MODAL LOGIC

AN INTRODUCTION

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PREFACE

This book is an introductory text in modal logic, the logic of necessity and possibility. It is intended for readers with the equivalent of a first course in formal logic, and it is designed to be used as a basic text in courses at the advanced undergraduate or beginning graduate level. The material in the book can easily be covered in a full-year course; with selectivity most of the material can be covered in a single term.

There are three parts to the book. Part I consists of two chapters, meant to introduce the reader to the subject of modal logic and to furnish a sufficient background for the parts that follow. Chapter 1 is a relatively informal examination of *S5*, one of the best-known systems of modal logic. Chapter 2 – 'Logical preliminaries' – contains almost everything needed for an understanding of the rest of the book. Some readers may prefer to go quickly through this chapter and then reread as necessary sections required in the context of succeeding chapters.

Part II comprises four chapters on standard models and normal systems of modal logic. The models, sometimes called 'Kripke models', are explained in chapter 3. In chapter 4 normal systems are presented from an axiomatic standpoint. Chapter 5 contains theorems on completeness and decidability, which bring together the model-theoretic and deductive-theoretic treatments of the preceding chapters. As an illustration of normal systems chapter 6 offers a discussion of deontic logic, the logic of obligation.

Part III is patterned like its predecessor, but here the topics are minimal models and classical modal logics. Thus chapter 7 is about the models (also known as 'neighborhood' or 'Scott-Montague' models), chapter 8 is an axiomatic account of the logics, and chapter 9 deals with completeness and decidability. Chapter 10 presents conditionality and (again) obligation by way of example.

An important feature of the book is the exercises that follow the sections of the chapters. These have been constructed both to consolidate understanding of the preceding material and to anticipate subsequent developments. They are an integral part of the text, and I have high hopes that the reader will attempt them as they appear.

I have appended to the text a short bibliography citing most of the works I found useful in writing this book. Many of these books and articles will take the reader farther afield to topics and results not treated here, and several contain good bibliographies.

I have a number of debts to record. First among these is to Lee Bowie, who several years ago suggested that we author a textbook in modal logic – I to write the chapters on propositional modal logic, he to write on quantification, identity, naming, and description. When it later became apparent that the material on propositional modal logic was bulky enough to warrant separate publication, Bowie graciously encouraged me to proceed alone.

In this connection I also want to express my gratitude to Richard Jeffrey and David Lewis, for their advice and support, and for recommending my project to Cambridge University Press and its distinguished editor Jeremy Mynott.

My debts to several of the works cited in the bibliography will perhaps be obvious to those already acquainted with the subject of modal logic. In particular I should mention Lemmon and Scott's *Introduction to modal logic* and Segerberg's *Essay in classical modal logic*.

The contents of chapters 6 and 10 are largely adapted from my papers 'Imperatives', 'Conditional obligation', and 'Basic conditional logic', cited in the bibliography. I wish to thank Krister Segerberg, editor of *Theoria*, Sören Stenlund, editor of *Logical theory and semantic analysis*, Richmond Thomason, editor of the *Journal of philosophical logic*, and the D. Reidel Publishing Company for permission to use this material.

Steven Kuhn and Audrey McKinney read much of my manuscript at different stages of its development, and I am grateful to them for criticism and advice.

Krister Segerberg has been a mainstay of counsel and encouragement for many years. I have learned a great deal about modal logic from Segerberg, and I have benefited enormously from conversations with him in the course of writing this book.

Among many others who have contributed in various ways to this book I would like to thank Roy Benton, Paul Golden, Deborah Mayo, and Robert Pelcovits.

Finally, I owe an enduring debt to Dana Scott, who introduced me to modal logic, who taught me how to think about it, and whose conception of the subject fundamentally influenced my own.

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PART I

1

INTRODUCTION

In this chapter we introduce the subject of modal logic by surveying some of the main features of the system of modal logic known as *S5*. This system is but one of many we shall study. Because it is one of the simplest, we choose it to begin with.

The system *S5* is determined semantically by an account of necessity and possibility that dates to the philosopher Leibniz: a proposition is *necessary* if it holds at all possible worlds, *possible* if it holds at some. The idea is that different things may be true at different possible worlds, but whatever holds true at every possible world is necessary, while that which holds at at least one possible world is possible.

In section 1.1 we develop this semantic idea by means of a definition of truth at a possible world in a model for a language of necessity and possibility. This leads to a definition of validity, and we set out some valid sentences and principles governing validity, as well as some examples of invalidity.

The totality of valid sentences forms the modal logic *S5*. In terms of the principles set out in section 1.1 it is possible to deduce all the valid sentences. Some evidence of this appears in section 1.2, where we take the principles in section 1.1 as axioms and rules of inference, formulate *S5* as a deductive system, and derive a number of further principles.

Sections 1.1 and 1.2 exemplify in miniature our approach to the study of modal logic throughout this book: first, semantically in terms of the notion of truth; second, syntactically by means of deductive systems.

The exposition in this chapter is quite casual, and intentionally so. The purpose, in part, is to acquaint the reader with many of the notions and notations used in the rest of the book; but formality is deferred to subsequent chapters. This leads to occasional wordiness, but not, it is hoped, to loss of intelligibility.

We study modal logic in the context of a language of necessity and possibility. The sentences of the language are of the following forms.

$$\mathbb{P}_0,\ \mathbb{P}_1,\ \mathbb{P}_2,\ \ldots$$

$$\top,\ \bot,\ \neg A,\ A \wedge B,\ A \vee B,\ A \rightarrow B,\ A \leftrightarrow B,\ \Box A,\ \Diamond A$$

Sentences of the form $\mathbb{P}_n$ (for $n = 0, 1, 2, \ldots$) are atomic. $\top$ is a constant for truth; $\bot$ is a constant for falsity. $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are signs of negation, conjunction, disjunction, conditionality, and biconditionality, respectively. $\Box$ is the necessity sign; $\Diamond$ is the possibility sign.

A more detailed account of the syntax of this language appears in section 2.1, but it is not essential for an understanding of the rest of this chapter.

1.1. Truth and possible worlds

According to the leibnizian idea, necessity is what is true at every possible world and possibility is what is true at some. Linguistically: a sentence of the form $\Box A$ – *necessarily* A – is true if and only if A itself is true at every possible world; and a sentence of the form $\Diamond A$ – *possibly* A – is true just in case A is true at some possible world.

The picture is of a collection of possible worlds – including our own, the real world – at which sentences of the language are variously true and false. Our purpose is to model this, and we do so by means of an infinite sequence of sets of possible worlds,

$$P_0, P_1, P_2, \ldots .$$

The intuition behind this modeling is that, for each natural number n, the set P_n collects just those possible worlds at which the corresponding atomic sentence $\mathbb{P}_n$ is true. In other words, the sequence $P_0, P_1, P_2, \ldots$ interprets the atomic sentences by stipulating at which possible worlds they are true (and, by omission, at which they are false): $\mathbb{P}_n$ is true at a possible world α if and only if α is in the set P_n.

More precisely, a model is a pair

$$\langle W, P \rangle$$

in which W is a set of possible worlds and P abbreviates an infinite sequence $P_0, P_1, P_2, \ldots$ of subsets of W. Note that W may contain possible worlds not in any of the sets P_n; indeed, any or all of these sets may be empty. Moreover, we do not require that the actual world appear in every model.

In terms of a possible world in a model we state the truth conditions for sentences according to their forms. Where A is a sentence and α is a possible world in a model $\mathcal{M} = \langle W, P \rangle$, we use the symbolism

$$\models^{\mathcal{M}}_{\alpha} A$$

as short for

A is true at α in $\mathcal{M}$.

The truth conditions are stated thus:

(1) $\models^{\mathcal{M}}_{\alpha} \mathbb{P}_n$ iff $\alpha \in P_n$, for $n = 0, 1, 2, \ldots$.
(2) $\models^{\mathcal{M}}_{\alpha} \top$.
(3) Not $\models^{\mathcal{M}}_{\alpha} \bot$.
(4) $\models^{\mathcal{M}}_{\alpha} \neg A$ iff not $\models^{\mathcal{M}}_{\alpha} A$.
(5) $\models^{\mathcal{M}}_{\alpha} A \wedge B$ iff both $\models^{\mathcal{M}}_{\alpha} A$ and $\models^{\mathcal{M}}_{\alpha} B$.
(6) $\models^{\mathcal{M}}_{\alpha} A \vee B$ iff either $\models^{\mathcal{M}}_{\alpha} A$ or $\models^{\mathcal{M}}_{\alpha} B$, or both.
(7) $\models^{\mathcal{M}}_{\alpha} A \rightarrow B$ iff if $\models^{\mathcal{M}}_{\alpha} A$ then $\models^{\mathcal{M}}_{\alpha} B$.
(8) $\models^{\mathcal{M}}_{\alpha} A \leftrightarrow B$ iff $\models^{\mathcal{M}}_{\alpha} A$ if and only if $\models^{\mathcal{M}}_{\alpha} B$.
(9) $\models^{\mathcal{M}}_{\alpha} \Box A$ iff for every β in $\mathcal{M}$, $\models^{\mathcal{M}}_{\beta} A$.
(10) $\models^{\mathcal{M}}_{\alpha} \Diamond A$ iff for some β in $\mathcal{M}$, $\models^{\mathcal{M}}_{\beta} A$.

Some discussion of this definition may be helpful.

Clause (1) reflects our remarks about the sets $P_0, P_1, P_2, \ldots$ in a model: an atomic sentence $\mathbb{P}_n$ is true at a possible world α just in case α is a member of the set P_n. According to clause (2), the truth constant $\top$ is always true at α. By (3), the falsity constant $\bot$ is always false at α. Clause (4) states that a negation $\neg A$ is true at α if and only if its negate A is false at α. The content of (5) is that a conjunction $A \wedge B$ is true at α just in case both its conjuncts, A and B, are. According to (6), a disjunction $A \vee B$ is true at α just when at least one of its disjuncts, A and B, is. Our intention in clause (7) is that a conditional $A \rightarrow B$ is to be understood as true at α just so long as it fails to be the case that its antecedent, A, is true at α while its consequent, B, is false. And, similarly, in (8) we intend that a biconditional $A \leftrightarrow B$ be accounted true at α just in case its members, A and B, are either both true at α or both false. Clause (9) formulates the leibnizian interpretation of necessity: a necessitation $\Box A$ is true at α if and only if its necessitate, A, is true at every possible world β in the model. Finally, according to (10), $\Diamond A$ is true at α just in case there is at least one possible world β in the model at which A is true.

A sentence true at every possible world in every model is said to be valid. We use the symbol $\models$ again – this time without subscripts or superscripts – and write

$$\models A$$

to mean that the sentence A is valid. More formally, then, the definition of validity may be expressed:

$$\models A \text{ iff for every model } \mathscr{M} \text{ and every possible world } \alpha \text{ in } \mathscr{M},\ \models^{\mathscr{M}}_{\alpha} A.$$

In asking after the logic of necessity and possibility we seek to know which sentences are valid – true, no matter how interpreted, at every possible world – and which are not. For example, as we shall see, every sentence of the form $\Box A \to A$ is valid, whereas not every sentence of the form $A \to \Box A$ is. In what follows we first set out some valid sentences and principles governing validity, enough to form an axiomatic basis for the derivation of all valid sentences. Then we mention some prominent cases of invalidity.

Let us begin our survey of validity with the principle just mentioned:

$$\text{T.}\quad \Box A \to A$$

According to T, whatever is necessary is so: *if necessarily* A, *then* A. To see that this schema – i.e. every sentence of this form – is valid, it is sufficient to prove that where α is any possible world in any model $\mathscr{M}$, $\models^{\mathscr{M}}_{\alpha} \Box A \to A$. And for this it will be enough to show that if $\models^{\mathscr{M}}_{\alpha} \Box A$ then $\models^{\mathscr{M}}_{\alpha} A$ (compare clause (7) in the definition of truth). So suppose that $\models^{\mathscr{M}}_{\alpha} \Box A$. By clause (9) of the truth definition, this means that $\models^{\mathscr{M}}_{\beta} A$ for every possible world β in $\mathscr{M}$. In particular, then, this holds for α, i.e. $\models^{\mathscr{M}}_{\alpha} A$.

Next let us consider the schema

$$5.\quad \Diamond A \to \Box \Diamond A.$$

The import of 5 is that what is possible is necessarily possible: *if possibly* A, *then necessarily possibly* A. To see that 5 is valid, suppose that $\models^{\mathscr{M}}_{\alpha} \Diamond A$, for possible world α in model $\mathscr{M}$. By clause (10) of the definition of truth this means that $\mathscr{M}$ has a possible world β such that $\models^{\mathscr{M}}_{\beta} A$. It follows from this (again by (10)) that no matter what possible world in the model we choose, $\Diamond A$ holds – i.e. $\models^{\mathscr{M}}_{\beta} \Diamond A$ for every possible world β in $\mathscr{M}$. But by clause (9) this means that $\models^{\mathscr{M}}_{\alpha} \Box \Diamond A$, which is what we wished to show.

The schemas T and 5 are rather special in that they do not hold in every system of modal logic we shall study. The next two principles are more widely accepted however; it is not until chapter 7 that they are called into question.

The first of these expresses a principle of distributivity of necessity with respect to the conditional:

$$\text{K.}\quad \Box(A \to B) \to (\Box A \to \Box B)$$

This means that if a conditional and its antecedent are both necessary, then so is the consequent. For the validity of K, suppose that α is a possible world in a model $\mathscr{M}$ such that both $\models^{\mathscr{M}}_{\alpha} \Box(A \to B)$ and $\models^{\mathscr{M}}_{\alpha} \Box A$. Then for every possible world β in $\mathscr{M}$, both $\models^{\mathscr{M}}_{\beta} A \to B$ and $\models^{\mathscr{M}}_{\beta} A$, from which it follows that for every possible world β in $\mathscr{M}$, $\models^{\mathscr{M}}_{\beta} B$. Thus $\models^{\mathscr{M}}_{\alpha} \Box B$.

The second principle corresponds to a rule of inference in the next section (RN, the rule of necessitation). It states that the necessitation of a valid sentence is itself always valid. In symbols:

If $\models A$, *then* $\models \Box A$.

For suppose that $\models A$, i.e. that $\models^{\mathscr{M}}_{\alpha} A$ for every possible world α in every model $\mathscr{M}$. Then for every possible world α in every model $\mathscr{M}$, $\models^{\mathscr{M}}_{\alpha} \Box A$, which is to say that $\models \Box A$.

The last specifically modal validity we wish to mention holds in every modal logic we shall discuss in this book.

$$\text{Df}\Diamond.\quad \Diamond A \leftrightarrow \neg\Box\neg A$$

This schema embodies the idea that what is possible is just what is not-necessarily-not. Its validity means that possibility is always expressible in terms of necessity and negation, and so is theoretically superfluous. In this sense $\Diamond$ is *definable* in terms of $\Box$ and $\neg$. Df$\Diamond$ is valid because to say that for some possible world β in a model $\mathscr{M}$, $\models^{\mathscr{M}}_{\beta} A$, is just to say that it is not the case that for every possible world β in $\mathscr{M}$ it is not the case that $\models^{\mathscr{M}}_{\beta} A$. Reference to clauses (4), (9), and (10) of the truth definition reveals that the former expression means that $\models^{\mathscr{M}}_{\alpha} \Diamond A$, while the latter expression means that $\models^{\mathscr{M}}_{\alpha} \neg\Box\neg A$. Hence the biconditional $\Diamond A \leftrightarrow \neg\Box\neg A$ holds at every possible world in every model.

Let us turn now to the relationship between our modal logic and ordinary propositional, or truth-functional, logic. The relationship is simple: the modal logic includes the propositional. In part, this means that every propositionally valid sentence is modally valid, i.e.:

If A *is a tautology, then* $\models$ A.

The explanation of this is as follows (there is a more careful account in chapter 2). By a *tautology* we mean a sentence true in every valuation of its propositionally atomic constituents. A sentence is *propositionally atomic* if it is either atomic in the ordinary sense ($\mathbb{P}_n$) or modal ($\Box$A or $\Diamond$A). A *valuation* is an assignment of truth values (truth and falsity) to the propositionally atomic sentences. The truth conditions in a valuation of the rest of the sentences in the language – those of the forms $\top$, $\bot$, $\neg$A, etc. – are determined just as they are by a possible world in a model. Thus $\top$ is true in every valuation, $\bot$ is false in every valuation, $\neg$A is true in a valuation if and only if A is false, and so on; compare clauses (2)–(8) in the definition of truth above.

In short, a valuation analyzes sentences semantically from the point of view of their truth-functional structure, counting as atomic the modal structure of sentences of the forms $\Box$A and $\Diamond$A, as well as those of the form $\mathbb{P}_n$. A sentence is a tautology, thus, if it comes out true no matter how truth values are assigned to its propositional atoms. For example, any sentence of the form $\Box$A$\to\Box$A is a tautology, since $\Box$A is propositionally atomic and such a conditional is true in a valuation whether $\Box$A is assigned truth or falsity.

Now observe that in any model $\mathscr{M}$ each possible world α is a valuation in the sense just explained, since α assigns truth or falsity to each sentence of the form $\mathbb{P}_n$, $\Box$A, and $\Diamond$A, i.e. to each propositionally atomic sentence. The world α assigns truth to a propositionally atomic sentence A when $\models^{\mathscr{M}}_{\alpha}$ A, and falsity otherwise.

To prove, finally, that every tautology is valid, assume that A is a tautology – that A is true in every valuation. Then A is true at every possible world in every model, i.e. $\models^{\mathscr{M}}_{\alpha}$ A for every possible world α in every model $\mathscr{M}$. This means that $\models$ A, i.e. that A is valid.

To say that all tautologies are valid does not exhaust what is meant by saying that modal logic includes propositional logic. It means moreover that validity is preserved by propositionally correct patterns of inference. For example, the inference from A$\to$B and A to B is propositionally correct; whenever both A$\to$B and A are true in a valuation, so is B. Corresponding to this we have the principle that whenever a conditional and its antecedent are both valid, so is the consequent:

If $\models$ A$\to$B *and* $\models$ A, *then* $\models$ B.

This emerges as the rule of inference modus ponens, MP, in the next section. To prove the principle, suppose that both $\models$ A$\to$B and $\models$ A. This means that for every possible world α in every model $\mathscr{M}$, $\models^{\mathscr{M}}_{\alpha}$ A$\to$B, and

that for every possible world α in every model $\mathscr{M}$, $\models^{\mathscr{M}}_{\alpha}$ A. It follows at once that for every possible world α in every model $\mathscr{M}$, $\models^{\mathscr{M}}_{\alpha}$ B, i.e. that $\models$ B.

In terms of this principle and the fact that every tautology is valid we can prove that validity is preserved by propositionally correct patterns of inference generally. For suppose that it is propositionally correct to infer a sentence A from sentences $A_1, \ldots, A_n$, i.e. that A is true in every valuation in which all of $A_1, \ldots, A_n$ are. Then the sentence

$$A_1 \to (\ldots (A_n \to A) \ldots)$$

is a tautology. Thus this sentence is valid. Hence if each of $A_1, \ldots, A_n$ is valid, then by applying the modus ponens principle n times we arrive at the result that A is valid.

For example, it is propositionally correct to infer A$\to$C from A$\to$B and B$\to$C (we leave it for the reader to check that A$\to$C is true in any valuation that verifies both A$\to$B and B$\to$C). So the sentence (A$\to$B)$\to$((B$\to$C)$\to$(A$\to$C)) is a tautology. Hence if $\models$ A$\to$B, then by the principle of modus ponens, $\models$ (B$\to$C)$\to$(A$\to$C). And so $\models$ A$\to$C, if also $\models$ B$\to$C.

This ends our short survey of valid sentences and principles governing validity. Let us turn now to some examples of invalidity.

To begin, the schema A$\to\Box$A – the converse of T – is not valid. To see this, let α and β be distinct possible worlds, let $W = \{\alpha, \beta\}$, and let $P_n = \{\alpha\}$ for every natural number n (i.e. $n = 0, 1, 2, \ldots$). Then $\mathscr{M} = \langle W, P \rangle$ is a model in which $\models^{\mathscr{M}}_{\alpha}$ $\mathbb{P}_0$ (since P_0 contains α) and not $\models^{\mathscr{M}}_{\alpha}$ $\Box\mathbb{P}_0$ (since there is a world in $\mathscr{M}$, viz. β, not in P_0). Thus not $\models^{\mathscr{M}}_{\alpha}$ $\mathbb{P}_0\to\Box\mathbb{P}_0$, which proves that the schema A$\to\Box$A is not valid. We say in this case that $\mathscr{M}$ is a countermodel to A$\to\Box$A.

Notice that if A$\to\Box$A were valid it would mean that whatever is the case is so necessarily. Indeed, if this schema were valid, then given the validity of T, the biconditional A$\leftrightarrow\Box$A would be valid, so that truth and necessity would be the same. The reader should contrast the invalidity of A$\to\Box$A with the correctness of the principle of necessitation, that if A is valid so is $\Box$A. This will help in understanding the difference between theorems and rules of inference in the next section.

