u*)' and the modal sign 'N' in the original sentence may exchange places.

It seems natural to apply the same result to the case in which not all values of '*u*' are expressible in *S*, that is to say, to interpret a sentence of the form '(*u*)[N(. . *u* . .)]' in any case, irrespective of the number and expressibility of the values of '*u*', as meaning the same as 'N[(*u*)(. . *u* . .)]'. In particular, we shall construct the semantical rules of the system S_2 in such a way that any two sentences of the forms just stated are L-equivalent (§ 41). In S_2 '*u*' must, of course, be an individual variable.

Since a modal system contains not only extensional but also intensional contexts, a designator may, in general, be replaced by another one only if they are not merely equivalent but L-equivalent. Thus, in general, we have to take into consideration the intensions of the designators, not merely their extensions. Similarly, we have to consider for a given variable its value-intensions in the first place. If the system contains variables of the type of sentences, say '*p*', '*q*', etc., then a quantifier with a variable of this kind occurring in a modal sentence must be interpreted as referring to propositions, not to truth-values. For example, the sentence '(∃*p*)(∼N*p*)' must be understood as saying that there is a non-necessary proposition. It would hardly make sense to interpret it as saying that there is a non-necessary truth-value, because there are propositions with the same truth-value such that one of them fulfils the matrix '∼N*p*', while another one does not. This interpretation in terms of propositions seems generally accepted. C. I. Lewis, as well as the other logicians who have discussed his systems of modal logic or have constructed new ones, have used interpretations in terms of propositions. If variables of the type of predicators of degree one occur in a modal system, it is clear that they must be interpreted analogously in terms of properties, not of classes. Here, again, I think that most logicians would agree;

however, modal sentences with variables of this kind have not been discussed frequently.

In my view the situation with respect to individual variables is quite analogous, although this is usually not recognized. I think that individual variables in modal sentences, for example, in S_2, must be interpreted as referring, not to individuals, but to individual concepts. The difficulties which would otherwise arise will be explained later (§ 43). Thus a sentence of the form '$(x)(..x..)$' in S_2 is to be interpreted as referring to all individual concepts. Therefore, we now have to study the question as to what is to be regarded as the totality of all individual concepts with respect to S_2.

We shall assume for the following discussions that the individual constants in S_2 are L-determinate (§ 19), that is to say that they are interpreted by the rules of designation as referring to positions in an ordered domain and that any two different constants refer to different positions. [For this purpose, it would be more natural to construct S_2 on the basis of S_3 (§ 18) rather than of S_1. The reason for taking S_1 as the basis is merely the possibility of using the earlier examples. But we must then suppose that, for example, the rule of designation for 's' does not use the phrase 'the man who was known by the name of 'Walter Scott'', but rather: 'the man who was born at such and such a place at such and such a time'; and even this formulation would not be entirely adequate.] Consequently, we take any sentence of the form '$a \equiv b$' as L-false. However, $\equiv$-sentences with one or two descriptions (for example, '$(\imath x)(Axw) \equiv s$') are still, in general, factual.

A description $\mathfrak{A}_i$ in S_2, say '$(\imath x)(..x..)$', characterizes one of the individual positions with the help of the property expressed by the matrix '$..x..$'. If exactly one position has this property, then this position is the descriptum; otherwise, a^* is the descriptum (§ 8). Thus for the determination of the descriptum, the extension of $\mathfrak{A}_i$, factual investigation is required (unless the description is L-determinate). On the other hand, the intension of $\mathfrak{A}_i$, the individual concept expressed by $\mathfrak{A}_i$, must be something that can be determined by logical analysis alone. In order to understand more clearly what kind of entity an individual concept is, let us see what we can find out about the description $\mathfrak{A}_i$ by logical analysis alone. Suppose a state-description $\mathfrak{R}_n$ in S_2 is given (which is an infinite class of sentences in S_2). Then the question of whether or not there is exactly one individual position in $\mathfrak{R}_n$ fulfilling the matrix '$..x..$'—in other words, whether or not there is exactly one substitution instance of

the matrix with an individual constant which holds in $\mathfrak{R}_n$—is a purely logical question. If the answer is in the affirmative, the descriptum of $\mathfrak{A}_i$ with respect to $\mathfrak{R}_n$ is represented by that one individual constant; otherwise it is represented by 'a^*'. Thus the description $\mathfrak{A}_i$ assigns to every state-description exactly one individual constant; any individual constant may be assigned to several state-descriptions. If $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are L-equivalent and hence express the same individual concept, then both assign to any state-description the same individual constant. Therefore, we might say that an individual concept with respect to S_2 is an assignment of exactly one individual to every state (which is a proposition expressed by a state-description). However, we shall actually take not these states but the state-descriptions; and not the individuals but the individual constants. The latter is possible because we have assumed that these constants are L-determinate and that there is a one-one correlation between the individuals and the individual constants. Thus we shall take any assignment of exactly one individual constant to each state-description in S_2 (in other words, any function from state-descriptions to individual constants) as representing an individual concept with respect to S_2. Only a small part (a denumerable class) of the individual concepts represented by assignments of this kind are expressible by descriptions in S_2. Now we decide to take as values of the individual variables in S_2 not only the individual concepts *expressible* by descriptions *in* S_2 but all individual concepts represented by assignments of the kind described; we call them individual concepts *with respect to* S_2. In the next section we shall lay down the semantical rules for S_2 in accord with this decision; a universal quantifier will be interpreted as referring to all individual concepts with respect to S_2.

Some remarks may, incidentally, be made concerning the interpretation of variables of other than individual type. Let S be a modal system which also contains propositional variables 'p', etc., and variables 'f', etc., for properties of level one, that is, properties of individuals. As values for propositional variables we should take not only those propositions which are expressed by sentences in S, but all propositions with respect to S. They are represented by the ranges in S, that is, the classes of state-descriptions in S. And as values for 'f', etc., we should take not only those properties which are expressed by predicators (including lambda-expressions) in S, but all properties with respect to S. Since the attribution of a property to an individual results in a proposition, we may regard a property as an assignment of exactly one proposition to each individual. Therefore, we may represent the properties with respect to *S* by the assignments of ranges (classes of state-descriptions) in *S* to the individual constants in *S*. Similarly, assignments of ranges in *S* to ordered pairs of individual constants in *S* may be taken as representing the relations with respect to *S* as values of relation variables in *S*. [In analogy to the rules of ranges for matrices containing individual variables in S_2, which will be given in the next section, rules for variables of other types in *S* might be stated as follows: (i) The matrix '*p*' holds in the state-description $\mathfrak{R}_n$ for a certain range as value if and only if $\mathfrak{R}_n$ belongs to this range. (ii) The matrix '*fa*' holds in $\mathfrak{R}_n$ for a given assignment of the kind described as value of '*f*' if and only if $\mathfrak{R}_n$ belongs to that range which is assigned to 'a'.]

§ 41. Semantical Rules for the Modal System S_2

On the basis of our previous decisions concerning the interpretation of 'N' (§ 39) and of the individual variables in S_2 (§ 40), we lay down semantical rules for S_2. The most important rules are the rules of ranges, which are here somewhat more complicated than for S_1 because individual concepts rather than individuals must here be taken as values of the variables. The L-concepts for S_2 have the same definitions as for S_1. Some examples of L-true modal sentences in S_2 are given.