Another example of invalidity is the schema $\Box$(A$\lor$B)$\to$($\Box$A$\lor\Box$B). The model $\mathscr{M} = \langle W, P \rangle$ in which $W = \{\alpha, \beta\}$, $P_0 = \{\alpha\}$, and $P_n = \{\beta\}$ for $n > 0$ is a countermodel to this schema. For $\models^{\mathscr{M}}_{\alpha}$ $\mathbb{P}_0$ and $\models^{\mathscr{M}}_{\beta}$ $\mathbb{P}_1$, which means that $\models^{\mathscr{M}}_{\alpha}$ $\mathbb{P}_0 \lor \mathbb{P}_1$ and $\models^{\mathscr{M}}_{\beta}$ $\mathbb{P}_0 \lor \mathbb{P}_1$. So $\models^{\mathscr{M}}_{\alpha}$ $\Box(\mathbb{P}_0 \lor \mathbb{P}_1)$, since the disjunction $\mathbb{P}_0 \lor \mathbb{P}_1$ is true at every possible world in $\mathscr{M}$. On the other hand, not $\models^{\mathscr{M}}_{\alpha}$ $\mathbb{P}_1$ and also not $\models^{\mathscr{M}}_{\beta}$ $\mathbb{P}_0$. So neither $\models^{\mathscr{M}}_{\alpha}$ $\Box\mathbb{P}_0$ nor $\models^{\mathscr{M}}_{\alpha}$ $\Box\mathbb{P}_1$, and hence not $\models_\alpha^{\mathcal{M}} \Box\mathbb{P}_0 \vee \Box\mathbb{P}_1$. Therefore, not $\models_\alpha^{\mathcal{M}} \Box(\mathbb{P}_0 \vee \mathbb{P}_1) \to (\Box\mathbb{P}_0 \vee \Box\mathbb{P}_1)$, i.e. $\mathcal{M}$ is a countermodel to the schema.

An even simpler way to see the invalidity of the schema $\Box(A \vee B) \to (\Box A \vee \Box B)$ is to consider the instance $\Box(\mathbb{P}_0 \vee \neg\mathbb{P}_0) \to (\Box\mathbb{P}_0 \vee \Box\neg\mathbb{P}_0)$. The disjunction $\mathbb{P}_0 \vee \neg\mathbb{P}_0$ is a tautology, so it is valid – and hence so is its necessitation, $\Box(\mathbb{P}_0 \vee \neg\mathbb{P}_0)$. Thus it is sufficient to show that the disjunction $\Box\mathbb{P}_0 \vee \Box\neg\mathbb{P}_0$ is not valid. The model described above in connection with $A \to \Box A$ does the job, as the reader may verify.

We have just shown that the necessity sign does not distribute into a disjunction; the validity of K, above, means that $\Box$ does distribute into a conditional. As a final example of invalidity, we describe a countermodel to $\Diamond(A \to B) \to (\Diamond A \to \Diamond B)$, thus showing that the possibility sign does not distribute into a conditional. The model is $\mathcal{M} = \langle W, P \rangle$, where $W = \{\alpha, \beta\}$, $P_0 = \{\alpha\}$, and $P_n = \varnothing$ (the empty set) for $n > 0$. We leave it for the reader to check that $\models_\alpha^{\mathcal{M}} \Diamond(\mathbb{P}_0 \to \mathbb{P}_1)$ and $\models_\alpha^{\mathcal{M}} \Diamond\mathbb{P}_0$, but not $\models_\alpha^{\mathcal{M}} \Diamond\mathbb{P}_1$. This being so, it follows that not $\models_\alpha^{\mathcal{M}} \Diamond(\mathbb{P}_0 \to \mathbb{P}_1) \to (\Diamond\mathbb{P}_0 \to \Diamond\mathbb{P}_1)$.

This concludes our semantical exposition of the modal logic *S5*.

EXERCISES

1.1. Prove that the following schemas are valid.

(*a*) $\Box A \to \Diamond A$
(*b*) $A \to \Box\Diamond A$
(*c*) $\Box A \to \Box\Box A$
(*d*) $\Diamond\top$
(*e*) $A \to \Diamond A$
(*f*) $\Diamond\Box A \to A$
(*g*) $\Diamond\Diamond A \to \Diamond A$
(*h*) $\Diamond\Box A \to \Box A$

1.2. Prove that the following schemas are valid.

(*a*) $\Box\top$
(*b*) $\Box(A \wedge B) \to (\Box A \wedge \Box B)$
(*c*) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$
(*d*) $\neg\Diamond\bot$
(*e*) $(\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$
(*f*) $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

1.3. Prove that the schema $\Box A \leftrightarrow \neg\Diamond\neg A$ is valid.

1.4. Prove each of the following.

(*a*) If $\models (A \wedge B) \to C$, then $\models (\Box A \wedge \Box B) \to \Box C$.
(*b*) If $\models A \to B$, then $\models \Box A \to \Box B$.
(*c*) If $\models A \leftrightarrow B$, then $\models \Box A \leftrightarrow \Box B$.
(*d*) If $\models A \to B$, then $\models \Diamond A \to \Diamond B$.
(*e*) If $\models A \leftrightarrow B$, then $\models \Diamond A \leftrightarrow \Diamond B$.

1.5. Prove that for any $n \geqslant 0$, if $\models (A_1 \wedge \ldots \wedge A_n) \to A$, then $\models (\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A$. (When $n = 0$ this just means if $\models A$ then $\models \Box A$.)

1.6. Referring to the model $\mathcal{M}$ defined in connection with showing the invalidity of $\Diamond(A \to B) \to (\Diamond A \to \Diamond B)$ (see the penultimate paragraph of section 1.1), verify that $\models_\alpha^{\mathcal{M}} \Diamond(\mathbb{P}_0 \to \mathbb{P}_1)$ and $\models_\alpha^{\mathcal{M}} \Diamond\mathbb{P}_0$, but not $\models_\alpha^{\mathcal{M}} \Diamond\mathbb{P}_1$.

1.7. Prove that the following schemas are invalid (i.e. that each has an invalid instance).

(*a*) $\Diamond A \to A$
(*b*) $\Diamond A \to \Box A$
(*c*) $\Box\Diamond A \to A$
(*d*) $(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$
(*e*) $(\Box A \to \Box B) \to \Box(A \to B)$

1.8. For each of the following, decide whether or not it is valid, and prove it.

(*a*) $\Box\Box A \to \Box A$
(*b*) $\Box\Diamond A \to \Diamond A$
(*c*) $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
(*d*) $\Diamond\Box A \to \Box\Diamond A$
(*e*) $\Box\Diamond A \to \Diamond\Box A$
(*f*) $(\neg\Diamond A \wedge \Diamond B) \to \Diamond(\neg A \wedge B)$
(*g*) $\Diamond A \to \Diamond\Diamond A$
(*h*) $\Box A \to \Diamond\Box A$
(*i*) $(\Box A \vee \Box B) \to \Box(A \vee B)$
(*j*) $\Box\bot$

1.9. Suppose that in every model there is just one possible world and prove that under this assumption the schema $A \to \Box A$ is valid.

hence not $\models^{\mathscr{M}}_{\alpha} \Box\mathbb{P}_0 \vee \Box\mathbb{P}_1$. Therefore, not $\models^{\mathscr{M}}_{\alpha} \Box(\mathbb{P}_0 \vee \mathbb{P}_1) \rightarrow (\Box\mathbb{P}_0 \vee \Box\mathbb{P}_1)$, i.e. $\mathscr{M}$ is a countermodel to the schema.

An even simpler way to see the invalidity of the schema $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$ is to consider the instance $\Box(\mathbb{P}_0 \vee \neg\mathbb{P}_0) \rightarrow (\Box\mathbb{P}_0 \vee \Box\neg\mathbb{P}_0)$. The disjunction $\mathbb{P}_0 \vee \neg\mathbb{P}_0$ is a tautology, so it is valid – and hence so is its necessitation, $\Box(\mathbb{P}_0 \vee \neg\mathbb{P}_0)$. Thus it is sufficient to show that the disjunction $\Box\mathbb{P}_0 \vee \Box\neg\mathbb{P}_0$ is not valid. The model described above in connection with $A \rightarrow \Box A$ does the job, as the reader may verify.

We have just shown that the necessity sign does not distribute into a disjunction; the validity of K, above, means that $\Box$ does distribute into a conditional. As a final example of invalidity, we describe a countermodel to $\Diamond(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$, thus showing that the possibility sign does not distribute into a conditional. The model is $\mathscr{M} = \langle W, P \rangle$, where $W = \{\alpha, \beta\}$, $P_0 = \{\alpha\}$, and $P_n = \varnothing$ (the empty set) for $n > 0$. We leave it for the reader to check that $\models^{\mathscr{M}}_{\alpha} \Diamond(\mathbb{P}_0 \rightarrow \mathbb{P}_1)$ and $\models^{\mathscr{M}}_{\alpha} \Diamond\mathbb{P}_0$, but not $\models^{\mathscr{M}}_{\alpha} \Diamond\mathbb{P}_1$. This being so, it follows that not $\models^{\mathscr{M}}_{\alpha} \Diamond(\mathbb{P}_0 \rightarrow \mathbb{P}_1) \rightarrow (\Diamond\mathbb{P}_0 \rightarrow \Diamond\mathbb{P}_1)$.

This concludes our semantical exposition of the modal logic *S5*.

EXERCISES

1.1. Prove that the following schemas are valid.

(*a*) $\Box A \rightarrow \Diamond A$
(*b*) $A \rightarrow \Box\Diamond A$
(*c*) $\Box A \rightarrow \Box\Box A$
(*d*) $\Diamond\top$
(*e*) $A \rightarrow \Diamond A$
(*f*) $\Diamond\Box A \rightarrow A$
(*g*) $\Diamond\Diamond A \rightarrow \Diamond A$
(*h*) $\Diamond\Box A \rightarrow \Box A$

1.2. Prove that the following schemas are valid.

(*a*) $\Box\top$
(*b*) $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$
(*c*) $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
(*d*) $\neg\Diamond\bot$
(*e*) $(\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$
(*f*) $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$

1.3. Prove that the schema $\Box A \leftrightarrow \neg\Diamond\neg A$ is valid.

1.4. Prove each of the following.

(*a*) If $\models (A \wedge B) \rightarrow C$, then $\models (\Box A \wedge \Box B) \rightarrow \Box C$.
(*b*) If $\models A \rightarrow B$, then $\models \Box A \rightarrow \Box B$.
(*c*) If $\models A \leftrightarrow B$, then $\models \Box A \leftrightarrow \Box B$.
(*d*) If $\models A \rightarrow B$, then $\models \Diamond A \rightarrow \Diamond B$.
(*e*) If $\models A \leftrightarrow B$, then $\models \Diamond A \leftrightarrow \Diamond B$.

1.5. Prove that for any $n \geqslant 0$, if $\models (A_1 \wedge \ldots \wedge A_n) \rightarrow A$, then $\models (\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box A$. (When $n = 0$ this just means if $\models A$ then $\models \Box A$.)

1.6. Referring to the model $\mathscr{M}$ defined in connection with showing the invalidity of $\Diamond(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$ (see the penultimate paragraph of section 1.1), verify that $\models^{\mathscr{M}}_{\alpha} \Diamond(\mathbb{P}_0 \rightarrow \mathbb{P}_1)$ and $\models^{\mathscr{M}}_{\alpha} \Diamond\mathbb{P}_0$, but not $\models^{\mathscr{M}}_{\alpha} \Diamond\mathbb{P}_1$.

1.7. Prove that the following schemas are invalid (i.e. that each has an invalid instance).

(*a*) $\Diamond A \rightarrow A$
(*b*) $\Diamond A \rightarrow \Box A$
(*c*) $\Box\Diamond A \rightarrow A$
(*d*) $(\Diamond A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$
(*e*) $(\Box A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

1.8. For each of the following, decide whether or not it is valid, and prove it.

(*a*) $\Box\Box A \rightarrow \Box A$
(*b*) $\Box\Diamond A \rightarrow \Diamond A$
(*c*) $\Diamond(A \wedge B) \rightarrow (\Diamond A \wedge \Diamond B)$
(*d*) $\Diamond\Box A \rightarrow \Box\Diamond A$
(*e*) $\Box\Diamond A \rightarrow \Diamond\Box A$
(*f*) $(\neg\Diamond A \wedge \Diamond B) \rightarrow \Diamond(\neg A \wedge B)$
(*g*) $\Diamond A \rightarrow \Diamond\Diamond A$
(*h*) $\Box A \rightarrow \Diamond\Box A$
(*i*) $(\Box A \vee \Box B) \rightarrow \Box(A \vee B)$
(*j*) $\Box\bot$

1.9. Suppose that in every model there is just one possible world and prove that under this assumption the schema $A \rightarrow \Box A$ is valid.

1.10. Consider a structure $\mathscr{M} = \langle W, R, P\rangle$ in which W and P are as they are in a model, and R is an equivalence relation on W. That is, W is a set of possible worlds, P is an infinite sequence $P_0, P_1, P_2, \ldots$ of subsets of W, and R is a binary relation on W that is reflexive (for every α in $\mathscr{M}$, $\alpha R\alpha$) and euclidean (for every α, β, and γ in $\mathscr{M}$, if $\alpha R\beta$ and $\alpha R\gamma$, then $\beta R\gamma$). Structures of this sort are models for *S5*, where the truth conditions for non-modal sentences are given as usual (i.e. (1)–(8) in the fourth paragraph of section 1.1) and those for sentences of the forms $\Box$A and $\Diamond$A are given by:

$$(9')\ \vDash_\alpha^{\mathscr{M}} \Box\mathrm{A} \text{ iff for every } \beta \text{ in } \mathscr{M} \text{ such that } \alpha R\beta,\ \vDash_\beta^{\mathscr{M}} \mathrm{A}.$$

$$(10')\ \vDash_\alpha^{\mathscr{M}} \Diamond\mathrm{A} \text{ iff for some } \beta \text{ in } \mathscr{M} \text{ such that } \alpha R\beta,\ \vDash_\beta^{\mathscr{M}} \mathrm{A}.$$

Intuitively, R is a relation that relates a world to those that are possible with respect to it; $\alpha R\beta$ means that the world β is possible with respect to the world α. Thus according to (9′) $\Box$A is true at α just in case A is true at all worlds possible with respect to α; and according to (10′) $\Diamond$A is true at α just in case A is true at some world possible with respect to α. Obviously, these models represent a generalization of the analysis of necessity and possibility in section 1.1: it is no longer assumed that every world is possible with respect to every other world.

As before, validity means truth at every possible world in every model.

Show that these models are adequate for an analysis of the system *S5* by proving that the schemas T, 5, K, and Df$\Diamond$ are all valid, that any tautology is valid, that $\Box$A is valid if A is, and that if A→B and A are valid so is B.

Hint: The validity of T depends upon the reflexivity of R, and the validity of 5 depends on the euclideanness of R. Nothing special is needed for the others.

1.11. (This exercise presupposes an acquaintance with elementary quantificational logic.) The reader may have noticed an analogy between the signs of necessity and possibility, $\Box$ and $\Diamond$, on the one hand, and the universal and existential quantifiers, $\forall$ and $\exists$, on the other. $\Box$A is true at a possible world just in case A holds at *every* world; $\Diamond$A is true at a world if and only if A holds at *some* world.

Let us specify a language of elementary quantificational logic by stipulating that its formulas are of the following forms:

$$\mathbb{P}_n(\alpha),\quad \top,\quad \bot,\quad \neg\mathrm{A},\quad \mathrm{A}\wedge\mathrm{B},\quad \mathrm{A}\vee\mathrm{B},\quad \mathrm{A}\to\mathrm{B},$$
$$\mathrm{A}\leftrightarrow\mathrm{B},\quad \forall\alpha\mathrm{A},\quad \exists\alpha\mathrm{A},$$

where $\mathbb{P}_n$ is one of denumerably many one-place predicates and α is a variable, so that $\mathbb{P}_n(\alpha)$ is an atomic formula.

We define the mapping τ, from the language of modal logic to the quantificational language, as follows.

(1) $\tau(\mathbb{P}_n) = \mathbb{P}_n(\alpha)$, for $n = 0, 1, 2, \ldots$.
(2) $\tau(\top) = \top$.
(3) $\tau(\bot) = \bot$.
(4) $\tau(\neg\mathrm{A}) = \neg\tau(\mathrm{A})$.
(5) $\tau(\mathrm{A}\wedge\mathrm{B}) = \tau(\mathrm{A})\wedge\tau(\mathrm{B})$.
(6) $\tau(\mathrm{A}\vee\mathrm{B}) = \tau(\mathrm{A})\vee\tau(\mathrm{B})$.
(7) $\tau(\mathrm{A}\to\mathrm{B}) = \tau(\mathrm{A})\to\tau(\mathrm{B})$.
(8) $\tau(\mathrm{A}\leftrightarrow\mathrm{B}) = \tau(\mathrm{A})\leftrightarrow\tau(\mathrm{B})$.
(9) $\tau(\Box\mathrm{A}) = \forall\alpha\tau(\mathrm{A})$.
(10) $\tau(\Diamond\mathrm{A}) = \exists\alpha\tau(\mathrm{A})$.

Thus τ associates with each sentence A in the modal language a unique formula τ(A) in the quantificational language by replacing each atomic sentence $\mathbb{P}_n$ by $\mathbb{P}_n(\alpha)$ and putting $\forall\alpha$ and $\exists\alpha$ respectively for occurrences of $\Box$ and $\Diamond$. For example, let us calculate the results of applying τ to $\Box\mathbb{P}_0\to\mathbb{P}_0$ and to $\Diamond\mathbb{P}_0\to\Box\Diamond\mathbb{P}_0$.

$$\begin{aligned}\tau(\Box\mathbb{P}_0\to\mathbb{P}_0) &= \tau(\Box\mathbb{P}_0)\to\tau(\mathbb{P}_0)\\ &= \forall\alpha\tau(\mathbb{P}_0)\to\tau(\mathbb{P}_0)\\ &= \forall\alpha\mathbb{P}_0(\alpha)\to\mathbb{P}_0(\alpha)\end{aligned}$$

$$\begin{aligned}\tau(\Diamond\mathbb{P}_0\to\Box\Diamond\mathbb{P}_0) &= \tau(\Diamond\mathbb{P}_0)\to\tau(\Box\Diamond\mathbb{P}_0)\\ &= \tau(\Diamond\mathbb{P}_0)\to\forall\alpha\tau(\Diamond\mathbb{P}_0)\\ &= \exists\alpha\tau(\mathbb{P}_0)\to\forall\alpha\exists\alpha\tau(\mathbb{P}_0)\\ &= \exists\alpha\mathbb{P}_0(\alpha)\to\forall\alpha\exists\alpha\mathbb{P}_0(\alpha)\end{aligned}$$

It should be apparent that τ is in effect a specification of the truth conditions of modal sentences at a possible world α in a model. The transformation shows that $\Box\mathbb{P}_0\to\mathbb{P}_0$ holds at α just in case $\forall\alpha\mathbb{P}_0(\alpha)\to\mathbb{P}_0(\alpha)$ is true, and that $\Diamond\mathbb{P}_0\to\Box\Diamond\mathbb{P}_0$ holds at α just in case $\exists\alpha\mathbb{P}_0(\alpha)\to\forall\alpha\exists\alpha\mathbb{P}_0(\alpha)$ is true. Generally, we can see that a modal sentence A is valid just when τ(A) is; i.e.

$\vDash$ A iff τ(A) is a valid formula of elementary quantificational logic.

For example, the instances of T and 5 above are valid, and so are their transformations. So τ provides a way of investigating questions of validity in the modal language.

(*a*) Apply τ to K, Df$\Diamond$, and selected tautologies, to see that their transformations are valid quantificational formulas.

(*b*) Show that if $\tau(A)$ is a valid formula of elementary quantificational logic so is $\tau(\Box A)$, and that if $\tau(A \to B)$ and $\tau(A)$ are quantificationally valid so is $\tau(B)$.

(*c*) Use τ on the schemas in exercises 1.1–1.3, 1.7, and 1.8.

(*d*) Show that the principles in exercises 1.4 and 1.5 hold with respect to quantificational validity and transformations of the schemas.

(*e*) Explain how models $\mathscr{M} = \langle W, P \rangle$ for the modal language serve equally well for the quantificational language.

1.2. The system *S5*

In this section we examine necessity and possibility in *S5* from an axiomatic point of view. We begin with an axiomatization based on the principles in the preceding section. That is, we adopt as axioms, or basic theorems, all sentences of the following forms.

T.	$\Box A \to A$
5.	$\Diamond A \to \Box \Diamond A$
K.	$\Box(A \to B) \to (\Box A \to \Box B)$
Df$\Diamond$.	$\Diamond A \leftrightarrow \neg \Box \neg A$
PL.	A, where A is a tautology

And we assume the following rules of inference.

RN. $\dfrac{A}{\Box A}$

MP. $\dfrac{A \to B,\ A}{B}$

By a theorem, generally, we mean any sentence that can be proved on the basis of the axioms and rules of inference. (Axioms are automatically theorems.) Where A is a sentence, we also write

$$\vdash A$$

to mean that A is a theorem.

Note that a rule of inference is properly understood as meaning that its conclusion is a theorem if each of its hypotheses is. For example, the rule RN means that $\vdash \Box A$ whenever $\vdash A$.

Before moving to proofs of genuinely modal theorems and further rules of inference, let us see again that *S5*, now formulated as a deductive system, includes propositional logic. Here this means that we can derive within the system the rule of inference

RPL. $\dfrac{A_1, \ldots, A_n}{A}$ $(n \geqslant 0)$,

where the inference from $A_1, \ldots, A_n$ to A is propositionally correct.

The proof that the rule RPL holds is like that for the analogous result in section 1.1. We show that if the inference from $A_1, \ldots, A_n$ to A is propositionally correct and each of $A_1, \ldots, A_n$ is a theorem, then A is a theorem, too. The supposition that the inference is propositionally correct means that A is true in every valuation in which each of $A_1, \ldots, A_n$ is, which in turn means that the sentence

$$A_1 \to (\ldots (A_n \to A) \ldots)$$

is a tautology (*PL*), which means that it is a theorem. If each of $A_1, \ldots, A_n$ is a theorem, then by n applications of the rule MP, so is A.

We may illustrate RPL, as in section 1.1, with the rule of inference sometimes called hypothetical syllogism:

$$\frac{A \to B, \quad B \to C}{A \to C}$$

Because $A \to C$ is true in every valuation in which both $A \to B$ and $B \to C$ are, the sentence $(A \to B) \to ((B \to C) \to (A \to C))$ is a tautology and hence a theorem. Thus if both $\vdash A \to B$ and $\vdash B \to C$, successive applications of MP yield first that $\vdash (B \to C) \to (A \to C)$ and then that $\vdash A \to C$. So this rule is covered by RPL.

The rule MP is obviously also a special case of RPL, but it should be noted that RPL covers the axioms *PL* as well. For when $n = 0$, RPL is the rule

$$\frac{}{A},$$

where the inference to A is propositionally correct.

And this simply means that A is a theorem whenever A is true in every valuation, i.e. whenever A is a tautology. Thus it is a matter of indifference whether we adopt *PL* and MP, on the one hand, or simply RPL, on the other, in our axiomatization of *S5*. We choose *PL* and MP here because this is closer to the traditional approach; in the rest of the book we use RPL.

In any case, we shall hereinafter freely make use of tautologies and propositionally correct patterns of inference in deducing theorems and deriving rules of inference. Wherever we do, we signal this by *PL* (for 'propositional logic').

Turning now to specifically modal principles, let us begin by proving that the schema

$$\mathrm{T}\Diamond.\quad A \to \Diamond A$$

– whatever is so is possibly so – is a theorem of *S5*. First we note that as a special case of the axiom T we have that $\vdash \Box\neg A \to \neg A$. By *PL*, it follows from this that $\vdash A \to \neg\Box\neg A$. In view of the axiom Df$\Diamond$, i.e. that $\vdash \Diamond A \leftrightarrow \neg\Box\neg A$, we may infer by *PL* that $\vdash A \to \Diamond A$.

We can put this discursive proof that T$\Diamond$ is a theorem more neatly as an annotated sequence of theorems:

1. $\Box\neg A \to \neg A$	T
2. $A \to \neg\Box\neg A$	1, *PL*
3. $\Diamond A \leftrightarrow \neg\Box\neg A$	Df$\Diamond$
4. $A \to \Diamond A$	2, 3, *PL*

The annotations are meant to indicate the reasoning involved as the proof proceeds. Thus line 1 is justified as an instance of T, line 2 comes from line 1 by *PL* (i.e. RPL), line 3 is a statement of Df$\Diamond$, and line 4 is inferred from lines 2 and 3 by *PL* (again, RPL). This way of setting out proofs is perspicuous and often useful, especially where the discursive mode is lengthy or tortuous. (But notice that line 2 might have been omitted, since line 4 follows from lines 1 and 3 by *PL*. We prefer the longer proof here for the sake of perspicuity.)

Next let us show that whatever is necessary is possible, i.e. that the schema

$$\mathrm{D}.\quad \Box A \to \Diamond A$$

is a theorem of *S5*. The proof is simple: Since both $\Box A \to A$ and $A \to \Diamond A$ are theorems, by *PL* (in fact, hypothetical syllogism) so is $\Box A \to \Diamond A$.

Likewise, using T$\Diamond$, we may show that the schema

$$\mathrm{B}.\quad A \to \Box\Diamond A$$

– whatever is so is necessarily possibly so – is a theorem of *S5*.

1. $\Diamond A \to \Box\Diamond A$	5
2. $A \to \Diamond A$	T$\Diamond$
3. $A \to \Box\Diamond A$	1, 2, *PL*

Before going on to prove more theorems of *S5*, it will be convenient to derive two further rules of inference.

$$\mathrm{RM}.\quad \frac{A \to B}{\Box A \to \Box B}$$

$$\mathrm{RE}.\quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

The rule RM may be understood as asserting that a proposition is necessary if it is implied by a necessary proposition. To show that *S5* has this rule we argue that its conclusion is a theorem if its hypothesis is, as follows.

1. $A \to B$	hypothesis
2. $\Box(A \to B)$	1, RN
3. $\Box(A \to B) \to (\Box A \to \Box B)$	K
4. $\Box A \to \Box B$	2, 3, *PL*

Given RM, it is easy to derive the rule RE (which says in effect that equivalent propositions are equally necessary). We leave the derivation as an exercise for the reader.

Now let us prove that *S5* has the theorem

$$\mathrm{Df}\Box.\quad \Box A \leftrightarrow \neg\Diamond\neg A,$$

i.e. that necessity is definable in terms of possibility and negation. Our proof uses just *PL*, RE, and the definability of possibility in terms of necessity and negation, Df$\Diamond$.

1. $\Diamond\neg A \leftrightarrow \neg\Box\neg\neg A$	Df$\Diamond$
2. $\Box\neg\neg A \leftrightarrow \neg\Diamond\neg A$	1, *PL*
3. $A \leftrightarrow \neg\neg A$	*PL*
4. $\Box A \leftrightarrow \Box\neg\neg A$	3, RE
5. $\Box A \leftrightarrow \neg\Diamond\neg A$	2, 4, *PL*

(Notice that line 2 might have been omitted, since line 5 follows from lines 1 and 4 by *PL*.)

Dual to the theorem B, *S5* has the theorem

$$\text{B}\Diamond.\quad \Diamond\Box A \to A,$$

which means that whatever is possibly necessary is simply so. By way of proof, note first that in virtue of B, $\vdash \neg A \to \Box\Diamond\neg A$, and so by *PL*, $\vdash \neg\Box\Diamond\neg A \to A$. Thus it is sufficient to show that $\vdash \Diamond\Box A \leftrightarrow \neg\Box\Diamond\neg A$. Our proof of this demonstrates the usefulness of being able to call upon Df$\Box$ and RE, as well as Df$\Diamond$:

1. $\Box A \leftrightarrow \neg\Diamond\neg A$	Df$\Box$
2. $\neg\Box A \leftrightarrow \Diamond\neg A$	1, *PL*
3. $\Box\neg\Box A \leftrightarrow \Box\Diamond\neg A$	2, RE
4. $\neg\Box\neg\Box A \leftrightarrow \neg\Box\Diamond\neg A$	3, *PL*
5. $\Diamond\Box A \leftrightarrow \neg\Box\neg\Box A$	Df$\Diamond$
6. $\Diamond\Box A \leftrightarrow \neg\Box\Diamond\neg A$	4, 5, *PL*

Here again our proof is spelled out in more detail than is necessary; line 6 follows by *PL* from lines 3 and 5.

By a similar argument we can also show that *S5* contains the following dual of the axiom 5.

$$5\Diamond.\quad \Diamond\Box A \to \Box A$$

For as a special case of 5, $\vdash \Diamond\neg A \to \Box\Diamond\neg A$, and hence $\vdash \neg\Box\Diamond\neg A \to \neg\Diamond\neg A$, by *PL*. Then 5$\Diamond$ follows by *PL*, using Df$\Box$ and the theorem on line 6 above. The import of 5$\Diamond$ is, of course, that a proposition is necessary if it is at least possibly necessary.

We come now to the schema

$$4.\quad \Box A \to \Box\Box A.$$

According to 4, whatever is necessary is necessarily necessary. We may prove that 4 is a theorem of *S5* as follows.

1. $\Diamond\Box A \to \Box A$	5$\Diamond$
2. $\Box\Diamond\Box A \to \Box\Box A$	1, RM
3. $\Box A \to \Box\Diamond\Box A$	B
4. $\Box A \to \Box\Box A$	2, 3, *PL*

Corresponding to the theorem 4 is the dual schema

$$4\Diamond.\quad \Diamond\Diamond A \to \Diamond A,$$

according to which whatever is possibly possible is possible simply. To show that 4$\Diamond$ is a theorem of *S5* we would argue that because of 4 and *PL*, $\vdash \neg\Box\Box\neg A \to \neg\Box\neg A$, and then prove that $\vdash \Diamond\Diamond A \leftrightarrow \neg\Box\Box\neg A$. For 4$\Diamond$ follows from these by *PL*. We leave the actual proof, however, as an exercise for the reader.

The system *S5* has the following noteworthy rule of inference.

$$\text{RK.}\quad \frac{(A_1 \wedge \ldots \wedge A_n) \to A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A} \quad (n \geqslant 0)$$

RK expresses a general rule of modal consequence: a proposition is necessary if it is a consequence of a collection of propositions each of which is necessary. The condition that $n \geqslant 0$ is intentional, for we make the convention that in the absence of antecedents – when $n = 0$ – the conditionals are identified with their consequents, A and $\Box$A. Thus when $n = 0$ we have the rule RN as a special case of RK. Moreover, when $n = 1$ the rule RK becomes RM.

A proper proof that *S5* has the rule RK proceeds by induction on the number n of conjuncts in the antecedents. The basis of the induction, where $n = 0$, is trivial, since in this case RK is RN, a basic rule in the axiomatization of *S5*. For the inductive part of the proof we suppose – as an *inductive hypothesis* – that the rule holds for any number of conjuncts in the antecedents up to (but not including) some number $n > 0$, and show from this that it holds when the number of conjuncts is exactly n. The argument for this is as follows. Suppose that

$$(A_1 \wedge \ldots \wedge A_n) \to A$$

is a theorem. By *PL* this is equivalent to

$$(A_1 \wedge \ldots \wedge A_{n-1}) \to (A_n \to A).$$

By the inductive hypothesis the rule RK applies to this theorem, since the number of conjuncts in the antecedent is less than n. Thus we have the theorem

$$(\Box A_1 \wedge \ldots \wedge \Box A_{n-1}) \to \Box(A_n \to A).$$

Now from this and the axiom K, in the form

$$\Box(A_n \to A) \to (\Box A_n \to \Box A),$$

we infer by *PL* the theorem

$$(\Box A_1 \wedge \ldots \wedge \Box A_{n-1}) \to (\Box A_n \to \Box A),$$

which is equivalent by *PL* to

$$(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A.$$

This completes the inductive part of the proof. It follows now that the rule RK holds for any number $n \geqslant 0$ of conjuncts in the antecedents, since it holds for $n = 0$ and also for any $n > 0$ whenever it holds up to n.

Notice that only *PL*, RN, and K are used in the derivation of the rule RK. Moreover, using only *PL* and RK we can prove RN (trivially) and K (by RK on the tautology $((A \to B) \wedge A) \to B$ we get the theorem $(\Box(A \to B) \wedge \Box A) \to \Box B$, which is equivalent to K by *PL*). The moral of this is that we could equally well have chosen RK instead of RN and K in our axiomatization of *S5*.

Another special case of RK, when $n = 2$, is the rule of inference

$$\text{RR.}\quad \frac{(A \wedge B) \to C}{(\Box A \wedge \Box B) \to \Box C},$$

which expresses a limited principle of consequence (a proposition is necessary if it follows from a pair of propositions each of which is necessary). A direct proof of RR – using *PL*, RM, and K – can also be had, and it may illuminate the inductive part of the proof above for RK; we leave it as an exercise.

Three further theorems are worth mention.

N. $\Box\top$

M. $\Box(A \wedge B) \to (\Box A \wedge \Box B)$

C. $(\Box A \wedge \Box B) \to \Box(A \wedge B)$

Proofs of N, M, and C – using RN, RM, and RR, respectively – are not hard to find, so we leave them as exercises.

It is clear from our results in section 1.1 that every theorem is valid: all the axioms, T, 5, K, Df$\Diamond$, and *PL*, are valid, and the rules of inference RN and MP preserve validity. In short, the axiomatization is *sound*. It is moreover *complete*: every valid sentence is a theorem. This may not be so obvious, however, and it is not until chapter 5 that we are in a position to prove it.

We thus have two ways of characterizing the modal logic *S5* – one semantic, the other deductive. It bears emphasis, moreover, that the set of principles T, 5, K, Df$\Diamond$, *PL*, RN, and MP is not the only selection that provides an axiomatization – a deductive characterization – of *S5*. We have seen already, for example, that the rule RPL would do just as well as MP plus *PL*, and that RK could take the place of RN plus K. Such alterations result in equivalent, alternative axiomatizations of *S5* – equivalent since the axioms and rules of inference of each are derivable

from the others, so that any sentence provable in one axiomatization is provable in the others.

Let us conclude this section with yet another axiomatization of *S5*, one of the best known. It is formulated on the basis of propositional logic by means of the rule RN together with the schemas T, B, 4, K, and Df$\Diamond$ as axioms. In other words, this axiomatization differs from the one with which we began only by having B and 4 as axioms in place of 5. For the sake of exposition we dub the set of theorems axiomatized in this way *S5′*.

Clearly, every theorem of *S5′* is also a theorem of *S5*, since every axiom and rule of *S5′* can be (and has been) proved in *S5*. Showing the reverse, that *S5′* includes *S5*, boils down to proving that the schema 5 is a theorem of *S5′* – i.e. that 5 can be derived on the basis of T, B, 4, K, Df$\Diamond$, *PL*, RN, and MP. To this end, observe that *S5′* has the rule RPL (because of *PL* and MP), that *S5′* has the rule RM (because of RPL, RN, and K), and that *S5′* has the theorem 4$\Diamond$ (exercise). So we may argue as follows.

1.	$\Diamond\Diamond A \to \Diamond A$	4$\Diamond$
2.	$\Box\Diamond\Diamond A \to \Box\Diamond A$	1, RM
3.	$\Diamond A \to \Box\Diamond\Diamond A$	B
4.	$\Diamond A \to \Box\Diamond A$	2, 3, *PL*

According to the last line the schema 5 is indeed a theorem of *S5′*. Therefore, the two axiomatizations are equivalent.

EXERCISES

Except where otherwise noted use any theorem or rule of inference established in section 1.2, and any theorems and rules established in previous exercises.

1.12. Derive the rule of inference RE in *S5*.

1.13. Derive the following rules of inference in *S5*.

$$(a)\ \frac{A \to B}{\Diamond A \to \Diamond B} \qquad (b)\ \frac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}$$

1.14. Derive the rule of inference RR in *S5* using only *PL*, RM, and K.

1.15. Prove that N, M, and C are theorems of *S5*.

1.16. Prove that the following schemas are theorems of *S5*.

(*a*) $\neg\Diamond\bot$

(*b*) $(\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$

(*c*) $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

1.17. Prove that the following schemas are theorems of *S5*.

(*a*) $(\Box A \vee \Box B) \to \Box(A \vee B)$
(*b*) $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
(*c*) $\Box(A \vee B) \to (\Diamond A \vee \Box B)$
(*d*) $(\Box A \wedge \Diamond B) \to \Diamond(A \wedge B)$

1.18. Derive the following rules of inference in *S5*.

(*a*) $\dfrac{A}{\Diamond A}$ (*b*) $\dfrac{\neg A}{\neg\Diamond A}$

1.19. Prove that the sentence $\Diamond\top$ is a theorem of *S5*.

1.20. Prove that the schema $4\Diamond$ is a theorem of *S5*.

1.21. Prove that the following schemas are theorems of *S5*.

(*a*) $\Box A \leftrightarrow \Box\Box A$
(*b*) $\Diamond A \leftrightarrow \Diamond\Diamond A$
(*c*) $\Diamond A \leftrightarrow \Box\Diamond A$
(*d*) $\Box A \leftrightarrow \Diamond\Box A$

1.22. Prove that the schema $\Diamond\Box A \to \Box\Diamond A$ is a theorem of *S5*.

1.23. Derive the following rules of inference in *S5*.

(*a*) $\dfrac{\Diamond A \to B}{A \to \Box B}$ (*b*) $\dfrac{A \to \Box B}{\Diamond A \to B}$

1.24. Prove that the schema $4\Diamond$ is a theorem of *S5′* (see the last paragraph of section 1.2).

1.25. Prove that *S5* is equivalently axiomatized if in the original axiomatization the axiom 5 is deleted in favor of the schema $A \to \Box\Box\Diamond A$.

1.26. Prove that *S5* is equivalently axiomatized if in the original axiomatization the axiom K is replaced by the schemas N, M, and C, and the rule of inference RN is replaced by RE.

1.27. We say that a system of modal logic is *consistent* when it does not contain $\bot$ as a theorem. It is clear that the system *S5* is consistent: the axiomatization is sound – i.e. every theorem is valid – and $\bot$ is not valid. Below we argue the consistency of *S5* in another way.

Let the mapping ϵ on the set of sentences be defined by the following clauses.

(1) $\epsilon(\mathbb{P}_n) = \mathbb{P}_n$, for $n = 0, 1, 2, \ldots$.
(2) $\epsilon(\top) = \top$.

(3) $\epsilon(\bot) = \bot$.
(4) $\epsilon(\neg A) = \neg\epsilon(A)$.
(5) $\epsilon(A \wedge B) = \epsilon(A) \wedge \epsilon(B)$.
(6) $\epsilon(A \vee B) = \epsilon(A) \vee \epsilon(B)$.
(7) $\epsilon(A \to B) = \epsilon(A) \to \epsilon(B)$.
(8) $\epsilon(A \leftrightarrow B) = \epsilon(A) \leftrightarrow \epsilon(B)$.
(9) $\epsilon(\Box A) = \epsilon(A)$.
(10) $\epsilon(\Diamond A) = \epsilon(A)$.

Thus ϵ is an 'erasure' transformation. It erases all occurrences of the modal operators $\Box$ and $\Diamond$ in a sentence A, but leaves A otherwise intact.

Now let us see that ϵ transforms axioms of *S5* into tautologies, and rules of inference of *S5* into rules of propositional logic. Clearly the erasure of a tautology (*PL*) is always a tautology. Moreover:

T. $\epsilon(\Box A \to A) = \epsilon(A) \to \epsilon(A)$.
5. $\epsilon(\Diamond A \to \Box\Diamond A) = \epsilon(A) \to \epsilon(A)$.
K. $\epsilon(\Box(A \to B) \to (\Box A \to \Box B)) = (\epsilon(A) \to \epsilon(B)) \to (\epsilon(A) \to \epsilon(B))$.
Df$\Diamond$. $\epsilon(\Diamond A \leftrightarrow \neg\Box\neg A) = \epsilon(A) \leftrightarrow \neg\neg\epsilon(A)$.

The schemas on the right-hand side of these identities are all tautologies; so the erasure of any non-propositional axiom is a tautology. Finally, under erasure the rules of inference RN and MP become

$$\frac{\epsilon(A)}{\epsilon(A)} \quad \text{and} \quad \frac{\epsilon(A) \to \epsilon(B),\ \epsilon(A)}{\epsilon(B)}.$$

The first of these is merely a rule of repetition, and the second is just MP again.

It follows that under the mapping ϵ every theorem of *S5* is transformed into a tautology. Therefore, since $\epsilon(\bot)$ – i.e. $\bot$ – is not a tautology, $\bot$ is not a theorem of *S5*. So we have proved the consistency of *S5* once again.

(*a*) Apply ϵ to T, 5, K, Df$\Diamond$, and selected instances of *PL* to see that their erasures are tautologies.
(*b*) Show that if $\epsilon(A)$ is a tautology so is $\epsilon(\Box A)$, and that if $\epsilon(A \to B)$ and $\epsilon(A)$ are tautologies so is $\epsilon(B)$.
(*c*) Use ϵ on the schemas in exercises 1.15–1.17, 1.19–1.22, and 1.25.

(*d*) Show that the rules of inference in exercises 1.12–1.14, 1.18, and 1.23 hold under erasure.

(*e*) Consider the system that results when the schema $A \to \Box A$ is added as an axiom to *S5*. This system is not sound, since $A \to \Box A$ is not valid (see section 1.1), but it is consistent. Use the erasure transformation to prove this.

(*f*) Prove the consistency of the system that results when the schema $\Box\Diamond A \to \Diamond\Box A$ is added as an axiom to *S5*. (Is this system sound?)

(*g*) Referring to exercise 1.11, observe that the transformation τ can be employed like ϵ to prove the consistency of *S5*, since $\bot$ is not quantificationally valid. How do the schemas $A \to \Box A$ and $\Box\Diamond A \to \Diamond\Box A$ fare under τ?

1.28. Sometimes there is confusion about the meaning of rules of inference. For example, because of RN it might be thought that $A \to \Box A$ is a theorem. Similarly the rules RM and RE might mistakenly be regarded as evidence for the theoremhood of the schemas

$$(A \to B) \to (\Box A \to \Box B) \quad \text{and} \quad (A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B).$$

Dispel this illusion by showing that neither schema is valid.

1.29. Show that $A \to \Box A$ is a theorem of the system that results when the schema $(A \to B) \to (\Box A \to \Box B)$ is added as an axiom to *S5*. Is this true if $(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$ is added to *S5*? What about the consistency of these systems?

1.30. Prove that $A \to \Box A$ is a theorem of the system that results when the schema $\Box\Diamond A \to \Diamond\Box A$ is added as an axiom to *S5*.

2

LOGICAL PRELIMINARIES

This chapter is an introduction to most of the concepts we shall use in studying modal logic.

In section 2.1 we set out most of the syntactic concepts. Section 2.2 introduces semantic concepts: the general idea of a model, truth conditions for non-modal sentences, and definitions of truth in a model and validity in a class of models. Filtrations of models are described in section 2.3. In section 2.4 the idea of a system of modal logic is explained, along with such relevant notions as theoremhood, deducibility, and consistency. Axiomatizability is discussed in section 2.5. Maximal sets of sentences and Lindenbaum's lemma occupy section 2.6. In section 2.7 we define determination and explain our approach, using canonical models, to proofs of determination. Finally in section 2.8 we outline our method of proving the decidability of systems of modal logic.