The signs of the modal system S_2 comprise those of S_1 and, in addition, the modal sign 'N'. In S_1, compound designators and designator matrices are formed out of atomic matrices with the help of the following means: the ordinary (i.e., nonmodal) connectives, quantifiers, the iota-operator, and the lambda-operator. In S_2 a rule of formation for 'N' is added, which says that, if '. . .' is any matrix, 'N(. . .)' is a matrix.

Now we have to construct the rules of ranges for S_2. The state-descriptions in S_2 are the same as in S_1 (§ 2), because S_2 does not contain any new descriptive constants. If we had only sentences without variables, we could simply take the rules of ranges for S_1 (see the examples in § 2, omitting the rule for a universal sentence) and add the following rule:

41-1. $N(\mathfrak{S}_i)$ holds in every state-description if $\mathfrak{S}_i$ holds in every state-description; otherwise, $N(\mathfrak{S}_i)$ holds in no state-description.

This rule is clearly in accord with our convention 39-3 (see 2-2 and 2-4). However, in order to accommodate sentences with variables, we have to use, instead, more complicated rules of ranges. They must apply not only to sentences, like the rules of ranges for S_1 (§ 2), but to matrices, and they

must refer to values of the individual variables occurring in the matrix. According to our analysis in the preceding section, we take as values of the variables all individual concepts with respect to S_2; every one of these concepts is represented by an assignment of individual constants to state-descriptions. Suppose that we have chosen as a value of the variable '*x*' occurring in the atomic matrix '*Px*' an assignment of this kind and that the individual constant assigned to a given state-description $\mathfrak{R}_n$ is 'b'. Then the question of whether the matrix '*Px*' for the chosen value of '*x*' holds in $\mathfrak{R}_n$ means simply whether the sentence 'Pb' holds in $\mathfrak{R}_n$; and this is, of course, the case if 'Pb' belongs to $\mathfrak{R}_n$ (compare the example (1) of the rules of ranges for S_1 in § 2). This analysis suggests the first of the subsequent rules of ranges (41-2a). The other rules are analogous to the rules of ranges for S_1 (§ 2), together with the rule 41-1 for 'N', except that the present rules apply to matrices and therefore have to refer to assignments as values of the free variables.[3] Note that sentences are matrices without free variables (§ 1); therefore, these rules apply also to sentences, in which case the references to values are dropped.

41-2. ***Rules of ranges*** for the modal system S_2. Let $\mathfrak{A}_i$ be a matrix and $\mathfrak{R}_n$ be a state-description in S_2. By a value of a variable we mean any assignment of the kind described earlier.

a. Let $\mathfrak{A}_i$ be of atomic form. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the individual variables occurring in $\mathfrak{A}_i$, if and only if $\mathfrak{R}_n$ contains the atomic sentence formed from $\mathfrak{A}_i$ by substituting for every free variable the constant assigned to $\mathfrak{R}_n$ by the value of the variable.

b. Let $\mathfrak{A}_i$ be an $\equiv$-matrix with individual signs (constants or variables). $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the variables occurring in $\mathfrak{A}_i$, if the individual constant for the left side (that is, either

[3] The system MFL described in [Modalities], § 9, is similar to, but somewhat simpler than, our present system S_2. Sentences of the form 'a = b' in MFL are regarded as L-false, like the corresponding sentences of the form 'a ≡ b' in S_2; this shows that the individual constants in MFL are, in terms of our present theory, L-determinate like those in S_2. The state-descriptions are the same in both systems. The differences are as follows: MFL does not contain lambda-expressions and individual descriptions; this difference is not essential, since both kinds of expressions in S_2 can be eliminated, as we have seen. More essential is the difference in the interpretation of individual variables. A universal sentence '(*x*)(. . *x* . .)' in MFL is regarded as L-equivalent to the class of substitution instances of the matrix '. . *x* . .' with all individual constants; thus, in terms of our present theory, the universal quantifier refers to all L-determinate individual concepts and to no others. A universal quantifier in S_2, on the other hand, refers to all individual concepts (with respect to S_2). This wider range of values for the individual variables in S_2 seems more adequate; but it makes necessary the somewhat more complicated form of the rules of ranges as given in the text, while the rules of ranges for MFL are as simple as those for S_1, together with the rule 41-1 for 'N'.

the individual constant standing on the left side or the individual constant assigned to $\mathfrak{R}_n$ by the value of the variable standing on the left side) is the same as that for the right side.

c. Let $\mathfrak{A}_i$ be $\sim\mathfrak{A}_j$. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the variables occurring freely in $\mathfrak{A}_i$, if $\mathfrak{A}_j$ does not hold in $\mathfrak{R}_n$ for these values.

d. Let $\mathfrak{A}_i$ be $\mathfrak{A}_j \vee \mathfrak{A}_k$. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the free variables, if either $\mathfrak{A}_j$ or $\mathfrak{A}_k$ or both hold in $\mathfrak{R}_n$ for these values.

e. Let $\mathfrak{A}_i$ be $\mathfrak{A}_j \bullet \mathfrak{A}_k$. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the free variables, if both $\mathfrak{A}_i$ and $\mathfrak{A}_k$ hold in $\mathfrak{R}_n$ for these values.

f. Let $\mathfrak{A}_i$ consist of a universal quantifier followed by the matrix $\mathfrak{A}_j$ as its scope. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the variables occurring freely in $\mathfrak{A}_i$ (hence not including the variable occurring in the initial quantifier), if $\mathfrak{A}_j$ holds in $\mathfrak{R}_n$ for every value of the variable of the initial quantifier and the given values of the other free variables.

g. Let $\mathfrak{A}_i$ be $\mathrm{N}(\mathfrak{A}_j)$. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the free variables, if $\mathfrak{A}_j$ holds in every state-description for these values.

The following two theorems are simple consequences of these rules; they may be used instead of the rules for the determination of the range of a nonmodal matrix or sentence in S_2.

41-3. Let $\mathfrak{A}_i$ be a matrix of any form without 'N' in S_2. $\mathfrak{A}_i$ holds in $\mathfrak{R}_n$ for given values of the free variables, if and only if the sentence formed from $\mathfrak{A}_i$ by substituting for every free variable the constant assigned to $\mathfrak{R}_n$ by the value of the variable holds in $\mathfrak{R}_n$.

41-4. If a sentence in S_2 does not contain 'N', then it holds in S_2 in the same state-descriptions as in S_1.

In order to avoid certain complications, which cannot be explained here, it seems advisable to admit in S_2 only descriptions which do not contain 'N'. But any description may, of course, occur within the scope of an 'N'. The smallest matrix in which a description occurs (in the primitive notation) is always a nonmodal context, because the description must be an argument expression either of a primitive predicator constant or of '$\equiv$'. This smallest matrix is then taken as the context '- -$(\imath x)$(. . x . .) - -', which can be transformed into 8-2. In this way every description can be eliminated. Since L-equivalent sentences are L-interchangeable also within modal contexts, according to the second principle of interchangeability (12-2), the result of the elimination is L-equivalent to the original sentence; or, rather, we lay down a rule to the effect that any sentence containing descriptions holds in the same state-descriptions as the sentence resulting from the described elimination of the descriptions, and hence the two sentences become L-equivalent.