As the need arises the reader may wish to return to various sections of this chapter, for important definitions and theorems.

2.1. Syntax

This section is devoted to a recital of the basic syntactic concepts for the language of modal logic, many of which the reader has likely gleaned from chapter 1. The ideas are very simple. The few formal definitions we offer may be helpful, but they are not essential; we state them mainly for the sake of completeness and future reference.

Sentences. The language is founded on a denumerable set of *atomic sentences*:

$$\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots$$

These are the simplest sentences.

The non-atomic *molecular sentences* are formed by means of nine *syntactic operations*, or *operators*:

$$\top, \bot, \neg, \land, \lor, \to, \leftrightarrow, \Box, \Diamond$$

(*l*) reflexive transitive
(*m*) reflexive euclidean

Except for (*a*), the proofs use theorems 3.19 and 3.20, in addition to the results in section 3.5. For (*m*), note exercise 3.31.

3.75. Give an example of a non-incestual filtration of an incestual standard model.

4

NORMAL SYSTEMS OF MODAL LOGIC

This chapter is devoted to studying, from a purely deductive standpoint, a class of systems of modal logic we call *normal*.

In section 4.1 we first define the class of normal systems. Then we derive a number of theorems and rules of inference common to all normal modal logics and use some of them to formulate alternative deductive characterizations of such systems. Theorems on replacement, negation, and duality are proved in section 4.2 for normal modal logics (they hold more generally for all classical systems, as we discover in chapter 8). These results provide rules and theorems that serve to facilitate derivations.

The smallest normal system of modal logic we call K. Thus every normal system of modal logic is a K-system. (The converse is false; not all K-systems are normal.) To simplify naming normal systems we write

$$KS_1 \ldots S_n$$

to denote the normal modal logic obtained by taking the schemas $S_1, \ldots, S_n$ as theorems. In other words:

$$KS_1 \ldots S_n = \text{the smallest normal system of modal logic containing (every instance of) the schemas } S_1, \ldots, S_n.$$

So, for example, $KT4$ is the smallest normal system produced by treating the schemas T and 4 as theorems in a normal modal logic. (It is also denoted by $K4T$; the order of the schema names is irrelevant.) As the limiting case, where there are no schemas, the definition yields K as the smallest normal system.

In section 4.3 we begin a survey of the normal extensions of K containing various combinations of the schemas D, T, B, 4, and 5. This continues in section 4.4 with an account of the numbers of distinct modalities present in certain of these systems.

The chapter concludes with section 4.5, which contains some theorems bout maximal sets of sentences in normal modal logics. These results igure importantly with regard to some theorems in chapter 5.

4.1. Normal systems

As we learned in chapter 2, a system of modal logic is a set of sentences containing all tautologies and closed under the rule of inference RPL. We characterize normal systems of modal logic in terms of the schema

Df$\Diamond$. $\Diamond A \leftrightarrow \neg\Box\neg A$

and the rule of inference

RK. $\dfrac{(A_1 \wedge \ldots \wedge A_n) \to A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A}$ $(n \geqslant 0)$.

DEFINITION 4.1. A system of modal logic is *normal* iff it contains Df$\Diamond$ and is closed under RK.

Beginning with theorem 4.2 and continuing with theorem 4.4 we register some of the more important rules and theorems present in all normal systems of modal logic. Many of these are familiar from chapter 1. In theorems 4.3 and 4.5 we record some alternative ways of characterizing normal modal logics.

THEOREM 4.2. *Every normal system of modal logic has the following rules of inference and theorems.*

RN. $\dfrac{A}{\Box A}$

RM. $\dfrac{A \to B}{\Box A \to \Box B}$

RR. $\dfrac{(A \wedge B) \to C}{(\Box A \wedge \Box B) \to \Box C}$

RE. $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

N. $\Box\top$

M. $\Box(A \wedge B) \to (\Box A \wedge \Box B)$

C. $(\Box A \wedge \Box B) \to \Box(A \wedge B)$

R. $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

K. $\Box(A \to B) \to (\Box A \to \Box B)$

Proof. Let Σ be a normal system of modal logic. By theorem 2.13 propositional logic is a part of Σ, a fact we take advantage of frequently and casually.

For RN, RM, and RR. These rules of inference are simply RK for $n = 0$, 1, and 2, respectively.

For RE. Suppose that $\vdash_\Sigma A \leftrightarrow B$. Then by *PL* both $\vdash_\Sigma A \to B$ and $\vdash_\Sigma B \to A$. By RM in each case, $\vdash_\Sigma \Box A \to \Box B$ and $\vdash_\Sigma \Box B \to \Box A$. Hence by *PL* again, $\vdash_\Sigma \Box A \leftrightarrow \Box B$.

For N. By *PL*, $\vdash_\Sigma \top$. Hence by RN, $\vdash_\Sigma \Box\top$.

For M. By *PL*, $\vdash_\Sigma (A \wedge B) \to A$ and $\vdash_\Sigma (A \wedge B) \to B$. So by RM, $\vdash_\Sigma \Box(A \wedge B) \to \Box A$ and $\vdash_\Sigma \Box(A \wedge B) \to \Box B$. By *PL* again, $\vdash_\Sigma \Box(A \wedge B) \to (\Box A \wedge \Box B)$.

For C. By *PL*, $\vdash_\Sigma (A \wedge B) \to (A \wedge B)$. Hence by RR, $\vdash_\Sigma (\Box A \wedge \Box B) \to \Box(A \wedge B)$.

For R. This is just the biconditional of M and C.

For K. By *PL*, $\vdash_\Sigma ((A \to B) \wedge A) \to B$. So by RR, $\vdash_\Sigma (\Box(A \to B) \wedge \Box A) \to \Box B$. Therefore by *PL*, $\vdash_\Sigma \Box(A \to B) \to (\Box A \to \Box B)$.

As in chapter 1, proofs like these for theorem 4.2 can often be stated more perspicuously as annotated sequences of theorems. For example, the proof above for K can be presented thus:

1.	$((A \to B) \wedge A) \to B$	*PL*
2.	$(\Box(A \to B) \wedge \Box A) \to \Box B$	1, RR
3.	$\Box(A \to B) \to (\Box A \to \Box B)$	2, *PL*

On top of propositional logic, the schema Df$\Diamond$ and the rule RK provide an axiomatic basis for normal systems of modal logic. Together with Df$\Diamond$ the rules and theorems listed in theorem 4.2 provide a number of alternative bases for – i.e. alternative ways of characterizing – normal systems. We select just four for attention in the next theorem; some others appear in the exercises.

THEOREM 4.3. *Let Σ be a system of modal logic containing* Df$\Diamond$. *Then:*

(1) *Σ is normal iff it contains* K *and is closed under* RN.

(2) *Σ is normal iff it contains* N *and is closed under* RR.

(3) *Σ is normal iff it contains* N *and* C *and is closed under* RM.

(4) *Σ is normal iff it contains* N, C, *and* M *and is closed under* RE.

Proof. Let Σ be a system containing Df$\Diamond$. Theorem 4.2 takes care of left-to-right in each case, so we show only right-to-left.

For (1). We need to show that if Σ contains K and is closed under RN, then Σ is closed under RK; i.e. that for $n \geqslant 0$,

if $\vdash_\Sigma (A_1 \wedge \ldots \wedge A_n) \to A$,
then $\vdash_\Sigma (\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A$.

The proof is by induction on n and is like that for theorem 3.3 (2) (recall lemmas 1 and 2 there). With this hint we leave the details to the reader.

For (2). Suppose Σ contains N and is closed under RR. In view of (1) it is enough to show that Σ contains K and is closed under RN. As to K, see the proof of theorem 4.2. For RN:

1. A	hypothesis
2. $(\top \wedge \top) \to A$	1, *PL*
3. $(\Box\top \wedge \Box\top) \to \Box A$	2, RR
4. $\Box\top$	N
5. $\Box A$	3, 4, *PL*

Note that line 1 means that $\vdash_\Sigma A$, so that RR is applicable at line 2.

For (3). Suppose Σ contains N and C and is closed under RM. Given (2), we need only show that Σ is closed under RR. Thus:

1. $(A \wedge B) \to C$	hypothesis
2. $\Box(A \wedge B) \to \Box C$	1, RM
3. $(\Box A \wedge \Box B) \to \Box(A \wedge B)$	C
4. $(\Box A \wedge \Box B) \to \Box C$	2, 3, *PL*

For (4). If Σ contains N, C, and M and is closed under RE, it is sufficient, given (3), to show that Σ is closed under RM. We leave this as an exercise.

With the exception of Df$\Diamond$, the rules and theorems so far have featured the necessity operator. The next theorem catalogues some rules and theorems of normal systems in which the possibility operator predominates.

THEOREM 4.4. *Every normal system of modal logic has the following rules of inference and theorems.*

RK$\Diamond$. $\dfrac{A \to (A_1 \vee \ldots \vee A_n)}{\Diamond A \to (\Diamond A_1 \vee \ldots \vee \Diamond A_n)}$ $(n \geqslant 0)$

RN$\Diamond$. $\dfrac{\neg A}{\neg\Diamond A}$

RM$\Diamond$. $\dfrac{A \to B}{\Diamond A \to \Diamond B}$

RR$\Diamond$. $\dfrac{A \to (B \vee C)}{\Diamond A \to (\Diamond B \vee \Diamond C)}$

RE$\Diamond$. $\dfrac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}$

Df$\Box$. $\Box A \leftrightarrow \neg\Diamond\neg A$

N$\Diamond$. $\neg\Diamond\bot$

M$\Diamond$. $(\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$

C$\Diamond$. $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

R$\Diamond$. $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$

K$\Diamond$. $(\neg\Diamond A \wedge \Diamond B) \to \Diamond(\neg A \wedge B)$

Proof. Let Σ be a normal system.

For RK$\Diamond$. Suppose that $\vdash_\Sigma A \to (A_1 \vee \ldots \vee A_n)$. Then by *PL*, $\vdash_\Sigma (\neg A_1 \wedge \ldots \wedge \neg A_n) \to \neg A$. By applying RK, $\vdash_\Sigma (\Box\neg A_1 \wedge \ldots \wedge \Box\neg A_n) \to \Box\neg A$. Hence by *PL* again, $\vdash_\Sigma \neg\Box\neg A \to (\neg\Box\neg A_1 \vee \ldots \vee \neg\Box\neg A_n)$. Therefore by Df$\Diamond$ and *PL*, $\vdash_\Sigma \Diamond A \to (\Diamond A_1 \vee \ldots \vee \Diamond A_n)$.

For RN$\Diamond$, RM$\Diamond$, and RR$\Diamond$. These are the rule RK$\Diamond$ for $n = 0$, 1, and 2, respectively. (For RN$\Diamond$, recall that when $n = 0$ the conditionals in RK$\Diamond$ are identified with the negations of their antecedents.)

For RE$\Diamond$. The proof uses RM$\Diamond$ and is like that for RE in theorem 4.3. Exercise.

For Df$\Box$. Compare the proof of this in section 1.2, and note that it uses only *PL*, Df$\Diamond$, and RE.

For N$\Diamond$. By *PL*, $\vdash_\Sigma \neg\bot$. So by RN$\Diamond$, $\vdash_\Sigma \neg\Diamond\bot$.

For M$\Diamond$. By *PL*, $\vdash_\Sigma A \to (A \vee B)$ and $\vdash_\Sigma B \to (A \vee B)$. Hence by RM$\Diamond$, $\vdash_\Sigma \Diamond A \to \Diamond(A \vee B)$ and $\vdash_\Sigma \Diamond B \to \Diamond(A \vee B)$. By *PL*, $\vdash_\Sigma (\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$.

For C$\Diamond$. The proof uses RR$\Diamond$ and the tautology $(A \vee B) \to (A \vee B)$. Exercise.

For R$\Diamond$. This is the biconditional of M$\Diamond$ and C$\Diamond$.

For K$\Diamond$:

1. $B \to (A \vee (\neg A \wedge B))$	*PL*
2. $\Diamond B \to (\Diamond A \vee \Diamond(\neg A \wedge B))$	1, RR$\Diamond$
3. $(\neg\Diamond A \wedge \Diamond B) \to \Diamond(\neg A \wedge B)$	2, *PL*

The reader should appreciate the parallels between the proofs above and those for the corresponding rules and theorems in theorem 4.2. We have developed this analogy intentionally, for the sake of simplicity and also to enhance the reader's ability to create such proofs on his own. There are of course other ways of doing this. As an example, let us prove again that R$\Diamond$ is a theorem of all normal modal logics, as follows.

1.	$\Box(\neg A\wedge\neg B)\leftrightarrow(\Box\neg A\wedge\Box\neg B)$	R
2.	$(\neg A\wedge\neg B)\leftrightarrow\neg(A\vee B)$	*PL*
3.	$\Box(\neg A\wedge\neg B)\leftrightarrow\Box\neg(A\vee B)$	2, RE
4.	$\Box\neg(A\vee B)\leftrightarrow(\Box\neg A\wedge\Box\neg B)$	1, 3, *PL*
5.	$\neg\Box\neg(A\vee B)\leftrightarrow(\neg\Box\neg A\vee\neg\Box\neg B)$	4, *PL*
6.	$\Diamond(A\vee B)\leftrightarrow(\Diamond A\vee\Diamond B)$	5, Df$\Diamond$ and *PL*

Other alternative proofs of rules and theorems are suggested in the exercises.

The characterization of normal systems of modal logic in terms of Df$\Diamond$ and RK and in theorem 4.3 may be said to be *necessity-based*, inasmuch as $\Box$ is treated as though it were primitive and $\Diamond$ is introduced only definitionally, through Df$\Diamond$. In the next theorem we turn this around by using rules and theorems from theorem 4.4 to give five characterizations of normal systems that are *possibility-based* and introduce necessity definitionally via Df$\Box$.

THEOREM 4.5. *Let Σ be a system of modal logic containing* Df$\Box$. *Then:*

(1) Σ *is normal iff it is closed under* RK$\Diamond$.

(2) Σ *is normal iff it contains* K$\Diamond$ *and is closed under* RN$\Diamond$.

(3) Σ *is normal iff it contains* N$\Diamond$ *and is closed under* RR$\Diamond$.

(4) Σ *is normal iff it contains* N$\Diamond$ *and* C$\Diamond$ *and is closed under* RM$\Diamond$.

(5) Σ *is normal iff it contains* N$\Diamond$, C$\Diamond$, *and* M$\Diamond$ *and is closed under* RE$\Diamond$.

Proof. Let Σ be a system containing Df$\Box$. The left-to-right cases are covered by theorem 4.4, so we need show only the converses.

For (1). Suppose that Σ is closed under RK$\Diamond$. We wish to prove first that Σ is closed under RK. The argument is analogous to that given for RK$\Diamond$, using RK, in theorem 4.4. Thus:

1.	$(A_1\wedge\ldots\wedge A_n)\rightarrow A$	hypothesis
2.	$\neg A\rightarrow(\neg A_1\vee\ldots\vee\neg A_n)$	1, *PL*
3.	$\Diamond\neg A\rightarrow(\Diamond\neg A_1\vee\ldots\vee\Diamond\neg A_n)$	2, RK$\Diamond$
4.	$(\neg\Diamond\neg A_1\wedge\ldots\wedge\neg\Diamond\neg A_n)\rightarrow\neg\Diamond\neg A$	3, *PL*
5.	$(\Box A_1\wedge\ldots\wedge\Box A_n)\rightarrow\Box A$	4, Df$\Box$ and *PL*

Next we must show that Σ contains Df$\Diamond$. The argument for this is like that suggested for Df$\Box$ in theorem 4.4, if as a lemma it is shown first that Σ is closed under RE$\Diamond$. This is left to the reader as an exercise. If thus Σ contains Df$\Diamond$ and is closed under RK, then by definition 4.1 it is normal.

The proofs for parts (2)–(5) parallel those for (1)–(4) in theorem 4.3.

For (2). Suppose that Σ contains K$\Diamond$ and is closed under RN$\Diamond$. In view of (1), just proved, it is enough to show that Σ is closed under RK$\Diamond$, i.e. that for $n\geqslant 0$,

$$\text{if } \vdash_\Sigma A\rightarrow(A_1\vee\ldots\vee A_n), \text{ then } \vdash_\Sigma \Diamond A\rightarrow(\Diamond A_1\vee\ldots\vee\Diamond A_n).$$

The proof is by induction on n. Where $n=0$, we need to show that if $\vdash_\Sigma\neg A$, then $\vdash_\Sigma\neg\Diamond A$. This is just RN$\Diamond$. So suppose as an inductive hypothesis that the rule holds for $k<n$. Then we reason as follows.

1.	$A\rightarrow(A_1\vee\ldots\vee A_n)$	hypothesis
2.	$(\neg A_1\wedge A)\rightarrow(A_2\vee\ldots\vee A_n)$	1, *PL*
3.	$\Diamond(\neg A_1\wedge A)\rightarrow(\Diamond A_2\vee\ldots\vee\Diamond A_n)$	2, inductive hypothesis
4.	$(\neg\Diamond A_1\wedge\Diamond A)\rightarrow\Diamond(\neg A_1\wedge A)$	K$\Diamond$
5.	$(\neg\Diamond A_1\wedge\Diamond A)\rightarrow(\Diamond A_2\vee\ldots\vee\Diamond A_n)$	3, 4, *PL*
6.	$\Diamond A\rightarrow(\Diamond A_1\vee\ldots\vee\Diamond A_n)$	5, *PL*

For (3). Suppose Σ contains N$\Diamond$ and is closed under RR$\Diamond$. Given (2), we need only show that Σ contains K$\Diamond$ and is closed under RN$\Diamond$. The proof of K$\Diamond$ appears in the proof of theorem 4.4. For RN$\Diamond$:

1.	$\neg A$	hypothesis
2.	$A\rightarrow(\bot\vee\bot)$	1, *PL*
3.	$\Diamond A\rightarrow(\Diamond\bot\vee\Diamond\bot)$	2, RR$\Diamond$
4.	$\neg\Diamond\bot$	N$\Diamond$
5.	$\neg\Diamond A$	3, 4, *PL*

For (4). Suppose Σ contains $N\Diamond$ and $C\Diamond$ and is closed under $RM\Diamond$. Given (3), it is sufficient to prove that Σ is closed under $RR\Diamond$. Exercise (compare the proof of theorem 4.3 (3)).

For (5). Suppose that Σ contains $N\Diamond$, $C\Diamond$, and $M\Diamond$ and is closed under $RE\Diamond$. In view of (4), it will do just to show that Σ is closed under $RM\Diamond$. Thus:

1. $A \to B$	hypothesis
2. $(A \vee B) \leftrightarrow B$	1, *PL*
3. $\Diamond(A \vee B) \leftrightarrow \Diamond B$	2, $RE\Diamond$
4. $(\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$	$M\Diamond$
5. $\Diamond A \to \Diamond B$	3, 4, *PL*

Many principles of normal systems can be generalized modally. For example, for every $k \geqslant 0$ every normal modal logic is closed under the rule of inference

$$\mathrm{RK}^k.\quad \frac{(A_1 \wedge \ldots \wedge A_n) \to A}{(\Box^k A_1 \wedge \ldots \wedge \Box^k A_n) \to \Box^k A} \quad (n \geqslant 0).$$

This should be evident, for the conclusion of the rule will follow from the hypothesis by k applications of the rule RK. More formally, it may be proved quite simply by induction on k. When $k = 0$ the hypothesis and the conclusion of the rule are the same, so of course the inference is good in this case. And from an inductive hypothesis that the rule holds whenever it has fewer than k $\Box$s, it follows by RK that it holds also when the number is k. That is to say, we may argue the inductive part of the proof as follows.

1. $(A_1 \wedge \ldots \wedge A_n) \to A$	hypothesis
2. $(\Box^{k-1}A_1 \wedge \ldots \wedge \Box^{k-1}A_n) \to \Box^{k-1}A$	1, inductive hypothesis
3. $(\Box\Box^{k-1}A_1 \wedge \ldots \wedge \Box\Box^{k-1}A_n) \to \Box\Box^{k-1}A$	2, RK
4. $(\Box^k A_1 \wedge \ldots \wedge \Box^k A_n) \to \Box^k A$	3, definition 2.3

Therefore, the rule RK^k holds in any normal system, for every $k \geqslant 0$.

The schema $Df\Diamond$ likewise generalizes along the modal dimension. For every $k \geqslant 0$ the schema

$$Df\Diamond^k.\quad \Diamond^k A \leftrightarrow \neg\Box^k\neg A$$

is a theorem of any normal modal logic. Here, too, a simple inductive argument suffices. For the basis, note that $Df\Diamond^k$ is a tautology when $k = 0$. For the inductive part, assume that the schema is a theorem whenever the number of $\Box$s and $\Diamond$s is less than k. Then we argue as follows.

1. $\Diamond^{k-1}A \leftrightarrow \neg\Box^{k-1}\neg A$	inductive hypothesis
2. $\Diamond\Diamond^{k-1}A \leftrightarrow \Diamond\neg\Box^{k-1}\neg A$	1, $RE\Diamond$
3. $\Box\Box^{k-1}\neg A \leftrightarrow \neg\Diamond\neg\Box^{k-1}\neg A$	$Df\Box$
4. $\Diamond\Diamond^{k-1}A \leftrightarrow \neg\Box\Box^{k-1}\neg A$	2, 3, *PL*
5. $\Diamond^k A \leftrightarrow \neg\Box^k\neg A$	4, definition 2.3

It should be apparent, given RK^k and $Df\Diamond^k$, that similar generalizations of all the principles in theorems 4.2 and 4.4 are part of any normal system of modal logic. More precisely, the results of putting $\Box^k$ and $\Diamond^k$ for $\Box$ and $\Diamond$ throughout these principles yield theorems and rules of inference that belong to every normal system, for every $k \geqslant 0$. Because we will need some of these principles later on (especially in chapter 5), we record this formally.

THEOREM 4.6. *Every normal system of modal logic has the principles* RK^k, $Df\Diamond^k$, RN^k, RM^k, RR^k, RE^k, N^k, M^k, C^k, R^k, K^k, $RK\Diamond^k$, $RN\Diamond^k$, $RM\Diamond^k$, $RR\Diamond^k$, $RE\Diamond^k$, $Df\Box^k$, $N\Diamond^k$, $M\Diamond^k$, $C\Diamond^k$, $R\Diamond^k$, *and* $K\Diamond^k$, *for every* $k \geqslant 0$.

Given the proofs above for RK^k and $Df\Diamond^k$, the reader can easily construct proofs for the remaining principles by attending to the proofs of theorems 4.2 and 4.4. Separate inductive proofs are also possible in each case.

EXERCISES

Where appropriate, freely make use of theorems and rules of inference established in section 4.1 and, farther along, the results of previous exercises.

4.1. Complete the proof of theorem 4.3 (parts (1) and (4)). (For (4), note that $A \to B$ is *PL*-equivalent to $A \leftrightarrow (A \wedge B)$.)

4.2. Complete the proof of theorem 4.4 by showing that every normal system has the rule $RE\Diamond$ and the theorem $C\Diamond$.

4.3. Complete the proof of theorem 4.5 (parts (1) and (4)).

4.4. Prove some of the parts of theorem 4.6.

4.5. Let Σ be a system of modal logic containing Df$\Diamond$. Prove:

(*a*) Σ is normal iff it is closed under RR and RN.
(*b*) Σ is normal iff it contains C and is closed under RM and RN.
(*c*) Σ is normal iff it contains N and K and is closed under RM.
(*d*) Σ is normal iff it contains C and M and is closed under RE and RN.
(*e*) Σ is normal iff it contains N and R and is closed under RE.
(*f*) Σ is normal iff it contains R and is closed under RE and RN.
(*g*) Σ is normal iff it contains N and K and is closed under RE.

4.6. Let Σ be a system of modal logic containing Df$\Box$. Prove:

(*a*) Σ is normal iff it is closed under RR$\Diamond$ and RN$\Diamond$.
(*b*) Σ is normal iff it contains C$\Diamond$ and is closed under RM$\Diamond$ and RN$\Diamond$.
(*c*) Σ is normal iff it contains N$\Diamond$ and K$\Diamond$ and is closed under RM$\Diamond$.
(*d*) Σ is normal iff it contains C$\Diamond$ and M$\Diamond$ and is closed under RE$\Diamond$ and RN$\Diamond$.
(*e*) Σ is normal iff it contains N$\Diamond$ and R$\Diamond$ and is closed under RE$\Diamond$.
(*f*) Σ is normal iff it contains R$\Diamond$ and is closed under RE$\Diamond$ and RN$\Diamond$.
(*g*) Σ is normal iff it contains N$\Diamond$ and K$\Diamond$ and is closed under RE$\Diamond$.

4.7. Prove that the following schemas are theorems of any normal system.

(*a*) $\Box A \to \Box(B \to A)$
(*b*) $\Box\neg A \to \Box(A \to B)$
(*c*) $\Diamond\top \leftrightarrow \neg\Box\bot$
(*d*) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
(*e*) $\Box(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$
(*f*) $\Box(A \leftrightarrow B) \to (\Diamond A \leftrightarrow \Diamond B)$
(*g*) $(\Box A \vee \Box B) \to \Box(A \vee B)$
(*h*) $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
(*i*) $(\Box A \wedge \Diamond B) \to \Diamond(A \wedge B)$

(*j*) $\Box(A \vee B) \to (\Diamond A \vee \Box B)$
(*k*) $\Diamond(A \to B) \vee \Box(B \to A)$
(*l*) $\Diamond(A \to B) \leftrightarrow (\Box A \to \Diamond B)$
(*m*) $\Diamond\top \leftrightarrow (\Box A \to \Diamond A)$
(*n*) $(\Diamond A \to \Box B) \to \Box(A \to B)$
(*o*) $(\Diamond A \to \Box B) \to (\Box A \to \Box B)$
(*p*) $(\Diamond A \to \Box B) \to (\Diamond A \to \Diamond B)$

4.8. Prove that the following schemas are theorems of any normal system (for any $n \geqslant 2$).

(*a*) $\Box(A_1 \wedge \ldots \wedge A_n) \leftrightarrow (\Box A_1 \wedge \ldots \wedge \Box A_n)$
(*b*) $\Diamond(A_1 \vee \ldots \vee A_n) \leftrightarrow (\Diamond A_1 \vee \ldots \vee \Diamond A_n)$
(*c*) $(\Box A_1 \vee \ldots \vee \Box A_n) \to \Box(A_1 \vee \ldots \vee A_n)$
(*d*) $\Diamond(A_1 \wedge \ldots \wedge A_n) \to (\Diamond A_1 \wedge \ldots \wedge \Diamond A_n)$
(*e*) $(\Box A_1 \wedge \ldots \wedge \Box A_{n-1} \wedge \Diamond A_n) \to \Diamond(A_1 \wedge \ldots \wedge A_n)$
(*f*) $\Box(A_1 \vee \ldots \vee A_n) \to (\Diamond A_1 \vee \ldots \vee \Diamond A_{n-1} \vee \Box A_n)$

4.9. Prove that the following sentences are theorems of any normal system whenever $m \leqslant n$.

(*a*) $\Diamond^n\top \to \Diamond^m\top$ (*b*) $\Box^m\bot \to \Box^n\bot$

4.10. Let Σ be any system of modal logic containing Df$\Diamond$ and satisfying the conditions that, for every $n \geqslant 0$,

(*a*) $\Box^n A \in \Sigma$ if $\vdash_{PL} A$,
(*b*) $\Box^n(\Box(A \to B) \to (\Box A \to \Box B)) \in \Sigma$,
(*c*) Σ is closed under the rule MP.

Prove that Σ is normal. (This boils down to a proof, by induction on n, that Σ is closed under the rule RN.)

4.11. Prove that every normal system has the following rule of inference, for any $k, m, n \geqslant 0$.

$$\frac{(A_1 \wedge \ldots \wedge A_m \wedge \Diamond^k B_1 \wedge \ldots \wedge \Diamond^k B_n) \to \bot}{(\Box A_1 \wedge \ldots \wedge \Box A_m) \to \Box^{k+1}\neg(B_1 \wedge \ldots \wedge B_n)}$$

4.12. Use the erasure transformation ϵ from exercise 1.27 to prove the consistency of the system K. (Alternatively, consider the mappings τ in exercises 1.11 and 3.16.)

4.13. Consider the following rules of inference.

$$(a)\ \frac{\Box A}{A} \qquad (d)\ \frac{\Diamond A}{A}$$

$$(b)\ \frac{\Box A \to \Box B}{A \to B} \qquad (e)\ \frac{\Diamond A \to \Diamond B}{A \to B}$$

$$(c)\ \frac{\Box A \leftrightarrow \Box B}{A \leftrightarrow B} \qquad (f)\ \frac{\Diamond A \leftrightarrow \Diamond B}{A \leftrightarrow B}$$

These rules hold for some normal systems, but not for all. To prove that they hold for the system K we first define the mapping σ, as follows.

(1) $\sigma(\mathbb{P}_n) = \mathbb{P}_n$, for $n = 0, 1, 2, \ldots$.
(2) $\sigma(\top) = \top$.
(3) $\sigma(\bot) = \bot$.
(4) $\sigma(\neg A) = \neg\sigma(A)$.
(5) $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$.
(6) $\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$.
(7) $\sigma(A \to B) = \sigma(A) \to \sigma(B)$.
(8) $\sigma(A \leftrightarrow B) = \sigma(A) \leftrightarrow \sigma(B)$.
(9) $\sigma(\Box A) = A$.
(10) $\sigma(\Diamond A) = A$.

So to speak, σ searches through a sentence – or schema – for its first, or outermost, occurrences of $\Box$ and $\Diamond$, and 'erases' them. Thus $\sigma(\Box A \to \Diamond A)$ is $A \to A$. Note that σ is not the same as ϵ in exercises 1.27 and 4.12: σ does not delete all occurrences of $\Box$ and $\Diamond$. For example, $\sigma(\Box A \to \Box\Box A)$ is $A \to \Box A$, not $A \to A$.

Now consider K as axiomatized by Df$\Diamond$, RK, and RPL. Prove by induction on the length (number of lines) of a proof, relative to this axiomatization, the following lemma.

If $\vdash_K A$, then $\vdash_K \sigma(A)$.

That is, prove that if A appears on the first line of a proof, then $\sigma(A)$ is also a K-theorem (this is the basis of the induction), and that, assuming that the result holds for all lines $k < n$, it holds as well for line n (this is the inductive step). (Take it for granted that the result holds with respect to RPL, i.e. that if A is a tautological consequence of $A_1, \ldots, A_n$, then $\sigma(A)$ is a tautological consequence of $\sigma(A_1), \ldots, \sigma(A_n)$.)

It follows from this lemma that K has rules (*a*)–(*f*). For example, for (*a*) we argue as follows. If $\vdash_K \Box A$, then by the lemma $\vdash_K \sigma(\Box A)$, and so – by the definition of σ – $\vdash_K A$.

Give the arguments for cases (*b*)–(*f*).

4.2. Replacement and duality

In this section we pause to state and prove some simple theorems about replacement and duality in normal modal logics. These principles function as theorems and rules of inference in every normal system, and where possible we present them as such. Their usefulness is illustrated by means of several examples.

THEOREM 4.7. *Every normal system of modal logic has the rule of replacement:*

$$\text{REP.}\quad \frac{B \leftrightarrow B'}{A \leftrightarrow A[B/B']}$$

(Recall from section 2.1 that $A[B/B']$ is any sentence that results from A by replacing zero or more occurrences of B, in A, by B′.)

Proof. Let Σ be a normal system, and suppose (throughout the proof) that $\vdash_\Sigma B \leftrightarrow B'$. Then what we wish to prove is that $\vdash_\Sigma A \leftrightarrow A[B/B']$.

We consider first the possibility that A and B are the same sentence. Then $A[B/B']$ is either A (when there is no replacement) or B′ (when A, i.e. B, is replaced by B′). In either case, $\vdash_\Sigma A \leftrightarrow A[B/B']$. For in the first case this is just $\vdash_\Sigma A \leftrightarrow A$, which is trivial; and in the second it is $\vdash_\Sigma B \leftrightarrow B'$, which is the assumption.

Thus we may assume henceforth that A and B are distinct.

The proof proceeds now by induction on the complexity of A. We give it for the cases in which A is (*a*) atomic, $\mathbb{P}_n$, (*b*) the falsum, $\bot$, (*c*) a conditional, $C \to D$, and (*d*) a necessitation, $\Box C$; the rest are left for the reader.

For (*a*). Given that $\mathbb{P}_n$ and B are distinct, $\mathbb{P}_n[B/B'] = \mathbb{P}_n$. So, $\vdash_\Sigma \mathbb{P}_n \leftrightarrow \mathbb{P}_n[B/B']$, trivially. So the theorem holds when A is atomic.

For (*b*). The argument is the same as for (*a*).

For the inductive cases (*c*) and (*d*) we make the hypothesis that the result holds for all sentences shorter than A.

For (*c*). By the inductive hypothesis, $\vdash_\Sigma C \leftrightarrow C[B/B']$ and $\vdash_\Sigma D \leftrightarrow D[B/B']$. It follows (by *PL*; the proof is left to the reader) that $\vdash_\Sigma (C \to D)$

$\leftrightarrow(C[B/B'] \to D[B/B'])$. But note that $(C \to D)[B/B'] = C[B/B'] \to D[B/B']$. Therefore, $\vdash_\Sigma (C \to D) \leftrightarrow (C \to D)[B/B']$. So the theorem holds when A is a conditional.

For (*d*). By the inductive hypothesis, $\vdash_\Sigma C \leftrightarrow C[B/B']$. By the rule RE it follows that $\vdash_\Sigma \Box C \leftrightarrow \Box(C[B/B'])$. However, $(\Box C)[B/B'] = \Box(C[B/B'])$. Therefore, $\vdash_\Sigma \Box C \leftrightarrow (\Box C)[B/B']$. So the theorem holds when A is a necessitation.

This ends the proof of theorem 4.7.

The use of the rule REP is illustrated in the following proof that the schema

$$\Diamond(A \to B) \leftrightarrow (\Box A \to \Diamond B)$$

is a theorem in any normal system of modal logic.

1.	$\Diamond(A \to B) \leftrightarrow \Diamond(\neg A \vee B)$	*PL* and REP
2.	$\leftrightarrow(\Diamond\neg A \vee \Diamond B)$	1, R$\Diamond$ and REP
3.	$\leftrightarrow(\neg\Diamond\neg A \to \Diamond B)$	2, *PL* and REP
4.	$\leftrightarrow(\Box A \to \Diamond B)$	3, Df$\Box$ and REP

This highly abbreviated proof needs some explanation. The justification of line 1 indicates a tacit use of REP in which, since $(A \to B) \leftrightarrow (\neg A \vee B)$ is a tautology, $\neg A \vee B$ replaces $A \to B$ in the tautology $\Diamond(A \to B) \leftrightarrow \Diamond(A \to B)$. In line 2, $\Diamond\neg A \vee \Diamond B$ replaces $\Diamond(\neg A \vee B)$ in line 1, in virtue of the theorem R$\Diamond$. Then in line 3, $\neg\Diamond\neg A \to \Diamond B$ replaces the tautologically equivalent $\Diamond\neg A \vee \Diamond B$ in line 2. Finally, in line 4 the theorem Df$\Box$ is used in replacing $\neg\Diamond\neg A$ by $\Box A$ in line 3.

Use of the rule REP is further illustrated in the proofs of theorems 4.8 and 4.10 below.

Let us turn now to the subject of duality (recall definition 2.4).

THEOREM 4.8. *Every normal system of modal logic has the following theorems and rules of inference, all referred to as* DUAL.

(1) $A \leftrightarrow \neg A^*$

(2) $\dfrac{A}{\neg A^*} \quad \dfrac{\neg A}{A^*}$

(3) $\dfrac{A \to B}{B^* \to A^*}$

(4) $\dfrac{A \leftrightarrow B}{A^* \leftrightarrow B^*}$

Proof. We assume throughout that Σ is a normal modal logic.

For (1). The proof here is by induction on the complexity of A. Let us treat the cases in which A is (*a*) atomic, $\mathbb{P}_n$, (*b*) the falsum, $\bot$, (*c*) a conjunction, $B \wedge C$, and (*d*) a necessitation, $\Box B$.

For (*a*):

1.	$\mathbb{P}_n \leftrightarrow \neg\neg\mathbb{P}_n$	*PL*
2.	$\leftrightarrow \neg\mathbb{P}_n^*$	1, definition 2.4(1)

So the theorem holds when A is atomic.

For (*b*):

1.	$\bot \leftrightarrow \neg\top$	*PL*
2.	$\leftrightarrow \neg\bot^*$	1, definition 2.4(3)

So the theorem holds when A is the falsum.

For the inductive cases (*c*) and (*d*), we make the hypothesis that the theorem holds for sentences shorter than A. Thus, $\vdash_\Sigma B \leftrightarrow \neg B^*$ and $\vdash_\Sigma C \leftrightarrow \neg C^*$.

For (*c*):

1.	$(B \wedge C) \leftrightarrow (\neg B^* \wedge \neg C^*)$	inductive hypothesis and REP
2.	$\leftrightarrow \neg(B^* \vee C^*)$	1, *PL* and REP
3.	$\leftrightarrow \neg(B \wedge C)^*$	2, definition 2.4(5)

So the theorem holds when A is a conjunction.

For (*d*):

1.	$\Box B \leftrightarrow \Box(\neg B^*)$	inductive hypothesis and REP
2.	$\leftrightarrow \neg\Diamond(B^*)$	1, Df$\Diamond$ and REP
3.	$\leftrightarrow \neg(\Box B)^*$	2, definition 2.4(9)

So the theorem holds when A is a necessitation.

This concludes the proof of (1). Parts (2)–(4) are corollaries.

For (2). It follows at once from (1) that if $\vdash_\Sigma A$, then $\vdash_\Sigma \neg A^*$. So Σ is closed under the first rule DUAL in (2). For the second, it is enough to note that (1) means that $\neg A \leftrightarrow A^*$ is always a theorem of Σ.

For (3). If $\vdash_\Sigma A \to B$, then $\vdash_\Sigma \neg A^* \to \neg B^*$, by (1) and REP. Hence by *PL*, $\vdash_\Sigma B^* \to A^*$. So Σ is closed under the rule DUAL in (3).

For (4). We leave this as an exercise.

This concludes the proof of theorem 4.8.

As an example of the use of DUAL, let us see that $\Diamond\top \leftrightarrow \neg\Box\bot$ is a theorem of every normal system. For by DUAL(1), $\Diamond\top \leftrightarrow \neg(\Diamond\top)^*$ is, and by definition 2.4, $(\Diamond\top)^* = \Box\bot$.

Similarly, using DUAL(2) one can show that every normal system Σ has $\neg(\Box A \wedge \Box\neg A)$ as a theorem if it has $\Diamond A \vee \Diamond\neg A$. (Note that neither schema is a theorem of every normal system, however.) For suppose that $\vdash_\Sigma \Diamond A \vee \Diamond\neg A$. Then also, $\vdash_\Sigma \Diamond\neg A \vee \Diamond\neg\neg A$. Hence by DUAL(2), $\vdash_\Sigma \neg(\Diamond\neg A \vee \Diamond\neg\neg A)^*$. But by definition 2.4 this means that $\vdash_\Sigma \neg(\Box(\neg A^*) \wedge \Box\neg(\neg A^*))$. So by DUAL(1) and REP, $\vdash_\Sigma \neg(\Box A \wedge \Box\neg A)$.

Finally, let us show that since $(\Box A \vee \Box B) \to \Box(A \vee B)$ is always a theorem of a normal system, so is $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$. The proof uses DUAL(3):

1.	$(\Box\neg A \vee \Box\neg B) \to \Box(\neg A \vee \neg B)$	theorem
2.	$(\Box(\neg A \vee \neg B))^* \to (\Box\neg A \vee \Box\neg B)^*$	1, DUAL(3)
3.	$\Diamond(\neg A^* \wedge \neg B^*) \to (\Diamond(\neg A^*) \wedge \Diamond(\neg B^*))$	2, definition 2.4
4.	$\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$	3, DUAL(1) and REP

In our last theorems of this section we state some simple principles concerning *duals of modalities*. Recall that a modality ϕ is a finite (possibly null) sequence of the operators $\neg$, $\Box$, and $\Diamond$, and that the dual of a modality ϕ – written ϕ^* – is the result of interchanging $\Box$ and $\Diamond$ throughout ϕ (see section 2.1).

THEOREM 4.9. *Let Σ be a normal system of modal logic. Then:*

(1) $\vdash_\Sigma \phi A \leftrightarrow \neg\phi^*\neg A$.

(2) $\vdash_\Sigma \phi A$ *iff* $\vdash_\Sigma \neg\phi^*\neg A$.

(3) $\vdash_\Sigma \phi A \to \psi A$, *for every* A, *iff* $\vdash_\Sigma \psi^* A \to \phi^* A$, *for every* A.

(4) $\vdash_\Sigma \phi A \leftrightarrow \psi A$, *for every* A, *iff* $\vdash_\Sigma \phi^* A \leftrightarrow \psi^* A$, *for every* A.

Proof. Let Σ be a normal system. The result is a corollary to theorem 4.8.

For (1):

1.	$\phi A \leftrightarrow \neg(\phi A)^*$	DUAL(1)
2.	$\leftrightarrow \neg\phi^*(A^*)$	1, definition of *
3.	$\leftrightarrow \neg\phi^*\neg A$	2, DUAL(1) and REP

For (2). This follows easily from (1). Exercise.

For (3):

$\vdash_\Sigma \phi A \to \psi A$, for every A, iff $\vdash_\Sigma \neg\phi^*\neg A \to \neg\psi^*\neg A$, for every A
– (1) and REP;

iff $\vdash_\Sigma \psi^*\neg A \to \phi^*\neg A$, for every A
– *PL*;

iff $\vdash_\Sigma \psi^* A \to \phi^* A$, for every A
– *PL* and REP.

For (4). This follows from (3). Exercise.

As an application of theorem 4.9 we see that a normal modal logic contains the schema 4, $\Box A \to \Box\Box A$, as a theorem just in case it contains the dual schema 4$\Diamond$, $\Diamond\Diamond A \to \Diamond A$. Similarly, the schema B, $A \to \Box\Diamond A$, is a theorem of a normal system if and only if its dual B$\Diamond$, $\Diamond\Box A \to A$, is. (Of course not every normal system contains these schemas as theorems.)

Our final theorem is a rather obvious consequence of the preceding one. We set it out primarily in order to simplify the discussion in the next few sections. Recall that an affirmative modality contains an even number of occurrences of $\neg$.

THEOREM 4.10. *Let Σ be a normal system of modal logic, and let ϕ and ψ be affirmative modalities. Then Σ has the schema*

S. $\phi A \to \psi A$

as a theorem iff Σ has any one of the following theorem and rules of inference.

S$\Diamond$. $\psi^* A \to \phi^* A$

RS. $\dfrac{A \to B}{\phi A \to \psi B}$

RS$\Diamond$. $\dfrac{A \to B}{\psi^* A \to \phi^* B}$

Proof. Let Σ be a normal modal logic, and let ϕ and ψ be affirmative modalities. For the sake of simplicity we assume that ϕ and ψ are in fact composed solely of the operators $\Box$ and $\Diamond$, so that $\neg$ does not appear.