Another point is worth noting. Although we interpret the individual variables in S_2 as referring to individual concepts, not to individuals, nevertheless a description in S_2 characterizes, not one individual concept, but mutually equivalent individual concepts—in other words, one individual. This follows from the rule just mentioned, which permits the transformation into 8-2. The first part of 8-2 says, in words: 'there is an individual concept y such that, for every individual concept x, x has the descriptional property if and only if x is *equivalent* (not 'L-equivalent' or 'identical'!) to y'; in other words, 'all individual concepts equivalent to y, and only these, have the property'; or, 'the *individual* y is the only individual which has the property'. This is as it should be, because the purpose of a description, even in a modal language, is to refer to one individual with the help of a property possessed by that individual alone. Nevertheless, the description has, of course, a unique intension, which is an individual concept. This individual concept is not the only one possessing the descriptional property, since, as we have seen, all equivalent ones do likewise; but it is uniquely determined by the descriptional property; as Frege puts it, it is not the individual but the way in which the description refers to the individual.

For lambda-expressions we do not impose the restriction stated for descriptions; they may also contain 'N'. Any lambda-operator can be eliminated in S_2 by conversion in the same way as in S_1 (§ 1). Here, again, a rule would be laid down saying that a sentence containing lambda-operators holds in the same state-descriptions as the sentence resulting from their elimination.

The L-concepts are defined for S_2 in the same way as for S_1 (§ 2). The following theorems give a few results, which hold on the basis of the rules of ranges stated above.

41-5. Any sentence of one of the following forms is L-true in S_2. (The variables 'p', 'q', . . 'f', do not occur in S_2 but are here used merely to describe forms of sentences in S_2. A sentence in S_2 is said to have one of the forms described if it is formed by substituting for 'p' or 'q' any sentence in S_2 and for 'fx' any matrix containing 'x' as the only free variable.)

a. $\text{'N}p \supset p\text{'}$.
b. $\text{'}p \supset \Diamond p\text{'}$.
c. $\text{'}(p \supset q) \supset (\text{N}p \supset \text{N}q)\text{'}$
d. $\text{'N}(p \bullet q) \equiv \text{N}p \bullet \text{N}q\text{'}$.
e. $\text{'}\Diamond(p \vee q) \equiv \Diamond p \vee \Diamond q\text{'}$.
f. $\text{'NN}p \equiv \text{N}p\text{'}$.
g. $\text{'N} \sim \text{N}p \equiv \sim \text{N}p\text{'}$.
h. $\text{'}\Diamond\Diamond p \equiv \Diamond p\text{'}$.
i. $\text{'}\Diamond \text{N}p \equiv \text{N}p\text{'}$.
j. $\text{'N}\Diamond p \equiv \Diamond p\text{'}$.
k. $\text{'}(x)\text{N}(fx) \equiv \text{N}(x)(fx)\text{'}$.
l. $\text{'}(\exists x)\text{N}(fx) \supset \text{N}(\exists x)(fx)\text{'}$.
m. $\text{'}(\exists x)\Diamond(fx) \equiv \Diamond(\exists x)(fx)\text{'}$.
n. $\text{'}\Diamond(x)(fx) \supset (x)\Diamond(fx)\text{'}$.

We see from these theorems that 'N' is quite similar to a universal quantifier and '$\Diamond$' to an existential quantifier. This seems plausible, since $\text{N}\mathfrak{S}_i$ is true if $\mathfrak{S}_i$ holds in every state-description, and $\Diamond\mathfrak{S}_i$ is true if $\mathfrak{S}_i$ holds in at least one state-description.

§ 42. Modalities in the Word Language

The problem of the translation of modal sentences of S_2 into the metalanguages M and M′ is discussed. It is shown that it is advisable to use for the translations either terms of intensions in M or neutral terms in M′. The use of terms of extensions within modal sentences in M is not in itself incorrect, provided that certain restrictions are observed; but it involves the danger of making wrong inferences by overlooking the restrictions.

We shall examine here the problem of the formulation of modal sentences in words and, in particular, the problem of the translation of modal sentences into our metalanguages M and M′. It is worth while to study this problem because, it seems to me, certain difficulties which have sometimes been found in connection with modal sentences are due chiefly to their inadequate or misleading formulation in the word language.

Since modal sentences, for instance, in S_2 or in a richer language with several types of variables, are not semantical, their translations are likewise not semantical sentences and hence belong to the nonsemantical part of M and M′ (this part of M′ was explained in §§ 34–36). As translation of 'N', we take 'it is necessary that'; hence, this is an intensional phrase.

We shall discuss three examples—A, B, and C. In A, we have predicators as argument expressions of '≡' or '≡'; in B, sentences; in C, individual expressions. Otherwise, the three examples are perfectly analogous. Therefore, we arrange them in three parallel columns. This facilitates the

comparison of corresponding expressions in the three examples and the recognition of their analogy.

Because of the perfect analogy, any one of the three examples would theoretically be sufficient. However, for practical reasons it seems advisable to give all three. The purpose of the analysis of the examples is to show that it is advisable to formulate modal sentences either in terms of intensions or in neutral terms, while formulation in terms of extensions involves certain dangers. Now this result is easily seen in the case of predicators; presumably, most readers will agree in this case. Then the analogy will make it easier to recognize the same situation in the case of sentences and, finally, in the case of individual expressions. In this last case the inhibitions against a translation in terms of intensions are strongest because it is not customary to speak of individual concepts. Therefore, here the help of the two other examples seems necessary for practical, psychological reasons, although theoretically the situation is here as clear and simple as in the first two cases.

The example A (the conjunction of 42-1A and 42-2aA) is similar to one given by Church;[4] our '$\sim$N(. . .)' corresponds to his '$\Diamond\sim$(. . .)'. In the example C, we use 'au' as abbreviation for '$(\iota x)(\text{A}xw)$'. In the translation of this description into the word language, we omit, for the sake of brevity, the phrase 'or a*, if there is not exactly one such individual' (as we did earlier, § 9).

The following sentences in S_2 are true but not L-true (see 3-7 and 9-2):

	A	B	C
42-1.	$\text{'F} \bullet \text{B} \equiv \text{H'}$.	$\text{'(F} \bullet \text{B)s} \equiv \text{Hs'}$.	$\text{'au} \equiv \text{s'}$.

Therefore, according to 39-1, prefixing 'N' yields false sentences; hence the following is true:

	A	B	C
42-2a.	$\text{'}\sim\text{N(F} \bullet \text{B} \equiv \text{H)'}$;	$\text{'}\sim\text{N[(F} \bullet \text{B)s} \equiv \text{Hs]'}$;	$\text{'}\sim\text{N(au} \equiv \text{s)'}$;

or, abbreviated with '≡' (39-6):

	A	B	C
42-2b.	$\text{'}\sim\text{(F} \bullet \text{B} \equiv \text{H)'}$.	$\text{'}\sim\text{[(F} \bullet \text{B)s} \equiv \text{Hs]'}$.	$\text{'}\sim\text{(au} \equiv \text{s)'}$.

Now let us examine the question of the translations of these sentences of S_2 into M. The first sentence, 42-1 (in each of the three examples), is a nonmodal sentence. It can be translated in two different ways, either into 42-3 in terms of intensions with the nonsemantical term 'equivalent' (see 5-3 and 5-5) or into 42-4 in terms of extensions with the identity phrase 'is the same as' (see 4-7 and 9-1):

[4] [Review Q.], p. 46.

42-3.

A	B	C
'The property Featherless Biped is equivalent to the property Human'.	'The proposition that Scott is a featherless biped is equivalent to the proposition that Scott is human'.	'The individual concept The Author Of Waverley is equivalent to the individual concept Walter Scott'.

42-4.

A	B	C
'The class Featherless Biped is the same as the class Human'.	'The truth-value that Scott is a featherless biped is the same as the truth-value that Scott is human'.	'The individual The Author Of Waverley is the same as the individual Walter Scott'.

For the modal sentences 42-2, however, the situation is different. First, we shall give the translation into M in terms of intensions. We base the translation 42-5 on the second of the two notations a and b given for 42-2, utilizing the fact that '≡' is a sign for the identity of intensions (§ 39). (For A, see 4-8; for B, 6-4; for C, § 9).

42-5.

A	B	C
'The property Featherless Biped is not the same as the property Human'.	'The proposition that Scott is a featherless biped is not the same as the proposition that Scott is human'.	'The individual concept The Author Of Waverley is not the same as the individual concept Walter Scott'.

This translation is adequate and unobjectionable. Not so, however, the following translation in terms of extensions; here we base the translation on the first notation 42-2a and regard '≡' as a sign for the identity of extensions (see remark on 5-3).

42-6.

A	B	C
'It is not necessary that the class Featherless Biped is the same as the class Human'.	'It is not necessary that the truth-value that Scott is a featherless biped is the same as the truth-value that Scott is human'.	'It is not necessary that the individual The Author Of Waverley is the same as the individual Walter Scott'.

Formulations of this kind might perhaps be admitted as sentences in M; if so, they would presumably be regarded as true and as correct translations of 42-2a. However, these formulations are dangerous; if we apply customary ways of thinking to them, we obtain false results. In the ordinary word language, we are accustomed to using the principle of interchangeability (24-3b) implicitly. If in any of the three examples we apply this principle to 42-6 on the basis of the true identity sentence 42-4, we obtain the following result, 42-7. This, however, if admitted at all as a sentence, will certainly be regarded as false.

42-7.

A	B	C
'It is not necessary that the class Human is the same as the class Human'.	'It is not necessary that the truth-value that Scott is human is the same as the truth-value that Scott is human'.	'It is not necessary that the individual Walter Scott is the same as the individual Walter Scott'.

These are instances of the antinomy of the name-relation in its second form, similar to our previous example (§ 31). In spite of this result, we may admit the formulations 42-6, provided that we are willing to prohibit the use of the principle of interchangeability in cases of nonextensional contexts. However, since the unrestricted use of this principle is customary and plausible, there would always be the danger of forgetting the prohibiting rule and using the principle inadvertently. Therefore, it seems more advisable to avoid formulations like 42-6 and, in general, formulations in terms of extensions within modal or other nonextensional contexts.

Now let us see how the given symbolic sentences of S_2 are to be translated into the neutral metalanguage M'. As explained earlier, there are no identity phrases in M'; instead, the terms 'equivalent' and 'L-equivalent' are applied in their nonsemantical use (see 34-8 and 34-9). As 'equivalent' is a direct translation of the symbol '≡', so is 'L-equivalent' of '≡'. (This shows again that the nonsemantical term 'L-equivalent' is intensional; this holds for all nonsemantical (absolute) L-terms, see [I], § 17.) Thus the translation of 42-1 into M' is as follows (see 34-10 and 34-13):

42-8.

A	B	C
'Featherless Biped is equivalent to Human'.	**a.** 'That Scott is a featherless biped, is equivalent to that Scott is human'.	'The Author Of Waverley is equivalent to Walter Scott'.
	b. 'Scott is a featherless biped if and only if Scott is human'.	

In B we add here the alternative form b because it sounds more natural (see end of § 34).

There are two ways of translating 42-2 into M′. The first is based on 42-2a and translates 'N' by 'it is necessary that'. (In B we use again the more natural phrase 'if and only if' instead of 'is equivalent to'; concerning the reason for the word order, see remark at the end of § 34.)

42-9a.

A	B	C
'It is not necessary that Featherless Biped is equivalent to Human'.	'That Scott is a featherless biped if and only if Scott is human, is not necessary'.	'It is not necessary that The Author Of Waverley is equivalent to Walter Scott'.

The second alternative is based on the notation 42-2b and translates '$\equiv$' by 'L-equivalent' (see 34-11):

42-9b.

A	B	C
'Featherless Biped is not L-equivalent to Human'.	'That Scott is a featherless biped, is not L-equivalent to that Scott is human'.	'The Author Of Waverley is not L-equivalent to Walter Scott'.

This translation does not involve any difficulty analogous to that connected with 42-6.

Thus the final result is as follows: It seems advisable to frame the formulation of modal and other nonextensional sentences in the word language, not in terms of extensions, but either (i) in terms of intensions or (ii) in neutral terms. Which of the two formulations (i) and (ii) one prefers is a matter of practical decision (see the discussion at the end of § 37). The formulation in neutral terms is simpler, but the nonsemantical

use of the terms 'equivalent' and 'L-equivalent' is not customary. Formulations in terms of intensions, like 42-5, are, in general, more customary, except for the reference to individual concepts in case C. But this reference will perhaps appear less strange if we recognize the essential analogy in 42-5 between C, on the one hand, and A and B, on the other.

§ 43. Modalities and Variables in the Word Language

Translations of symbolic modal sentences with variables into M and M′ are examined. The result is analogous to that in the preceding section. It is advisable to avoid terms of extensions and to use either terms of intensions in M or the neutral terms in M′. The translation in terms of propositions and properties is customary, but that in terms of individual concepts instead of individuals may at first appear strange.

We have seen earlier (§ 10) that, as a designator has both an extension and an intension, a variable has both value-extensions and value-intensions. Therefore, a sentence with a variable can be translated into M either in terms of its value-extensions or in terms of its value-intensions. Furthermore, it can be translated into M′ in neutral terms (§ 36). In analogy to the result in the preceding section, we shall find here that it is advisable to avoid the formulation in terms of value-extensions and to use either terms of value-intensions or neutral terms.

For the same reason as in the preceding section, we use here three analogous examples, A, B, and C. They are existential sentences with the variables 'f', 'p', and 'x' in a modal system S containing variables of these types and the modal sign 'N'.

The following sentences 43-1a and b differ only in their notation. In each of the three examples, A, B, and C, 43-1a is derived by existential generalization from the conjunction of the sentences 42-1 and 42-2a; and likewise 43-1b from 42-1 and 42-2b.

43-1a.

A	B	C
'$(\exists f)[(f \equiv \mathrm{H}) \bullet \sim\mathrm{N}(f \equiv \mathrm{H})]$'.	'$(\exists p)[(p \equiv \mathrm{Hs}) \bullet \sim\mathrm{N}(p \equiv \mathrm{Hs})]$'.	'$(\exists x)[(x \equiv \mathrm{s}) \bullet \sim\mathrm{N}(x \equiv \mathrm{s})]$'.

43-1b.

A	B	C
'$(\exists f)[(f \equiv \mathrm{H}) \bullet \sim(f \equiv \mathrm{H})]$'.	'$(\exists p)[(p \equiv \mathrm{Hs}) \bullet \sim(p \equiv \mathrm{Hs})]$'.	'$(\exists x)[(x \equiv \mathrm{s}) \bullet \sim(x \equiv \mathrm{s})]$'.