For S$\Diamond$. This follows from theorem 4.9 (3).

For RS. Suppose that $\vdash_\Sigma A \to B$. Then by repeated applications of the rules RM and RM$\Diamond$, $\vdash_\Sigma \psi A \to \psi B$. So if $\vdash_\Sigma \phi A \to \psi A$, then by *PL*, $\vdash_\Sigma \phi A \to \psi B$. Thus Σ has RS if it has S. Conversely, suppose Σ is closed under RS. Then $\vdash_\Sigma \phi A \to \psi A$, by RS on the tautology $A \to A$. So Σ has S if it has RS.

For RS$\Diamond$. Exercise.

Theorem 4.10 is illustrated by the fact that a normal system of modal logic has the schema 5, $\Diamond A \to \Box\Diamond A$, as a theorem if and only if it has its dual 5$\Diamond$, $\Diamond\Box A \to \Box A$, or either of these rules of inference:

$$\text{R5.}\quad \frac{A \to B}{\Diamond A \to \Box\Diamond B}$$

$$\text{R5}\Diamond.\quad \frac{A \to B}{\Diamond\Box A \to \Box B}$$

The theorems in this section afford the reader a handy means of recognizing theorems of normal systems. It is not so important, at this point, that the details of the several proofs be mastered and absorbed. It is worth remarking, however, that the proofs of theorems 4.7, 4.8, and 4.9 (but not 4.10) all depend ultimately only on *PL* and the presence of RE and Df$\Diamond$ (or RE$\Diamond$ and Df$\Box$) in normal systems of modal logic. This becomes important in chapter 8, where we return to these results.

EXERCISES

4.14. Complete the proof of theorem 4.7 (for the cases in which $A = \top$, $\neg C$, $C \wedge D$, $C \vee D$, $C \leftrightarrow D$, $\Diamond C$).

4.15. Complete the proof of theorem 4.8 (part (1) – for the cases in which $A = \top$, $\neg B$, $B \vee C$, $B \to C$, $B \leftrightarrow C$, $\Diamond B$ – and part (4)).

4.16. Complete the proof of theorem 4.9 (parts (2) and (4)).

4.17. Give the proof of theorem 4.10 for RS$\Diamond$.

4.18. Prove that if Σ is a system of modal logic closed under the rule REP, then Σ contains Df$\Diamond$ if and only if Σ contains Df$\Box$.

4.19. Prove that a system of modal logic is normal if it contains Df$\Diamond$, N, K, and is closed under REP.

4.20. Prove that $A \leftrightarrow A^{**}$ is a theorem of any normal modal logic.

4.21. Use REP and DUAL (and perhaps the result in the preceding exercise) to prove that N$\Diamond$, M$\Diamond$, C$\Diamond$, R$\Diamond$, and K$\Diamond$ are theorems of every normal system given that N, M, C, R, and K are. Then prove the reverse of this, i.e. that N etc. are theorems of every normal system given that N$\Diamond$ etc. are.

4.3. The schemas D, T, B, 4, and 5

The smallest normal system of modal logic, *K*, contains as theorems just what comes from Df$\Diamond$, RK, and propositional logic, nothing more. Thus we have canvassed the principal rules and theorems of the system *K* already in the preceding sections.

In this and the next section we are interested in the normal extensions of *K* obtained by adding as theorems the following schemas.

D. $\Box A \to \Diamond A$

T. $\Box A \to A$

B. $A \to \Box\Diamond A$

4. $\Box A \to \Box\Box A$

5. $\Diamond A \to \Box\Diamond A$

Including *K* itself there are just fifteen distinct normal systems produced by taking these schemas as theorems in all possible combinations. These systems appear on the diagram in figure 4.1.

The inclusions among the systems on the diagram are marked by lines: extensions of a system are reached by going in a rightward direction along the lines (for example, *KT* is shown to be an extension of *KD*). Most of the inclusions are obvious; some of those that are not we shall establish, and others are given as exercises. Likewise it is possible to show that each of the seventeen systems apparently not registered on the diagram is identical with one that is. Indeed, many of these identities are obvious from the diagram – for example, that *KDT* is the same as *KT*. The distinctness of the systems listed – and so the properness of the inclusions – is proved in chapter 5.

Historically the most important of these systems are *KD*, *KT*, *KTB*, *KT4*, and *KT5*. The first two are widely regarded as basic deontic and alethic modal logics, respectively, and are sometimes referred to simply as *D* and *T*. The other three systems – *KTB*, *KT4*, and *KT5* – are the well-known *Brouwersche* system (sometimes called *B*) and the Lewis systems *S4* and *S5*. Nevertheless, we approach these logics more analytically, by focusing on the systems *KD*, *KT*, *KB*, *K4*, *K5*, and their normal extensions. We begin with the following theorem about some alternative characterizations of these systems.

THEOREM 4.11. *Let Σ be a normal system of modal logic. Then:*

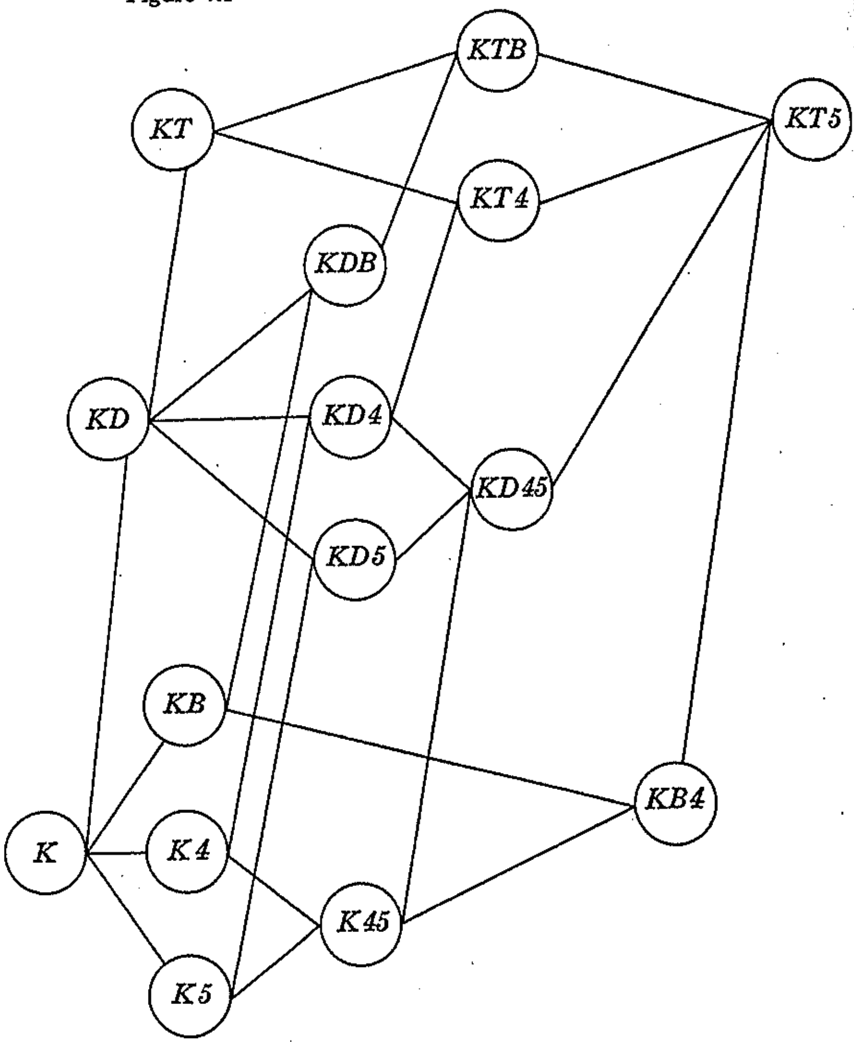
(1) *Σ is a KD-system iff it has* RD.

(2) *Σ is a KT-system iff it has any of* T$\Diamond$, RT, *and* RT$\Diamond$.

(3) Σ *is a KB-system iff it has any of* B$\Diamond$, RB, *and* RB$\Diamond$.
(4) Σ *is a K4-system iff it has any of* 4$\Diamond$, R4, *and* R4$\Diamond$.
(5) Σ *is a K5-system iff it has any of* 5$\Diamond$, R5, *and* R5$\Diamond$.

Proof. The theorem is an immediate consequence of theorem 4.10 and the fact that the modalities in the schemas D, T, B, 4, and 5 are all affirmative.

Figure 4.1



In what follows we freely make use of these theorems and rules of inference wherever appropriate. The reader should consult theorem 4.10 to ascertain their identities.

Now let us examine in turn each of the systems *KD*, *KT*, *KB*, *K4*, and *K5*, and their normal extensions.

Normal *KD*-systems. These come in many guises, as theorems 4.12 and 4.13 reveal.

THEOREM 4.12. *A normal system of modal logic is a KD-system iff it has any of the following theorems and rules of inference.*

RP. $\dfrac{A}{\Diamond A}$

P. $\Diamond\top$

O. $\Diamond A \vee \Diamond\neg A$

RP$\Box$. $\dfrac{\neg A}{\neg\Box A}$

P$\Box$. $\neg\Box\bot$

O$\Box$. $\neg(\Box A \wedge \Box\neg A)$

Proof. Suppose that Σ is a normal system.
For RP:

1.	A	hypothesis
2.	$\Box A$	1, RN
3.	$\Box A \to \Diamond A$	D
4.	$\Diamond A$	2, 3, *PL*

Thus Σ is closed under RP if it is a *KD*-system. For the reverse, suppose that Σ has RP. Then $\vdash_\Sigma \Diamond(A\to A)$, by RP on the tautology $A\to A$. To see from this that $\vdash_\Sigma \Box A\to\Diamond A$, it is enough to recall that the schema

$$\Diamond(A\to B)\leftrightarrow(\Box A\to\Diamond B)$$

is a theorem of every normal system (see following the proof of theorem 4.7). So Σ is a *KD*-system if it has the rule RP.

For P. Every normal modal logic has the theorem

$$\Diamond\top\leftrightarrow(\Box A\to\Diamond A)$$

(exercise 4.7(m)). So P is a theorem of Σ if and only if D is, which means that Σ is a *KD*-system just in case it contains P.

For O:

$$\vdash_\Sigma \Box A \to \Diamond A \text{ iff } \vdash_\Sigma \neg\Diamond\neg A \to \Diamond A \quad \text{– Df}\Box \text{ and REP;}$$
$$\text{iff } \vdash_\Sigma \Diamond A \vee \Diamond\neg A \quad \text{– } PL.$$

So Σ is a *KD*-system if and only if it contains O. More generally, this follows from the fact that every normal system has the theorem

$$(\Box A \to \Diamond B) \leftrightarrow (\Diamond B \vee \Diamond\neg A).$$

For RP$\Box$, P$\Box$, and O$\Box$. These principles are dual to RP, P, and O, and we leave the proofs as exercises. (See the examples after the proof of theorem 4.8.)

This completes our proof of theorem 4.12.

The theorem D admits of generalization along the modal dimension. To wit, for every $k > 0$ the schema

$$\text{D}^k. \quad \Box^k A \to \Diamond^k A$$

is a theorem of a normal *KD*-system. For D^k is D itself when $k = 1$. And if we suppose (as an inductive hypothesis) that the schema is a theorem when it has fewer than k $\Box$s and $\Diamond$s, then we can argue that D^k is, too. Thus:

1.	$\Box^{k-1}A \to \Diamond^{k-1}A$	inductive hypothesis
2.	$\Box\Box^{k-1}A \to \Diamond\Diamond^{k-1}A$	1, RD
3.	$\Box^k A \to \Diamond^k A$	2, definition 2.3

Therefore, D^k is a theorem of every normal *KD*-system, for every $k > 0$.

From this result one can readily perceive analogous generalizations of the theorems and rules of normal *KD*-systems in theorems 4.11 (1) and 4.12. That is to say, the results of putting $\Box^k$ and $\Diamond^k$ for $\Box$ and $\Diamond$ throughout RD, RP, P, O, RP$\Box$, P$\Box$, and O$\Box$ are all principles of any normal *KD*-system, for every $k > 0$.

Moreover, the reverse is true. If a normal system has D^k or any one of these generalizations – RD^k, RP^k, P^k, O^k, RP$\Box^k$, P$\Box^k$, and O$\Box^k$ – for any $k > 0$, then it is a *KD*-system. We may illustrate this by showing that if

$$\text{P}^k. \quad \Diamond^k \top$$

is a theorem of a normal modal logic, for $k > 0$, then so is P, $\Diamond\top$, and so the modal logic is a *KD*-system. The proof:

1.	$\Diamond^{k-1}\top \to \top$	*PL*
2.	$\Diamond\Diamond^{k-1}\top \to \Diamond\top$	1, RM$\Diamond$
3.	$\Diamond^k\top \to \Diamond\top$	2, definition 2.3
4.	$\Diamond^k\top$	P^k
5.	$\Diamond\top$	3, 4, *PL*

Therefore, a normal modal logic is a *KD*-system whenever it contains P^k, for any $k > 0$.

These proofs should be enough to convince the reader of the correctness of these alternative ways of characterizing normal *KD*-systems of modal logic. We state this formally as a theorem and leave the remaining proofs as exercises.

THEOREM 4.13. *A normal system of modal logic is a KD-system iff it has any of the theorems and rules of inference* D^k, RD^k, RP^k, P^k, O^k, RP$\Box^k$, P$\Box^k$, *and* O$\Box^k$, *for any* $k > 0$.

Normal *KT*-systems. The schema D is a theorem of any modal logic containing T and T$\Diamond$ ($\Box A \to \Diamond A$ follows by *PL* from $\Box A \to A$ and $A \to \Diamond A$). Therefore:

THEOREM 4.14. *Every normal KT-system is a KD-system.*

Thus all the principles mentioned in theorems 4.11 (1), 4.12, and 4.13 are present in any normal *KT*-system of modal logic. (This is not true the other way around, as we shall prove.)

The theorem T can be generalized modally; i.e. the schema

$$\text{T}^k. \quad \Box^k A \to A$$

is a theorem of every normal *KT*-system, for every $k > 0$. The inductive proof of this is left to the reader as an exercise. Thus, in virtue of theorem 4.10:

THEOREM 4.15. *Every normal KT-system has the theorems and rules of inference* T^k, T$\Diamond^k$, RT^k, *and* RT$\Diamond^k$, *for every* $k > 0$.

The name *T* for the logic *KT* derives from the designation *logique t* of Feys. The system is also called *M*, following von Wright.

Normal *KB*-systems. We begin by noting some recondite ways of characterizing systems of this kind.

THEOREM 4.16. *A normal system of modal logic is a KB-system iff it has any of the following theorems and rules of inference.*

X. $\Box(\Diamond A \to B) \to (A \to \Box B)$

X$\Diamond$. $\Box(A \to \Box B) \to (\Diamond A \to B)$

RX. $\dfrac{\Diamond A \to B}{A \to \Box B}$

RX$\Diamond$. $\dfrac{A \to \Box B}{\Diamond A \to B}$

Proof

For X:

1. $\Box(\Diamond A \to B) \to (\Box\Diamond A \to \Box B)$ K
2. $A \to \Box\Diamond A$ B
3. $\Box(\Diamond A \to B) \to (A \to \Box B)$ 1, 2, *PL*

So a normal *KB*-system contains X. For the reverse, note that $\Box(\Diamond A \to \Diamond A) \to (A \to \Box\Diamond A)$ is a special case of X and that the antecedent is a theorem by RN on the tautology $\Diamond A \to \Diamond A$. By *PL*, then, B is a theorem. So a normal logic is a *KB*-system if X is a theorem.

For X$\Diamond$. Exercise.

For RX:

1. $\Diamond A \to B$ hypothesis
2. $\Box\Diamond A \to \Box B$ 1, RM
3. $A \to \Box\Diamond A$ B
4. $A \to \Box B$ 2, 3, *PL*

So a normal *KB*-system has the rule RX. (Alternatively, if $\Diamond A \to B$ is a theorem so is $\Box(\Diamond A \to B)$ (by RN) – whence $A \to \Box B$ is a theorem by MP on X.) Conversely, a normal system closed under RX has the theorem $A \to \Box\Diamond A$, by RX on the tautology $\Diamond A \to \Diamond A$. So a normal modal logic is a *KB*-system if it has the rule RX.

For RX$\Diamond$. Exercise.

The theorem B can be generalized modally in two ways. In the first, the operators $\Box$ and $\Diamond$ are each iterated k times, for $k > 0$:

B^k. $A \to \Box^k\Diamond^k A$

To prove that every normal *KB*-system has B^k for every $k > 0$, notice first that B^k = B, for $k = 1$, and then suppose as an inductive hypothesis

that the schema is a theorem whenever it has fewer than k $\Box$s and $\Diamond$s. Then:

1. $\Diamond A \to \Box^{k-1}\Diamond^{k-1}\Diamond A$ inductive hypothesis
2. $A \to \Box\Box^{k-1}\Diamond^{k-1}\Diamond A$ 1, RX
3. $A \to \Box^k\Diamond^k A$ 2, definition 2.3 and exercise 2.6

In the second way of generalizing B, the modality $\Box\Diamond$ itself is iterated k times, for $k > 0$:

B( )^k. $A \to (\Box\Diamond)^k A$

We leave it to the reader to prove, inductively, that every normal *KB*-system contains B( )^k for every $k > 0$.

By means of theorem 4.10 the principles in theorem 4.11 (3) can similarly be generalized, and so can the rules RX and RX$\Diamond$ in theorem 4.26. We record all these generalizations formally.

THEOREM 4.17. *Every normal KB-system has the theorems and rules of inference* B^k, B$\Diamond^k$, RB^k, RB$\Diamond^k$, RX^k, RX$\Diamond^k$, B( )^k, B$\Diamond$( )^k, RB( )^k, *and* RB$\Diamond$( )^k, *for every* $k > 0$.

According to the next theorem the schema 4 is a theorem of a normal *KB*-system just in case the schema 5 is.

THEOREM 4.18. *A normal modal logic is a KB4-system iff it is a KB5-system.*

Proof. To show that 5 is a *KB4*-theorem we may argue as follows.

1. $\Diamond\Diamond A \to \Diamond A$ 4$\Diamond$
2. $\Diamond A \to \Box\Diamond A$ 1, RX

And to show that 4 is a *KB5*-theorem we may argue as follows.

1. $\Diamond\Box A \to \Box A$ 5$\Diamond$
2. $\Box A \to \Box\Box A$ 1, RX

In particular, then, the systems *KB4* and *KB5* are identical. (Our choice of the designation *KB4* in the diagram in figure 4.1 is thus somewhat arbitrary.)

The schema B is called the *Brouwersche* axiom for the curious reason that when it is stated equivalently as

$$A \to \neg\Diamond\neg\Diamond A$$

and the modality $\neg\Diamond$ is replaced by the intuitionistic negation sign, $\sim$, the result is

$$A \rightarrow \sim\sim A,$$

the intuitionistically valid version of the law of double negation. Brouwer was a leading exponent of intuitionism. So far as is known, however, Brouwer had no concern with the modal schema B; the name *Brouwersche* was given by Becker. The *Brouwersche system*, it should be noted, is *KTB*, not *KB*.

Normal *K4*-systems. The important modal generalization of the schema 4 is

$$4^k.\quad \Box A \rightarrow \Box^k \Box A.$$

This is a theorem of every normal *K4*-system, for any $k > 0$. The proof is left as an exercise. Hence by theorem 4.10:

THEOREM 4.19. *Every normal K4-system has the theorems and rules of inference* 4^k, $4\Diamond^k$, $R4^k$, *and* $R4\Diamond^k$, *for every* $k > 0$.

An interesting feature of normal *K4*-systems is that in them it is inconsistent to hold that every proposition is at least possibly possible, i.e. that the schema

$$\Diamond\Diamond A$$

is a theorem. For in conjunction with $4\Diamond$ this would lead to

$$\Diamond A,$$

and so in particular to

$$\Diamond\bot,$$

which conflicts with $N\Diamond$, $\neg\Diamond\bot$.

The schema 4 is often called the characteristic theorem of the system *S4*. But note that *S4* is *KT4*, which is stronger than *K4*.

Normal *K5*-systems. These all contain, for every $k > 0$, the schema

$$5^k.\quad \Diamond A \rightarrow \Box^k \Diamond A.$$

(Again we leave the proof to the reader.) Hence:

THEOREM 4.20. *Every normal K5-system has the theorems and rules of inference* 5^k, $5\Diamond^k$, $R5^k$, *and* $R5\Diamond^k$, *for every* $k > 0$.

As the diagram in figure 4.1 shows, the strongest normal system that can be formed using the schemas D, T, B, 4, and 5 is *KT5* – better known as the Lewis system *S5* – which we discussed in chapter 1. (Thus the schema 5 or $5\Diamond$ is often referred to as the characteristic theorem of *S5*.) There are many ways of axiomatizing *S5*. The next theorem gives the principal axiomatizations of *S5* using D, T, B, 4, and 5; of course duality (for example, putting $T\Diamond$ for T) provides many more possibilities.

THEOREM 4.21. *A normal modal logic is a KT5-system iff it has as theorems* (1) T, B, *and* 4, (2) D, B, *and* 4, *or* (3) D, B, *and* 5. *In particular, then, KT5 = KTB4 = KDB4 = KDB5.*

Proof. Part (1) was established in chapter 1. In light of this and theorems 4.14 and 4.18 it is then sufficient to show that T is a theorem of every normal *KDB4*-system. We leave the details of the reasoning as an exercise.