We shall now examine the possibilities for the translation of these sentences into M. If it were a question of an extensional existential sentence—for instance, 43-1a with the second conjunctive component omitted—then translations in terms of value-intensions and of value-extensions would be equally acceptable. This, however, is not the case for these modal sentences. We shall first give a translation in terms of value-intensions, in analogy to 42-3 and 42-5, taking notation 43-1b and translating '≡' by identity of intensions:

43-2.

A	B	C
'There is a property f which is equivalent to but not the same as the property Human'.	'There is a proposition p which is equivalent to but not the same as the proposition that Scott is human'.	'There is an individual concept x which is equivalent to but not the same as the individual concept Walter Scott'.

In each of the three examples, this sentence can be derived by existential generalization from the conjunction of 42-3 and 42-5.

Now we shall translate 43-1a in terms of value-extensions, in analogy to 42-4 and 42-6, translating '≡' by identity of extensions:

43-3.

A	B	C
'There is a class f which is the same but not necessarily the same as the class Human'.	'There is a truth-value p which is the same but not necessarily the same as the truth-value that Scott is human'.	'There is an individual x which is the same but not necessarily the same as the individual Walter Scott'.

In each of the three examples, this sentence can be derived by existential generalization from the conjunction of 42-4 and 42-6. We have seen in the preceding section that formulations of modal sentences in terms of extensions, like 42-6, are dangerous because they lead to the antinomy of the name-relation unless special restrictions are imposed and that it is therefore advisable to avoid these formulations. The same holds for formulations like 43-3.

The translation of 43-1 into neutral formulations in M′, in analogy to 42-8 and 42-9b, is as follows:

43-4.

A	B	C
'There is an f such that f is equivalent but not L-equivalent to Human'.	'There is a p such that p is equivalent but not L-equivalent to that Scott is human'.	'There is an x such that x is equivalent but not L-equivalent to Walter Scott'.

(Use of 'F-equivalent' as a nonsemantical term would provide a shorter formulation.) In each of the three examples this sentence can be derived by existential generalization from the conjunction of 42-8 and 42-9b. The formulations 43-4 are free of the dangers involved in 43-3.

Now let us compare the three examples, A, B, and C. Our proposal not to translate variables in modal sentences in terms of extensions seems quite natural in cases B and A. As remarked earlier (§ 40), it seems that all logicians interpret modal sentences in terms of propositions rather than of truth-values, and most of them use terms of properties rather than of classes. Only in case C does our interpretation deviate from the customary one. The reference to individual concepts may first appear somewhat strange; and the alternative translation in neutral terms (e.g., 43-4C), which avoids the reference to individual concepts, uses the unfamiliar terms 'equivalent' and 'L-equivalent'. However, I believe that, once we are aware of the perfect analogy between the three cases, we recognize the inadequacy of the formulations in terms of individuals; and the impression of strangeness which the formulation in terms of individual concepts and, to a lesser degree, the neutral formulation may first give will perhaps disappear. Modal sentences with variables are of a quite peculiar logical nature, and it should not be surprising that an adequate and correct rendering for them in the word language is not always possible in entirely customary and natural terms.

§ 44. Quine on Modalities

Quine's article [Notes] explained his view that, under customary conditions, modalities and quantification cannot be combined. A new statement by Quine is quoted here, in which he says that my language succeeds in combining modalities with quantification but only at the price of repudiating all extensions, for instance, classes and individuals. I try to show that my modal language does not exclude anything that is admitted by a corresponding extensional language.

Quine[5] illustrates the difficulty which we have called the antinomy of the name-relation by the following example among others (as mentioned above, § 31). We find as an arithmetical and hence logical truth:

(i) '9 is necessarily greater than 7'.

The following is a true statement of astronomy:

(ii) 'The number of planets = 9'.

[5] Quine [Notes] (18) p. 121, (15) p. 119, (23) p. 121.

If, in (i), '9' is replaced by 'the number of planets' in virtue of the true identity statement (ii), we obtain the false statement:

(iii) 'The number of planets is necessarily greater than 7'.

Quine's method for solving the antinomy has been explained earlier (§ 32, Method II). According to our method, the following sentence takes the place of (ii) in M′:

(iv) 'The number of planets is equivalent to 9'.

The sentences (i) and (iii) occur also in M′. But now it is not possible to infer the false sentence (iii) from the true sentence (i) together with (iv). According to the first principle of interchangeability (12-1), the expressions 'the number of planets' and '9' are interchangeable on the basis of (iv) in extensional contexts only, hence not in (i). Thus the difficulty disappears, and the designators occurring in nonextensional contexts still function, according to our conception, as normal designators.

An even more serious problem is raised by Quine's objection to modal sentences with variables. He discusses the following expression:

(v) 'There is something which is necessarily greater than 7'.

He says[6] that this expression "is meaningless. For, would 9, that is, the number of planets, be one of the numbers necessarily greater than 7? But such an affirmation would be at once true in the form . . . [our (i)] and false in the form . . . [our (iii)]." Quine does not regard (i) and (iii) as meaningless. As explained earlier (§ 32, Method II), he regards occurrences of designators in nonextensional contexts, e.g., '9' in (i) and 'the number of planets' in (iii), as "not purely designative"; in other words, these occurrences do not function as names, and hence the principle of interchangeability is not applicable. For the same reason, according to Quine's view, the rule of existential generalization is not applicable to these occurrences. Therefore, there is no valid inference from (i) to (v), and, moreover, (v) has no meaning and hence cannot be admitted as a sentence. Thus Quine arrives at the following conclusions, which are stated at the end of his paper: "A substantive word or phrase which designates an object may occur purely designatively in some contexts and not purely designatively in others. This second type of context, though not less "correct" than the first, is not subject to the law of substitutivity of identity nor to the laws of application and existential generalization. Moreover, no pronoun (or variable of quantification) within a context of this

[6] *Ibid.*, p. 124.

second type can refer back to an antecedent (or quantifier) prior to that context. This circumstance imposes serious restrictions, commonly unheeded, upon the significant use of modal operators, as well as challenging that philosophy of mathematics which assumes as basic a theory of attributes [i.e., properties] in a sense distinct from classes."[7]

To Quine's contexts of the second kind belong all those which we call nonextensional. He discusses, in particular, contexts within quotes and modal contexts. With respect to contexts within quotes his conclusions are no doubt correct. I cannot agree, however, with Quine's conclusion concerning modal contexts. We have combined modalities and variables both in symbolic object languages (§ 40) and in word formulations in our metalanguages (§ 43).

Church likewise does not accept Quine's result. He says in the review of Quine's paper that he "would question strongly the conclusion which the author draws that no variable within an intensional context . . . can refer back to a quantifier prior to that context . . . . The conclusion should rather be that in order to do this a variable must have an intensional range—a range, for instance, composed of attributes [properties] rather than classes."[8] Up to this point I am in agreement with Church. His solution is as follows: He distinguishes, like the system PM (see § 27), between class variables, e.g., 'α', and property variables, e.g., 'ϕ'. He takes as example a sentence which is essentially the same as a conjunction of 42-1A and 42-2aA. In distinction to Quine, he regards it as admissible to infer from this sentence by existential generalization an existential sentence; the latter, however, must not have the form '$(\exists\alpha)(\ldots\alpha\ldots)$' but rather the form '$(\exists\phi)(\ldots\phi\ldots)$'. It seems to me that this procedure is correct and, indeed, solves completely the difficulty pointed out by Quine. I believe, however, that there is a simpler way to achieve this. It is similar to that of Church but avoids the use of two kinds of variables for the same type. This use is, as explained earlier (§ 27), an unnecessary duplication. It is sufficient to use variables of one kind which are neutral in the sense that they have classes as value-extensions and properties as value-intensions; this is done in 43-1aA. The use of different variables for extensions and intensions within all types would lead in the case of Quine's example (v) to the introduction of variables for number concepts different from the variables for numbers. This, however, would be both unnecessary and unusual.

The problem of whether or not it is possible to combine modalities and

[7] *Ibid.*, p. 127.

[8] [Review Q.], p. 46.

variables in such a way that the customary inferences of the logic of quantification—in particular, specification and existential generalization—remain valid is, of course, of greatest importance. Any system of modal logic without quantification is of interest only as a basis for a wider system including quantification. If such a wider system were found to be impossible, logicians would probably abandon modal logic entirely. Therefore, it is essential to clarify the situation created by Quine's analysis and objections. For this reason I have asked Quine, who has read an earlier version of the manuscript of this book, for a statement of his present view on the problem mentioned and, in particular, his reaction to my method for combining modalities and variables as explained in the preceding section. With his kind permission, I am quoting here his statement in full:[9]

> Every language system, insofar at least as it uses quantifiers, assumes one or another realm of entities which it talks about. The determination of this realm is not contingent upon varying metalinguistic usage of the term 'designation' or 'denotation', since the entities are simply the values of the variables of quantification. This is evident from the meaning of the quantifiers '(x)', '(f)', '(p)', '$(\exists x)$', '$(\exists f)$', '$(\exists p)$' themselves: 'Every (or, Some) entity x (or f or p) is such that'. The question *what there is* from the point of view of a given language—the question of the *ontology* of the language—is the question of the range of values of its variables.
>
> Usually the question will turn out to be in part an a priori question regarding the nature and intended interpretation of the language itself, and in part an empirical question about the world. The general question whether for example individuals, or classes, or properties, etc., are admitted among the values of the variables of a given language, will be an a priori question regarding the nature and intended interpretation of the language itself. On the other hand, supposing individuals admitted among the values, the further question whether the values comprise any unicorns will be empirical. It is the former type of inquiry—ontology in a philosophical rather than empirical sense—that interests me here. Let us turn our attention to the ontology, in this sense, of your object language.
>
> An apparent complication confronts us in the so-called duality of M' as between intensional and extensional values of variables; for it would appear then that we must inquire into two alternative ontologies of the object language. This, however, I consider to be illusory; since the duality in question is a peculiarity only of a special metalinguistic idiom and not of the object language itself, there is nothing to prevent our examining the object language from the old point of view and asking what the values of its variables are in the old-fashioned non-dual sense of the term.
>
> It is now readily seen that those values are merely intensions, rather than extensions or both. For, we have:
>
> $(x)(x \equiv x)$,
>
> i.e., every entity is L-equivalent to itself. This is the same as saying that entities between which L-equivalence fails are distinct entities—a clear indication that the *values* (in the ordinary non-dual sense of the term) of the variables are properties rather than classes, propositions rather than truth-values, individual concepts rather than individuals. (I neglect the further possibility of distinctness among L-equivalent entities themselves, which would compel the entities to be somehow "ultra-intensional"; for it is evident that you have no cause in the present connection to go so far.)
>
> I agree that such adherence to an intensional ontology, with extrusion of extensional entities altogether from the range of values of the variables, is indeed an effective way of reconciling quantification and modality. The cases of conflict between quantification and modality depend on extensions as values of variables. In your object language we may unhesitatingly quantify modalities because extensions have been dropped from among the values of the variables; even the individuals of the concrete world have disappeared, leaving only their concepts behind them.
>
> I find this intensional language interesting, for it illustrates what it would be like to be able to give the modalities free rein. But this repudiation of the concrete and extensional is a more radical move, in general, than a mere comparison of 43-3 with 43-2 might suggest. The strangeness of the intensional language becomes more evident when we try to reformulate statements such as these:
>
> (1) The number of planets is a power of three,
> (2) The wives of two of the directors are deaf.
>
> In the familiar logic, (1) and (2) would be analyzed in part as follows:
>
> (3) $(\exists n)$ (n is a natural number . the number of planets $= 3^n$),
> (4) $(\exists x)(\exists y)(\exists z)(\exists w)[x$ is a director . y is a director . $\sim(x = y)$. z is wife of x . w is wife of y . z is deaf . w is deaf].
>
> But the formulation (3) depends on there being numbers (extensions, presumably classes of classes) as values of the bound variable; and the formulation (4) depends on there being persons (extensions, individuals) as values of the four bound variables. Failing such values, (3) and (4) would have to be reformulated in terms of number concepts and individual concepts. The logical predicate '=' of identity in (3) and (4) would thereupon have to give way to a logical predicate of extensional equivalence of concepts. The logical predicate 'is a natural number' in (3) would have to give way to a logical predicate having the sense 'is a natural-number-concept'. The empirical predicates 'is a director', 'is wife of', and 'is deaf', in (4), would have to give way to some new predicates whose senses are more readily imagined than put into words. These examples do not prove your language-structure inadequate, but they give some hint of the unusual character which a development of it adequate to general purposes would have to assume.

[9] The first two-thirds of Quine's statement as here quoted is dated October 23, 1945; the remainder January 1, 1946.

The first important point to be noticed in Quine's statement is that he agrees that the form of modal language explained in the present chapter "is indeed an effective way of reconciling quantification and modality". Some readers of Quine's article believed that it proved the impossibility of a logical system combining modalities with variables. Quine's statement now shows that this is not the case.

However, there are still some serious problems involved. Quine, while admitting the possibility of modal systems with quantification, believes that these systems have certain peculiar features which he regards as disadvantages. Let us now examine these problems.

I have previously explained (at the beginning of § 10) that I agree with Quine's view that an author who uses variables of some kind thereby indicates that he recognizes those entities which are values of the variables. (I have simultaneously expressed some doubts concerning the advisability of applying the term 'ontology' to this recognition; but for our present discussion we may leave aside this question.) It is the counterpart of this thesis that is of importance for our problem; it says that, if someone uses a language which does not contain any variables with certain entities as values, he thereby indicates that he does not recognize these entities or at least that he does not intend to speak about them as long as he restricts himself to the use of this language. In a certain sense, I can agree also with this thesis. As an example, let us compare the following two languages S_P and S_P'. Let S_P be the ordinary language of physics (§ 19). It contains variables which have real numbers, both rational and irrational, as values. Suppose somebody proposes another language S_P' for physics which contains variables for rational numbers, but no variables to whose values irrational numbers belong. Here I would be willing to say, like Quine, that the user of this language S_P' excludes or "repudiates" the irrational numbers and that these numbers "have disappeared" from the universe of discourse. Now Quine says that the variables in the modal language have as values only intensions, not extensions, and that therefore, as far as this language is concerned, all extensions, for example, classes and "the individuals of the concrete world", "have disappeared". With this I cannot agree. At the first glance, the situation here may seem to be similar to that in the example of the irrational numbers; but actually it is fundamentally different.