This is a good place to affirm the correctness of figure 4.1 with respect to the inclusions advertised there. For the most part this is a matter of definition – for example, *KTB* is obviously an extension of *KT*. For the rest, note that $KD \subseteq KT$, and so $KDB \subseteq KTB$ and $KD4 \subseteq KT4$, by theorem 4.15; that $K45 \subseteq KB4$ by theorem 4.18; and that by theorem 4.21 *KT5* is an extension of *KTB*, *KT4*, *KD45*, and *KB4*. Several of the seventeen systems apparently missing from figure 4.1 have already been mentioned, for example, in the alternative axiomatizations of *S5* in theorem 4.21. We leave it as an exercise for the reader to identify all the missing systems and locate them in figure 4.1.

We might also remark here that although the system *KD* results from the addition to *K* of D^k (or any other of the principles listed in theorem 4.13) for any $k > 0$, there is no analogous result with respect to the modal generalizations of T, B, 4, and 5, for $k > 1$. We shall be in a position to prove this in chapter 5.

The point of our analytical approach in this section may by now be apparent. It enables us to see better the individual contributions of the schemas D, T, B, 4, and 5 to more familiar modal logics such as *KTB*, *KT4*, and *KT5*. Two examples will make this clear. First, it is often pointed out that the rules of inference RX and $RX\Diamond$ are present in the *Brouwersche* system, *KTB*. But as we have seen, these rules are already in the modal logic *KB* (and hence in any normal *KB*-system); the theorem T has no bearing on the matter. Second, the result that the schema $\Diamond\Diamond A$ is inconsistent as an addition to the Lewis system *S4* (*KT4*) is frequently mentioned. But our analytical exposition shows this to be so with respect to *K4* and normal *K4*-systems generally. Here again the presence of the theorem T is of no consequence.

EXERCISES

4.22. Use theorem 4.10 to ascertain the identities of the schemas and rules of inference mentioned in theorem 4.11.

4.23. Complete the proof of theorem 4.12 (for RP$\Box$, P$\Box$, and O$\Box$).

4.24. Complete the proof of theorem 4.13.

4.25. Prove by induction that, for any $k > 0$, the schema T^k is a theorem of every normal *KT*-system (for theorem 4.15).

4.26. Complete the proof of theorem 4.16 (for X$\Diamond$ and RX$\Diamond$).

4.27. Prove by induction that, for any $k > 0$, the schema $B(\ )^k$ is a theorem of every normal *KB*-system (for theorem 4.17).

4.28. Prove by induction that, for any $k > 0$, the schema 4^k is a theorem of every normal *K4*-system (for theorem 4.19).

4.29. Prove by induction that, for any $k > 0$, the schema 5^k is a theorem of every normal *K5*-system (for theorem 4.20).

4.30. Complete the proof of theorem 4.21 by proving that the schema T is a theorem of any normal *KDB4*-system.

4.31. Identify and locate on the diagram in figure 4.1 the seventeen systems not already listed there.

4.32. Prove that a normal modal logic is a *KD*-system if and only if it has theorems of the form $\Diamond A$.

4.33. Consider the following schemas.

U.	$\Box(\Box A \to A)$	U$\Diamond$.	$\Box(A \to \Diamond A)$
4_c.	$\Box\Box A \to \Box A$	$4\Diamond_c$.	$\Diamond A \to \Diamond\Diamond A$
5_c.	$\Box\Diamond A \to \Diamond A$	$5\Diamond_c$.	$\Box A \to \Diamond\Box A$

Prove:

(*a*) U is a theorem of a normal system if and only if U$\Diamond$ is.

(*b*) 4_c (and hence $4\Diamond_c$) is a theorem of any normal *KU*-system.

(*c*) U (and hence 4_c) and 5_c (and hence $5\Diamond_c$) are theorems of any normal *KT*-system.

(*d*) D is a theorem of any normal *K5*$_c$-system.

4.34. Prove that the schema G, $\Diamond\Box A \to \Box\Diamond A$, is a theorem of any normal *KB*-system.

4.35. Prove that any normal *KB*-system is closed under the following rules of inference.

$$\frac{\Box A}{\Diamond\top \to A} \qquad \frac{\Diamond\top \to A}{\Box A}$$

4.36. Prove that a normal system of modal logic is a *K4*-system if and only if it has any of the following theorems.

(*a*) $\Box(A \to B) \to \Box(\Box A \to \Box B)$

(*b*) $(\Box A \vee \Box B) \to \Box(\Box A \vee \Box B)$

(*c*) $\Box(\Box(A \to B) \to C) \to \Box(\Box(A \to B) \to \Box C)$

4.37. Prove that a normal system of modal logic is a *K5*-system if and only if it has either of the following theorems.

(*a*) $\Box(\Box A \vee B) \to (\Box A \vee \Box B)$

(*b*) $(\Diamond A \wedge \Diamond B) \to \Diamond(\Diamond A \wedge B)$

4.38. Prove:

(*a*) U (and hence 4_c) is a theorem of any normal *K5*-system (see exercise 4.33).

(*b*) G is a theorem of any normal *K5*-system.

4.39. Prove that every normal *K5*-system contains the following theorems.

(*a*) $\Box(\Box A \leftrightarrow \Box\Box A)$	(*b*) $\Box(\Diamond A \leftrightarrow \Diamond\Diamond A)$
(*c*) $\Box(\Box A \leftrightarrow \Diamond\Box A)$	(*d*) $\Box(\Diamond A \leftrightarrow \Box\Diamond A)$

4.40. Referring to the preceding exercise, prove that every normal *K5*-system contains the following theorems.

(*a*) $\Box\Box A \leftrightarrow \Box\Box\Box A$	(*e*) $\Diamond\Diamond A \leftrightarrow \Diamond\Diamond\Diamond A$
(*b*) $\Box\Box A \leftrightarrow \Box\Diamond\Box A$	(*f*) $\Diamond\Diamond A \leftrightarrow \Diamond\Box\Diamond A$
(*c*) $\Diamond\Box A \leftrightarrow \Diamond\Box\Box A$	(*g*) $\Box\Diamond A \leftrightarrow \Box\Diamond\Diamond A$
(*d*) $\Diamond\Box A \leftrightarrow \Diamond\Diamond\Box A$	(*h*) $\Box\Diamond A \leftrightarrow \Box\Box\Diamond A$

4.41. Notice that the interiors of the four necessitations listed in exercise

4.39 are all theorems of any normal *KT5*-system. This suggests the following result (which leads at once to a solution for exercise 4.39). Where Σ is any normal *K5*-system:

$$\vdash_\Sigma \Box A, \text{ whenever A is a theorem of } KT5.$$

Prove this by induction on (the set of theorems of) *KT5*. For the basis, show that the necessitations of the axioms Df$\Diamond$, T, and 5 are theorems of Σ; for the inductive part, show that the set $\{A : \vdash_\Sigma \Box A\}$ is closed under the rules RPL and RK.

4.42. Prove that every normal *K5*-system contains the following theorems.

(*a*) $\Diamond\Box A \to \Diamond A$ (*b*) $\Box A \to \Box\Diamond A$
(*c*) $\Diamond\Box A \to \Box\Box A$ (*d*) $\Diamond\Diamond A \to \Box\Diamond A$

4.43. Prove:

(*a*) 5_c and $5\Diamond_c$ (see exercise 4.33) are theorems of any normal *KD4*-system.

(*b*) The schemas $\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A$ and $\Box\Diamond A \leftrightarrow \Box\Diamond\Box\Diamond A$ are theorems of any normal *KD4*-system.

4.44. Prove that every normal *KD5*-system contains the following theorems.

(*a*) $\Box\Box A \leftrightarrow \Diamond\Box A$ (*b*) $\Diamond\Diamond A \leftrightarrow \Box\Diamond A$

4.45. Prove that every normal *K45*-system contains the following theorems.

4!. $\Box A \leftrightarrow \Box\Box A$ 4$\Diamond$!. $\Diamond A \leftrightarrow \Diamond\Diamond A$

4.46. Consider the following schemas.

5!. $\Diamond A \leftrightarrow \Box\Diamond A$ 5$\Diamond$!. $\Box A \leftrightarrow \Diamond\Box A$

Prove that 4! and 4$\Diamond$! (see the preceding exercise) are theorems of any normal *K5!*- or normal *K5*$\Diamond$*!*-system.

4.47. Prove that 4!, 4$\Diamond$!, 5!, and 5$\Diamond$! are theorems of any normal *KD45*-system (see exercises 4.43(*a*), 4.45, and 4.46).

4.48. Let us say that a sentence A is ***fully modalized*** just in case every atomic sentence in A is within the scope of an occurrence of $\Box$ or $\Diamond$. Show that where Σ is any normal *KD45*-system and A is fully modalized:

$$\vdash_\Sigma A \leftrightarrow \Box A \quad \text{and} \quad \vdash_\Sigma A \leftrightarrow \Diamond A.$$

The proof is by induction on the complexity of A.

4.49. Consider the following rules of inference.

RFM. $\dfrac{A \to B}{A \to \Box B}$, where A is fully modalized

RFM$\Diamond$. $\dfrac{A \to B}{\Diamond A \to B}$, where A is fully modalized

Using the results in the preceding exercise, prove:

(*a*) A normal system is closed under RFM if and only if it is closed under RFM$\Diamond$.

(*b*) Every normal *KD45*-system is closed under RFM and RFM$\Diamond$.

(*c*) The schemas 4 and 5 are theorems of any normal *KD*-system closed under RFM or RFM$\Diamond$.

4.50. Prove that the schema $\Box(\Diamond A \to B) \leftrightarrow \Box(A \to \Box B)$ is a theorem of any normal *KB4*-system.

4.51. Prove that if any of the schemas B, $\Diamond A \to \Diamond\Box\Diamond A$, and $\Box\Diamond\Box A \to \Box A$ is a theorem of a normal *KT*-system, then so are the others.

4.52. Consider the following schemas.

T_c. $A \to \Box A$ T$\Diamond_c$. $\Diamond A \to A$
D_c. $\Diamond A \to \Box A$
F. $\Box(A \vee B) \to (\Box A \vee \Box B)$ F$\Diamond$. $(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$

Prove:

(*a*) T_c is a theorem of a normal system if and only if T$\Diamond_c$ is.

(*b*) If any one of D_c, $\Box A \vee \Box\neg A$, $\neg(\Diamond A \wedge \Diamond\neg A)$, F, and F$\Diamond$ is a theorem of a normal system, then so are all the others.

(*c*) D_c (and hence the rest in (*b*)) is a theorem of any normal *KT*$_c$-system.

(*d*) B, 4, 5, and G are theorems of any normal *KT*$_c$-system.

(*e*) T is a theorem of any normal *KDT*$_c$-system.

(*f*) T_c is a theorem of any normal *KD*$_c$*T*-system.

(*g*) 4 and 4_c are theorems of any normal *KD*$_c$*5*$_c$-system.

4.53. Consider the following schemas.

T!. $\Box A \leftrightarrow A$ T$\Diamond$!. $\Diamond A \leftrightarrow A$
D!. $\Box A \leftrightarrow \Diamond A$

Prove:

(*a*) T! is a theorem of a normal system if and only if $T\Diamond!$ is.

(*b*) D! is a theorem of any normal *KT!*-system.

(*c*) If any of 4, 4_c, 5, and 5_c is a theorem of a normal *KD!*-system, then so are the others.

4.54. Consider the sentences $\overline{P}$ ($\neg\Diamond\top$) and $\overline{P\Box}$ ($\Box\bot$). Prove:

(*a*) $\overline{P}$ is a theorem of a normal system if and only if $\overline{P\Box}$ is.

(*b*) T_c (and hence D_c, etc., B, 4, 5, and G) is a theorem of any normal $K\overline{P}$-system.

4.55. Prove that a normal system of modal logic is a *KG*-system if and only if it has either of the following theorems.

(*a*) $\Box\Diamond A \vee \Box\Diamond\neg A$

(*b*) $\neg(\Diamond\Box A \wedge \Diamond\Box\neg A)$

The system *KT4G* is known as *S4.2*. This system (properly) contains *S4* (*KT4*) and is (properly) contained in *S5* (*KT5*); see exercise 4.38.

4.56. Consider the following schemas.

H^{++}.	$\Diamond(\Box A \wedge \Diamond B) \to \Box(\Box A \vee \Diamond B)$
$H^{++}\Diamond$.	$\Diamond(\Diamond A \wedge \Box B) \to \Box(\Diamond A \vee \Box B)$
H^{+}.	$(\Box(\Box A \vee B) \wedge \Box(A \vee \Box B)) \to (\Box A \vee \Box B)$
$H^{+}\Diamond$.	$(\Diamond A \wedge \Diamond B) \to (\Diamond(\Diamond A \wedge B) \vee \Diamond(A \wedge \Diamond B))$
H.	$(\Box(A \vee B) \wedge \Box(\Box A \vee B) \wedge \Box(A \vee \Box B)) \to (\Box A \vee \Box B)$
$H\Diamond$.	$(\Diamond A \wedge \Diamond B) \to (\Diamond(A \wedge B) \vee \Diamond(\Diamond A \wedge B) \vee \Diamond(A \wedge \Diamond B))$
L^{++}.	$\Box(\Box A \to \Box B) \vee \Box(\Box B \to \Box A)$
$L^{++}\Diamond$.	$\Box(\Diamond A \to \Diamond B) \vee \Box(\Diamond B \to \Diamond A)$
L^{+}.	$\Box(\Box A \to B) \vee \Box(\Box B \to A)$
$L^{+}\Diamond$.	$\Box(A \to \Diamond B) \vee \Box(B \to \Diamond A)$
L.	$\Box(A \to (\Box B \to C)) \vee \Box(A \vee (\Box C \to B))$
$L\Diamond$.	$\Box(A \vee (B \to \Diamond C)) \vee \Box(A \to (C \to \Diamond B))$

Taken as theorems these schemas are all equivalent additions to any normal *KT4*-system; that is, any one of them is a theorem of a normal *KT4*-system if and only if all the others are. Thus the systems $KT4H^{++}$, …, $KT4L\Diamond$ are identical. This system is known as *S4.3*: it is (properly) contained in *S5* (*KT5*), and it (properly) contains *S4* (*KT4*). Indeed, *S4.3* is a (proper) extension of the system *S4.2* (*KT4G*) mentioned in exercise 4.55. Except for properness, these facts are all consequences of the following results (which are for the reader to prove) in conjunction with exercises 4.33 and 4.38.

(*a*) In any normal system H^{++} is a theorem if and only if $H^{++}\Diamond$ is.

(*b*) In any normal system H^{+} is a theorem if and only if $H^{+}\Diamond$ is.

(*c*) In any normal system H is a theorem if and only if $H\Diamond$ is.

(*d*) In any normal system L^{++} is a theorem if and only if $L^{++}\Diamond$ is.

(*e*) In any normal system L^{+} is a theorem if and only if $L^{+}\Diamond$ is.

(*f*) In any normal system L is a theorem if and only if $L\Diamond$ is.

(*g*) In any normal system H^{++} is a theorem if and only if L^{++} is.

(*h*) In any normal system H^{+} is a theorem if and only if L^{+} is.

(*i*) In any normal system H is a theorem if and only if L is.

(*j*) U is a theorem of any normal KH^{+}-system.

(*k*) H is a theorem of any normal KH^{+}-system.

(*l*) H^{+} is a theorem of any normal *KUH*-system.

(*m*) H^{+} is a theorem of any normal KUH^{++}-system.

(*n*) H^{++} is a theorem of any normal *K4H*-system.

In proving (*j*)–(*n*) it may help to restate some of them using (*a*)–(*i*). In any case, the foregoing yield the identity of $KT4H^{++}$, …, $KT4L\Diamond$ – i.e. *S4.3* – as the reader should confirm.

(*o*) H^{++} is a theorem of any normal *K5*-system.

This is enough, given the preceding results, to show that *S5* (*KT5*) is an extension of *S4.3*. See also exercise 4.37 with respect to H^{+} (and hence the rest).

(*p*) G is a theorem of any normal KH^{+}-system.

This is enough, given the preceding results, to show that *S4.3* is an extension of *S4.2*. But also:

(*q*) G is a theorem of any normal KDH^{++}-system.

4.57. Consider the following schemas.

$A \vee (\Diamond A \to \Box\Diamond A)$	$A \to (\Diamond\Box A \to \Box A)$
$(\Diamond A \to A) \vee (\Diamond A \to \Box\Diamond A)$	$(A \to \Box A) \vee (\Diamond\Box A \to \Box A)$

The two on the left we indifferently dub 5^-, the two on the right, $5^-\Diamond$. Prove:

(*a*) In any normal system 5^- is a theorem if and only if $5^-\Diamond$ is.

(*b*) 5^- is a theorem of any normal *K5*-system.

(*c*) The schema $\Box(\Box(A\to\Box A)\to A)\to\Box(\Diamond\Box A\to A)$ is a theorem of any normal *K5*$^-$-system.

Added as a theorem to *S4* (*KT4*) the schema 5^- produces a system called *S4.4*. By (*b*) *S4.4* is contained in *S5* (*KT5*) (in fact it is properly contained). *S4.4* does not, however, contain *S4.3* or *S4.2* (in exercises 4.55 and 4.56). Added as a theorem to *S4* the schema mentioned in (*c*) produces a system that Hughes and Cresswell call *S4.1*, which is therefore contained (in fact properly) in *S4.4*. But *S4.1* is not extended by *S4.2* or *S4.3*. N.B. this *S4.1* is not the same as that described by McKinsey; see the following exercise.

4.58. Consider the following schema.

$$\text{G}_c.\quad \Box\Diamond A\to\Diamond\Box A$$

Prove:

(*a*) In any normal system the schema $\Diamond(\Diamond A\to\Box A)$ is a theorem if and only if the schema G_c is.

(*b*) D is a theorem of any normal KG_c-system.

(*c*) In any normal system the schema $\Diamond(\Box A\leftrightarrow\Diamond A)$ is a theorem if and only if the schema G_c is.

$KT4G_c$ is the system called *S4.1* by McKinsey. It is clearly an extension of *S4* (*KT4*) – in fact a proper one (otherwise $\Diamond\Box A\leftrightarrow\Diamond\Box\Diamond A$ and $\Box\Diamond A\leftrightarrow\Box\Diamond\Box A$ would be theorems of *S4*, which they are not). But this *S4.1* is not included in *S5* (*KT5*) (if it were, D_c would be a theorem of *S5*, which it is not). *S4.1* is equivalently axiomatized by adding to *S4* any of the following schemas.

$$\Diamond\Box(A\to\Box A)\qquad\qquad \Diamond\Box(\Diamond A\to A)$$

$$\Box(A\vee B)\to(\Diamond\Box A\vee\Diamond\Box B)\qquad (\Box\Diamond A\wedge\Box\Diamond B)\to\Diamond(A\wedge B)$$

This is for the reader to prove.

4.59. Consider the following schema and rule of inference.

$$\text{Gr.}\quad \Box(\Box A\to A)\to\Box A\qquad\qquad \text{RGr.}\quad \frac{\Box A\to A}{A}$$

Prove:

(*a*) 4 is a theorem of any normal *KGr*-system.

(*b*) Every normal *KGr*-system is closed under RGr.

(*c*) Gr is a theorem of any normal *K4*-system that has the rule RGr.

Is the system *KGr* an extension of *S4* (*KT4*)? Is it included in *S5* (*KT5*)?

4.60. Use the erasure transformation ϵ from exercise 1.27 to prove the consistency of the fourteen systems beyond *K* on the diagram in figure 4.1. Identify some consistent normal modal logics for which ϵ cannot be deployed to prove consistency. Prove the consistency of these examples.

4.61. Amplify the proof of the lemma in exercise 4.13 to show that the systems *KD*, KD_c, and *KD!* have the rules (*a*)–(*f*) listed there.

4.62. Consider the rule (*a*) from exercise 4.13:

$$\frac{\Box A}{A}$$

Of the fifteen systems in figure 4.1 only *K*, *KD*, *KT*, *K4*, *KDB*, *KD4*, *KTB*, *KT4*, and *KT5* have this rule. Exercise 4.13 and 4.61 cover the first two cases, and it is obvious that any *KT*-system has the rule (if $\Box A$ is a theorem of a system containing $\Box A\to A$, then A is a theorem of the system). The cases of *K4* and *KD4* must await the developments in chapter 5. Prove that every normal *KDB*-system has the rule.

4.4. Modalities

A modality, once again, is any sequence of the operators $\neg$, $\Box$, and $\Diamond$, including the empty sequence $\cdot$. Within a system of modal logic two modalities ϕ and ψ are *equivalent* if and only if for every A the sentence

$$\phi A\leftrightarrow\psi A$$

is a theorem; otherwise ϕ and ψ are said to be *distinct*. For example, in the system *S4* we have the theorem $\Box A\leftrightarrow\Box\Box A$, so in *S4* the modalities $\Box$ and $\Box\Box$ are equivalent.

Theorems like $\Box A\leftrightarrow\Box\Box A$ are often called *reduction laws*, since in virtue of them one modality is reducible to another.

In some systems of modal logic it happens that every modality is equivalent to one or another in a finite class. For example, in the system *S5* every modality is equivalent to one of $\cdot$, $\Box$, $\Diamond$, or their negations, $\neg$,

$\neg\Box$, $\neg\Diamond$. Thus *S5* is said to have at most six distinct modalities (three affirmative, three negative). To see this it is sufficient to note that *S5* contains the following reduction laws.