In order to clarify the situation, we shall contrast in the following discussion our two language systems, the extensional language S_1 and the modal language S_2. We shall further consider the following two extended languages. The language S_1' is extensional like S_1 but contains additional kinds of variables, say 'f', 'g', etc., for which predicators of level one (and degree one) are substitutable, 'm', 'n', etc., for predicators of level two, and 'p', 'q', etc., for sentences. The language S_2' is constructed from S_1' by the addition of 'N'; hence it is a modal language like S_2. According to Quine's view, the values of 'f' in S_2' are not classes but properties, because '$(f)(f \equiv f)$' holds. In the extensional system S_1', on the other hand, we have only '$(f)(f \equiv f)$'. Therefore, Quine will presumably regard classes as the values of 'f' in this system, as he does for the variables of his extensional system ML (see above, § 25). Similarly, Quine says that the values of individual variables (e.g., 'x') in modal systems like S_2 and S_2' are individual concepts; on the other hand, he presumably regards individuals (concrete things or positions) as the values of individual variables in extensional systems like S_1 and S_1'. Now the decisive point is the following: As explained previously (§ 35), there is no objection against regarding designators in a modal language as names of intensions and regarding variables as having intensions as values, provided we are not misled by this formulation into the erroneous conception that the extensions have disappeared from the universe of discourse of the language. As explained earlier (§ 27), it is not possible for a predicator in an interpreted language to possess only an extension and not an intension or, in customary terms, to refer only to a class and not to a property. Similarly, it is impossible for a variable to be merely a class variable and not also a property variable. On the other hand, it is, of course, possible for a variable to have as values only properties and no relations, or only rational numbers and no irrational numbers. This shows the difference between the two cases. For example, the so-called class variables in the system PM′ (e.g., 'a') are, as we have seen (§ 27), also property variables, that is to say, they have properties as value intensions. The same holds now for variables like 'f' in S_1'. Languages of Quine's form ML′ or of Russell's form PM′ or of our form S_1' speak also about properties. The restriction of these extensional languages in comparison with modal languages like S_2' consists merely in the fact that whatever is said in any of these languages about a property is either true for all equivalent properties or false for all equivalent properties; in technical terms, all properties of properties expressible in these languages (by a matrix with a free variable of the kind mentioned) are extensional. This makes it possible to paraphrase all sentences of these languages in terms of classes. An analogous result holds for individual variables. These variables in an extensional language like S_1 and S_1' refer not only to individuals but also, and even primarily, to individual concepts. The restriction is again merely this: Whatever is said in these languages about individual concepts is either true for all equivalent individual concepts or false for all of them; in technical terms, it is extensional. Therefore, whatever is said in these languages about individual concepts can be paraphrased in terms of individuals.

Although the sentences of an extensional language (S_1 or S_1') can thus be interpreted as speaking about individuals and classes, they can be translated into the corresponding modal language (S_2 or S_2', respectively). This translation fulfils not only the requirement of L-equivalence but also the requirement of intensional isomorphism, the strictest requirement that any translation can fulfil (§ 14). Any given sentence in S_1' is translated into S_2' by that sentence itself, that is, by the same sequence of signs now taken as signs in S_2'. Any two corresponding designators, that is, any designator in S_1' and the same expression in S_2', are L-equivalent to one another. This follows from the following two results:

(i) The rules of designation for the descriptive signs are the same in both systems S_1' and S_2' (for example, the rules 1-2 for primitive predicators).

(ii) Any sentence in S_1' has the same range in both systems S_1' and S_2' (see 41-4 concerning S_1 and S_2). Since the range is the same, the truth-conditions are the same; therefore, the sentence means exactly the same in S_2' as in S_1'.

Thus the decisive difference between the situation here and that in the earlier example concerning the irrational numbers becomes clear. In the transition from S_P to S_P' the irrational numbers actually disappear, because a sentence in S_P of the form 'there is an irrational number such that . . .' is not translatable into S_P'. On the other hand, in the transition from an extensional to a modal language the individuals and classes do by no means disappear. A sentence in S_1 (or S_1') which says that there is an individual of a certain kind is translatable into S_2 (or S_2'); and a sentence in S_1' which says that there is a class of a certain kind is translatable into S_2'.

In order to illustrate this result by an example, let us take Quine's sentence (2). Since this sentence requires only individual variables, it can be translated into S_1. Let us assume that S_1 contains the following predicators, either as primitive signs or as defined in a suitable way: 'W' for the relation Wife, 'D' for the property Director, and 'F' for the property Deaf. Then (2) is translated into S_1 by the following sentence:

(5) '$(\exists x)(\exists y)(\exists z)(\exists w)[Dx \bullet Dy \bullet \sim(x \equiv y) \bullet Wzx \bullet Wwy \bullet Fz \bullet Fw]$'.

Now this same sentence is also the translation of (2) into S_2. It would be an error to think that it was necessary for the translation into S_2 either to use new predicators or to assign a new meaning to the old predicators, as though, for example, 'Dx' in S_1 said that the individual x has the property Director while 'Dx' in S_2 said that the individual concept x has a strange new property somehow analogous but not quite the same as the property Director. The matrix 'Dx' expresses in both languages the property Director; it may be defined in both languages in exactly the same way. Suppose a speaker X_1 uses the language S_1 and X_2 uses S_2. Then the question of whether a given full sentence, say 'Db', is true, may be decided by both speakers in the same way. Both confirm or disconfirm this sentence on the basis of observations of the person b, using the same empirical criteria for the property Director. Nothing in the semantical analysis of this sentence or in the procedure of empirical confirmation or in the expectation of possible future experiences implied by the sentence needs to be different for the two speakers. The same holds for the existential sentence (5) and for any other sentence occurring in both languages. Therefore, I cannot agree with the view that, while the speaker X_1 recognizes the individuals of the concrete world, they have disappeared for X_2, leaving only their concepts behind them.

The situation with respect to Quine's other example (1) is analogous, except that cardinal numbers are involved and therefore a variable of second level, say 'n', is used. We have seen earlier (§ 27) that, for the introduction of particular cardinal numbers and of the general concept of cardinal number, it is not necessary to use special class expressions and class variables, as Frege and Russell did; we may, instead, regard cardinal numbers as properties of second level or, rather, introduce cardinal number expressions as predicators of second level, whose intensions are properties of second level and whose extensions are classes of second level. Equality of cardinal numbers is then expressed with the help of '$\equiv$'. Thus we translated the sentence

(6) 'the number of planets = 9'

into the following sentence of S_1':

(7) '$Nc'P \equiv 9$'.

Similarly, Quine's sentence (1) can be translated into S_1' as follows, if we assume that exponentiation has been defined by a suitable procedure (analogous to that of Cantor or Russell, [P.M.], Vol. II, *116):

(8) '$(\exists n)[NC(n) \bullet Nc'P \equiv 3^n]$'.

(If we wish to say that n is finite, we may use the concept of inductive cardinal number with a definition analogous to Russell's). Here, again, the given sentence (1) can likewise be translated into the modal language S_2', namely, by the same sentence (8), hence without the use of any strange new concepts. The translation is by no means dependent upon the occurrence of class variables as distinct from property variables. '$NC(n)$' means in S_2', just as in S_1', that n is a cardinal number; thus in S_2', just as in S_1', sentences like '$NC(2)$' and '$NC(Nc'P)$' are L-true. That the sentence (8) has in S_2' the same factual content as in S_1' is seen by considerations similar to those concerning the previous example (5). The same astronomical observations confirm the sentence in the one as in the other language; it gives rise to the same expectations of future observations in both languages. Thus there cannot be any difference in meaning.

The preceding discussion shows that a modal language is not inadequate in comparison with the corresponding extensional language, that is to say that we can express in the former whatever is expressed in the latter. (So much Quine seems to admit.) We have seen, moreover, that the expressions used in a modal language for translations from the extensional language do not have any unusual character with respect to either their form or their meaning. Every designator and every sentence in the extensional language has exactly the same meaning in the modal language—more exactly speaking, it has both the same intension and the same extension. The world of concrete things and the conceptual world of numbers are dealt with in the modal language just as well as in the extensional one. In order to see correctly the functions of these languages, and generally of any languages, it is essential to abandon the old prejudice that a predicator must stand either for a class or for a property but cannot stand for both and that an individual expression must stand either for an individual or for an individual concept but cannot stand for both. To understand how language works, we must realize that every designator has both an intension and an extension.

§ 45. Conclusions

The main conclusions of the discussions in this book are briefly summarized. The difference between the two operations—understanding the meaning of a given expression and investigating whether and how it applies to the actual state of the world—suggests a distinction between two different semantical factors, which our method tries to explicate by the concepts of the intension and the extension of an expression.

The chief purpose of this book is to develop a method for the analysis of meaning in language, hence a semantical method. We may distinguish two operations with respect to a given linguistic expression, in particular, a (declarative) sentence and its parts. The first operation is the analysis of the expression with the aim of understanding it, of grasping its meaning. This operation is a logical or semantical one; in its technical form it is based on the semantical rules concerning the given expression. The second operation consists in investigations concerning the factual situation referred to by the given expression. Its aim is the establishment

of factual truth. This operation is not of a purely logical, but of an empirical, nature. We can distinguish two sides or factors in the given expression with regard to these two operations. The first factor is that side of the expression which we can establish by the first operation alone, that is, by understanding without using factual knowledge. This is what is usually called the meaning of the expression. In our method it is explicated by the technical concept of intension. The second factor is established by both operations together. Knowing the meaning, we discover by an investigation of facts to which locations, if any, the expression applies in the actual state of the world. This factor is explicated in our method by the technical concept of extension. Thus, for every expression which we can understand, there is the question of meaning and the question of actual application; therefore, the expression has primarily an intension and secondarily an extension.

The method of intension and extension stands in contrast to the customary method of the name-relation. The basic weakness of the latter method is its failure to realize the fundamental distinction between meaning and application. This leads to the conception that an expression must be the name of exactly one of the two semantical factors involved. For example, properties and classes are regarded as entities of equal standing; this leads to the view that a language ought to contain both names of properties and names of classes. This conception is the ultimate source of the various difficulties which we found involved in the method of the name-relation. They center around the well-known difficulty which we have called the antinomy of the name-relation. We have seen how the various methods of keeping the name-relation but avoiding the antinomy lead either to great complications in the language structure or to serious restrictions in the use of the language or in the application of the semantical method.

The formulations in terms of 'extension' and 'intension', 'class' and 'property', etc., seem to refer to two kinds of entities in each type. We have seen, however, that, in fact, no such duplication of entities is presupposed by our method and that those formulations involve only a convenient duplication of modes of speech. As it was shown to be unnecessary to use different expressions for classes and properties in a symbolic object language, it likewise turned out to be unnecessary to use those pairs of terms in the word language as a metalanguage. A new metalanguage was constructed, in which instead of the pair of phrases 'the class Human' and 'the property Human' only the neutral term 'Human' is used. It was shown that the ordinary formulations can be translated into this neutral metalanguage and that the latter language preserves all previous distinctions, though in different formulations.

Our semantical method also helps in the clarification of the problems of the modalities. It suggests a certain interpretation of the logical modalities which supplies a suitable basis for a system of modal logic. In particular, the distinction between intensions and extensions enables us to overcome the difficulties involved in combining modalities with quantified variables.

The different conceptions of other authors discussed in this book, for instance, those of Frege, Russell, Church, and Quine, concerning semantical problems, that is, problems of meaning, extension, naming, denotation, and the like, have sometimes been regarded as different theories so that one of them at most could be right while all others must be false. I regard these conceptions and my own rather as different methods, methods of semantical analysis characterized chiefly by the concepts used. Of course, once a method has been chosen, the question of whether or not certain results are valid on its basis is a theoretical one. But there is hardly any question of this kind on which I disagree with one of the other authors. Our differences are mainly practical differences concerning the choice of a method for semantical analysis. Methods, unlike logical statements, are never final. For any method of semantical analysis which someone proposes, somebody else will find improvements, that is, changes which will seem preferable to him and many others. This will certainly hold for the method which I have proposed here, no less than for the others.

Let me conclude our discussions by borrowing the words with which Russell concludes his paper.[10] It seems to me that his remarks, although written more than forty years ago, still apply to the present situation (except, perhaps, that instead of 'the true theory' I might prefer to say 'the best method'):

"Of the many other consequences of the view I have been advocating, I will say nothing. I will only beg the reader not to make up his mind against the view—as he might be tempted to do, on account of its apparently excessive complication—until he has attempted to construct a theory of his own on the subject of denotation. This attempt, I believe, will convince him that, whatever the true theory may be, it cannot have such a simplicity as one might have expected beforehand."

[10] [Denoting], p. 493.

SUPPLEMENT

This Supplement consists of five previously published articles. How they are related to the main body of the book is indicated in my Preface to the Second Edition. For the original places of their publication, see the starred items in the Bibliography.

A. EMPIRICISM, SEMANTICS, AND ONTOLOGY*

1. The Problem of Abstract Entities

Empiricists are in general rather suspicious with respect to any kind of abstract entities like properties, classes, relations, numbers, propositions, etc. They usually feel much more in sympathy with nominalists than with realists (in the medieval sense). As far as possible they try to avoid any reference to abstract entities and to restrict themselves to what is sometimes called a nominalistic language, i.e., one not containing such references. However, within certain scientific contexts it seems hardly possible to avoid them. In the case of mathematics, some empiricists try to find a way out by treating the whole of mathematics as a mere calculus, a formal system for which no interpretation is given or can be given. Accordingly, the mathematician is said to speak not about numbers, functions, and infinite classes, but merely about meaningless symbols and formulas manipulated according to given formal rules. In physics it is more difficult to shun the suspected entities, because the language of physics serves for the communication of reports and predictions and hence cannot be taken as a mere calculus. A physicist who is suspicious of abstract entities may perhaps try to declare a certain part of the language of physics as uninterpreted and uninterpretable, that part which refers to real numbers as space-time coordinates or as values of physical magnitudes, to functions, limits, etc. More probably he will just speak about all these things like anybody else but with an uneasy conscience, like a man who in his everyday life does with qualms many things which are not in accord with the high moral principles he professes on Sundays. Recently the problem of abstract entities has arisen again in connection with semantics, the theory

* I have made here some minor changes in the formulations to the effect that the term "framework" is now used only for the system of linguistic expressions, and not for the system of the entities in question.