(1) $\Box A \leftrightarrow \Box\Box A$

(2) $\Diamond A \leftrightarrow \Diamond\Diamond A$

(3) $\Box A \leftrightarrow \Diamond\Box A$

(4) $\Diamond A \leftrightarrow \Box\Diamond A$

Because of these every modality in *S5* reduces to one of the specified six.

An example may help to make this clear. Suppose we have the sentence

$$\neg\Box\Box\Diamond\neg\Box\Diamond A.$$

First we put the modality $\neg\Box\Box\Diamond\neg\Box\Diamond$ in a standard form by using Df$\Diamond$, Df$\Box$, and REP to bring the negation signs all to the outside – successively,

$$\neg\Box\Box\neg\Box\Box\Diamond A,\ \neg\Box\neg\Diamond\Box\Box\Diamond A,\ \neg\neg\Diamond\Diamond\Box\Box\Diamond A$$

– and then reducing the number of occurrences of $\neg$ to zero (as in this case) or one by *PL*:

$$\Diamond\Diamond\Box\Box\Diamond A$$

According to reduction law (2) the modality $\Diamond\Diamond$ can be replaced by $\Diamond$ alone, yielding

$$\Diamond\Box\Box\Diamond A$$

By (3), $\Diamond\Box$ becomes $\Box$:

$$\Box\Box\Diamond A$$

By (1), $\Box\Box$ becomes $\Box$:

$$\Box\Diamond A$$

And $\Box\Diamond$ reduces to $\Diamond$ by (4):

$$\Diamond A$$

Thus the modality $\neg\Box\Box\Diamond\neg\Box\Diamond$ is shown to be equivalent to $\Diamond$. In a similar way one can show the modality $\neg\Box\Diamond\neg\Box\Diamond\neg\Box$ to be equivalent to $\neg\Box$; details of the reduction are left to the reader.

Of course the presence of reduction laws can only put an upper bound on the number of distinct modalities in a system. To show that *S5* has at least – and hence exactly – six distinct modalities it is necessary to

establish that there are no further reduction laws in the system (for example, that $\Box A \leftrightarrow \Diamond A$ is not also a theorem). In general other means are required to fix a lower bound (possibly infinite) for the number of distinct modalities in a modal logic. We return to this point in chapter 5.

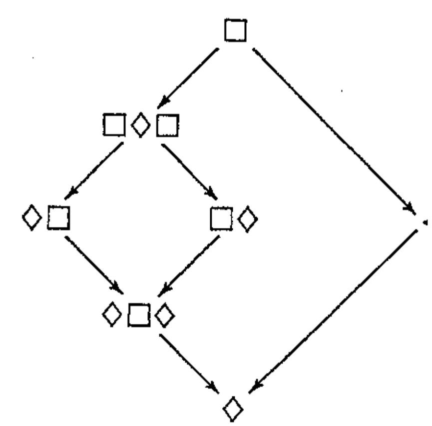
Systems of modal logic can have the same distinct modalities but differ with respect to the pattern of implications among them. *S5* and the system *KD45* provide an example of this; each has $\cdot$, $\Box$, $\Diamond$, and their negations, but the *S5*-theorems $\Box A \to A$ and $A \to \Diamond A$ are absent from *KD45*. The diagrams in figures 4.7 and 4.8 chart the differences (among the affirmative modalities; for their negations, reverse the arrows). The systems *KD5*, *K45*, and *KB4* provide another example; compare figures 4.4, 4.5, and 4.6. Moreover, systems may be different even though they have the same distinct modalities and the same patterns of implications among them. Some examples of this situation will be found in the exercises at the end of the section.

It turns out that of the normal systems that can be formed using D, T, B, 4, and 5, only seven have a finite number of distinct modalities: *KT4*, *K5*, *KD5*, *K45*, *KB4*, *KD45*, and *KT5*. The following theorems give the details.

THEOREM 4.22. *Every normal KT4-system has at most fourteen distinct modalities, viz.* $\cdot$, $\Box$, $\Diamond$, $\Box\Diamond$, $\Diamond\Box$, $\Box\Diamond\Box$, $\Diamond\Box\Diamond$, *and their negations, with implications among the affirmative seven as diagramed in figure 4.2.*

Proof. To show that a normal *KT4*-system has at most the specified

Figure 4.2. Modalities in normal *KT4*-systems.



fourteen distinct modalities it is sufficient to show the following reduction laws to be theorems of the system.

$\Box A \leftrightarrow \Box\Box A \qquad \Diamond A \leftrightarrow \Diamond\Diamond A$

$\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A \qquad \Box\Diamond A \leftrightarrow \Box\Diamond\Box\Diamond A$

For then every modality will be reducible to one of those specified (as the reader should confirm). The laws on the right are dual to those on the left, and $\Box A \leftrightarrow \Box\Box A$ is obvious in view of T and 4. So it suffices to establish $\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A$. For left-to-right:

1.	$\Box A \to \Diamond\Box A$	T$\Diamond$
2.	$\Box\Box A \to \Box\Diamond\Box A$	1, RM
3.	$\Box A \to \Box\Box A$	4
4.	$\Box A \to \Box\Diamond\Box A$	2, 3, *PL*
5.	$\Diamond\Box A \to \Diamond\Box\Diamond\Box A$	4, RM$\Diamond$

And for right-to-left:

1.	$\Box\Diamond\Box A \to \Diamond\Box A$	T
2.	$\Diamond\Box\Diamond\Box A \to \Diamond\Diamond\Box A$	1, RM$\Diamond$
3.	$\Diamond\Diamond\Box A \to \Diamond\Box A$	4$\Diamond$
4.	$\Diamond\Box\Diamond\Box A \to \Diamond\Box A$	2, 3, *PL*

For the eight implications diagramed in figure 4.2 we need consider only the top four; the others are duals. Of these four, two are T and one appears on line 4 of the first proof above. The remaining theorem, $\Box\Diamond\Box A \to \Box\Diamond A$, follows from T by RM$\Diamond$ and RM.

Thus the system *S4* – i.e. *KT4* itself – has at most fourteen distinct modalities. In chapter 5 we prove it has exactly that many.

THEOREM 4.23. *Every normal K5-system has at most fourteen distinct modalities, viz.* $\cdot$, $\Box$, $\Diamond$, $\Box\Box$, $\Diamond\Diamond$, $\Box\Diamond$, $\Diamond\Box$, *and their negations, with implications among the affirmative seven as diagramed in figure 4.3.*

Proof. For this result we require the following reduction laws.

$\Box\Box A \leftrightarrow \Box\Box\Box A \qquad \Diamond\Diamond A \leftrightarrow \Diamond\Diamond\Diamond A$

$\Box\Box A \leftrightarrow \Box\Diamond\Box A \qquad \Diamond\Diamond A \leftrightarrow \Diamond\Box\Diamond A$

$\Diamond\Box A \leftrightarrow \Diamond\Box\Box A \qquad \Box\Diamond A \leftrightarrow \Box\Diamond\Diamond A$

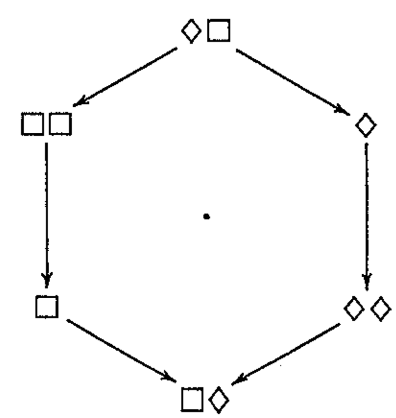
$\Diamond\Box A \leftrightarrow \Diamond\Diamond\Box A \qquad \Box\Diamond A \leftrightarrow \Box\Box\Diamond A$

These laws might be summed up by the motto ***Drop the middle modality***. The conditional halves of those on the left appear below on lines 2, 3, and 9–14; those on the right follow by duality. The last six lines below make implicit appeal to $(\Diamond A \to \Box B) \to (\Box A \to \Box B)$ and $(\Diamond A \to \Box B) \to (\Diamond A \to \Diamond B)$. Because both are theorems of any normal system (exercise 4.7(*o*, *p*)) we mark these steps *K*.

1.	$\Diamond\Box A \to \Box A$	5$\Diamond$
2.	$\Box\Diamond\Box A \to \Box\Box A$	1, RM
3.	$\Diamond\Diamond\Box A \to \Diamond\Box A$	1, RM$\Diamond$
4.	$\Diamond\Box A \to \Box\Diamond\Box A$	5
5.	$\Diamond\Box\Box A \to \Box\Box A$	5$\Diamond$
6.	$\Diamond\Box A \to \Box\Box A$	2, 4, *PL*
7.	$\Box\Diamond\Box A \to \Box\Box\Box A$	6, RM
8.	$\Diamond\Box A \to \Box\Box\Box A$	4, 7, *PL*
9.	$\Box\Box A \to \Box\Diamond\Box A$	4, *K*
10.	$\Diamond\Box A \to \Diamond\Diamond\Box A$	4, *K*
11.	$\Box\Box\Box A \to \Box\Box A$	5, *K*
12.	$\Diamond\Box\Box A \to \Diamond\Box A$	5, *K*
13.	$\Box\Box A \to \Box\Box\Box A$	8, *K*
14.	$\Diamond\Box A \to \Diamond\Box\Box A$	8, *K*

As to the six implications pictured in figure 4.3, first note that those on the right are duals of those on the left. Of the latter, $\Diamond\Box A \to \Box\Box A$

Figure 4.3: Modalities in normal *K5*-systems.



appears on line 6 above, and the remaining two follow from the theorems 5 and 5$\Diamond$ by the *K*-theorems mentioned in the last paragraph.

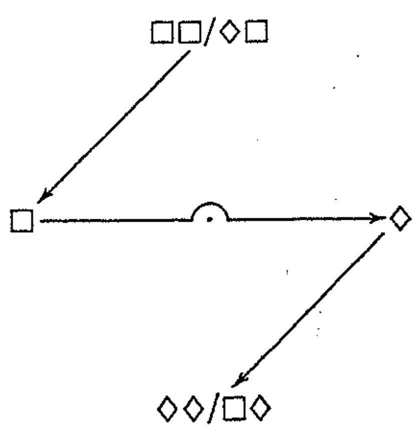
THEOREM 4.24. *Every normal KD5-system has at most ten distinct modalities, viz.* $\cdot$, $\Box$, $\Diamond$, *one from each of the pairs* $\Box\Box$, $\Diamond\Box$ *and* $\Diamond\Diamond$, $\Box\Diamond$, *and their negations, with implications among the affirmative five as diagramed in figure 4.4.*

Proof. By theorem 4.23 a normal *KD5*-system has at most the distinct modalities, and at least the implications among them, pictured in figure 4.3. But in addition to the reduction laws in the proof above, every normal *KD5*-system contains the laws $\Box\Box A \leftrightarrow \Diamond\Box A$ and $\Diamond\Diamond A \leftrightarrow \Box\Diamond A$; each is the theorem D in one direction, and the converses belong to any normal *K5*-system (as in figure 4.3). Finally, D gives the implication from $\Box$ to $\Diamond$. Hence the modalities and implications in figure 4.4.

THEOREM 4.25. *Every normal K45-system has at most ten distinct modalities, viz.* $\cdot$, $\Box$, $\Diamond$, $\Box\Diamond$, $\Diamond\Box$, *and their negations, with implications among the affirmative five as diagramed in figure 4.5.*

Proof. By theorem 4.23 it is sufficient to point out that a normal *K45*-system contains the reduction laws $\Box A \leftrightarrow \Box\Box A$ and $\Diamond A \leftrightarrow \Diamond\Diamond A$, so that we may delete $\Box\Box$ and $\Diamond\Diamond$ from the diagram in figure 4.3. The result is figure 4.5.

Figure 4.4. Modalities in normal *KD5*-systems.



THEOREM 4.26. *Every normal KB4-system has at most ten distinct modalities, viz.* $\cdot$, $\Box$, $\Diamond$, $\Box\Diamond$, $\Diamond\Box$, *and their negations, with implications among the affirmative five as diagramed in figure 4.6.*

Proof. Recall (theorem 4.18) that 5 is a theorem of any normal *KB4*-system, so that every such system is an extension of *K45*. By theorem 4.25, then, a normal *KB4*-system has at most the distinct modalities $\cdot$, $\Box$, $\Diamond$, $\Box\Diamond$, $\Diamond\Box$, and their negations. The only new elements are the implications involving the modalities $\cdot$ and $\neg$. Thus figure 4.6.

Figure 4.5. Modalities in normal *K45*-systems.

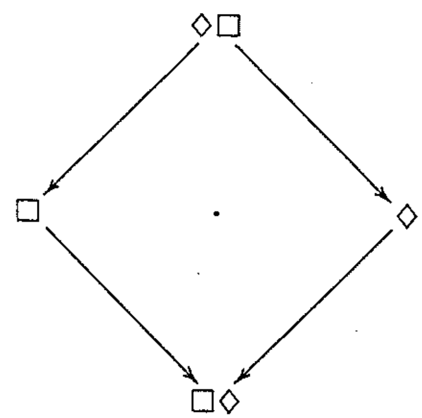
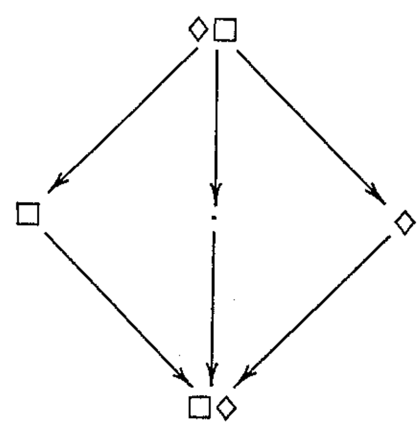


Figure 4.6. Modalities in normal *KB4*-systems.



THEOREM 4.27. *Every normal KD45-system has at most six distinct modalities, viz.* ·, □, ◇, *and their negations, with implications among the affirmative three as diagramed in figure 4.7.*

Proof. By the proofs for theorems 4.24 and 4.25 every normal *KD45*-system has the reduction laws

$$\Box A \leftrightarrow \Box\Box A \qquad \Diamond A \leftrightarrow \Diamond\Diamond A$$
$$\Box A \leftrightarrow \Diamond\Box A \qquad \Diamond A \leftrightarrow \Box\Diamond A$$

as well as the implication $\Box A \rightarrow \Diamond A$. Thus a system of this kind has at most the distinct modalities and at least the implication laid out in figure 4.7.

THEOREM 4.28. *Every normal KT5-system has at most six distinct modalities, viz.* ·, □, ◇, *and their negations, with implications among the affirmative three as diagramed in figure 4.8.*

Proof. By theorem 4.21 every normal *KT5*-system contains D and 4. So by theorem 4.27 every such system contains at most the distinct modalities ·, □, ◇, and their negations. In virtue of T and T◇ we may add arrows to and from · in figure 4.7. The result is figure 4.8. Alternatively, we may note that a normal *KT5*-system contains B and 4 (theorem 4.21) and so has all the reduction laws and implications had jointly by normal *KT4*- and normal *KB4*-systems. Applying theorems 4.22 and 4.26 – or, combining figures 4.2 and 4.6 – we arrive at the desired result.

Thus, as we said at the beginning of the section, both *KD45* and *KT5* (*S5*) have at most the six modalities ·, □, ◇, ¬, ¬□, and ¬◇. The moral about modalities in systems of this sort is that *iteration is vacuous*: any sequence of □s and ◇s can always be reduced to its innermost term.

Figure 4.7. Modalities in normal *KD45*-systems.

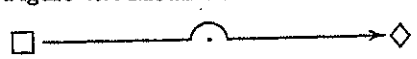
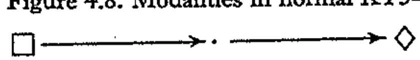


Figure 4.8. Modalities in normal *KT5*-systems.



EXERCISES

4.63. Show that the modalities ¬□◇¬□◇¬□ and ¬□ are equivalent in *S5* (*KT5*).

4.64. Using the reduction laws mentioned in the proof of theorem 4.22, show that every modality in a normal *KT4*-system is reducible to one of ·, □, ◇, □◇, ◇□, □◇□, ◇□◇, or the negation of one of these.

4.65. Prove that normal *KD4!*- and normal *KD4U*-systems have at most fourteen distinct modalities, viz. ·, □, ◇, □◇, ◇□, □◇□, ◇□◇, and their negations, with implications among the affirmative seven as diagramed in figure 4.9. (See exercises 4.33(*b*) and 4.43(*b*)).

4.66. Prove that normal *KT4G*- and normal *KT4H*+-systems have at most ten distinct modalities, viz. ·, □, ◇, □◇, ◇□, and their negations, with implications among the affirmative five as diagramed in figure 4.10. (See exercises 4.43, 4.55, and 4.56.)

4.67. Prove that normal *KD4H*+-systems have at most ten distinct modalities, viz. ·, □, ◇, □◇, ◇□, and their negations, with implications among the affirmative five as diagramed in figure 4.11. (See exercises 4.43 and 4.56.)

4.68. Prove that normal $KT4G_c$-systems have at most ten distinct modalities, viz. ·, □, ◇, □◇, ◇□, and their negations, with implications among the affirmative five as diagramed in figure 4.12. (See exercises 4.43 and 4.58.) Note that the modalities □◇□ and ◇□◇ are equi-

Figure 4.9. Modalities in normal *KD4!*- and normal *KD4U*-systems.

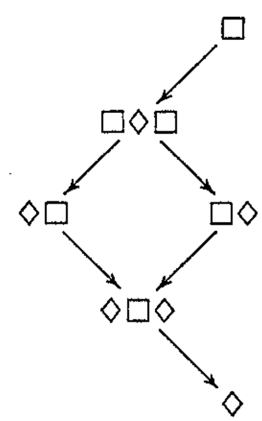


Figure 4.10. Modalities in normal *KT4G*- and normal $KT4H^{+}$-systems.

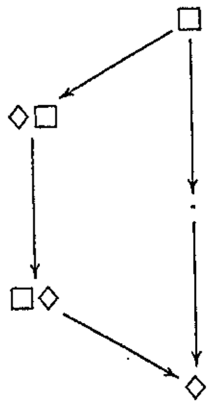


Figure 4.11. Modalities in normal $KD4H^{+}$-systems.

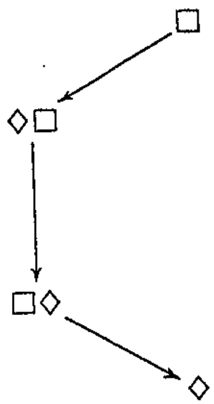
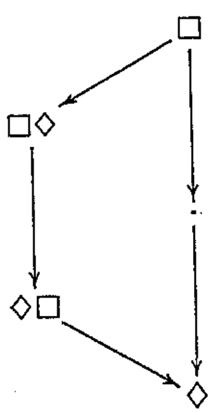


Figure 4.12. Modalities in normal $KT4G_c$-systems.



valent, respectively, to □◇ and ◇□ in normal $KT4G_c$-systems, whereas in normal *KT4G*- and normal $KT4H^{+}$-systems it is the other way around.

4.69. Prove that normal $K4G_c$-systems have at most ten distinct modalities, viz. ·, □, ◇, □◇, ◇□, and their negations, with implications among the affirmative five as diagramed in figure 4.13. (See exercises 4.43 and 4.58.) Compare the reduction laws here for □◇□ and ◇□◇ with those in normal $KD4H^{+}$-systems.

4.70. Prove that normal *K5!*-systems have at most six distinct modalities, viz. ·, □, ◇, and their negations, with implications among the affirmative three as in the diagram in figure 4.7 for normal *KD45*-systems. (See exercise 4.47.)

4.71. Prove that normal extensions of *KD!B*, $KD!B_c$, *KD!4*, $KD!4_c$, *KD!5*, and KD_c5_c have at most four distinct modalities, viz. ·, one of □ and ◇ and their negations. (See exercises 4.33 (*d*), 4.38, 4.52, and 4.53.)

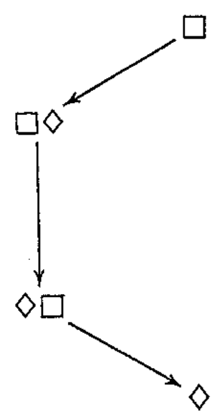
4.72. Prove that normal extensions of *KT!* (and hence KDT_c and KD_cT) have at most two distinct modalities, viz. · and ¬. (See exercises 4.52 and 4.53.)

4.73. Describe a normal modal logic that has just one (distinct!) modality.

4.5. Maximal sets in normal systems

We bring this chapter to a close by stating and proving a few theorems about maximal sets of sentences (section 2.6) in normal systems of modal

Figure 4.13. Modalities in normal $K4G_c$-systems.



logic. The importance of these theorems will become apparent in chapter 5, where they are useful in several proofs.

THEOREM 4.29. *Let* Γ *and* Δ *be maximal sets of sentences in a normal system* Σ. *Then:*

(1) $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$ *iff* $\{\Diamond A\colon A \in \Delta\} \subseteq \Gamma$;

and more generally, for any $k \geqslant 0$,

(2) $\{A\colon \Box^k A \in \Gamma\} \subseteq \Delta$ *iff* $\{\Diamond^k A\colon A \in \Delta\} \subseteq \Gamma$.

Proof. Let Γ and Δ be Σ-maximal sets of sentences, and suppose that Σ is normal. We prove (1) only; (2) is a simple generalization. For left-to-right, assume that $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$ and that A is in Δ. We wish to show that Γ contains $\Diamond A$. By theorem 2.18 (5), $\neg A$ is not in Δ. So $\Box\neg A$ is not in Γ, which means that $\neg\Box\neg A$ is a member of Γ. Because Σ is normal, Γ contains Df$\Diamond$, and so – by theorem 2.18 (9) – $\Diamond A$ is also in Γ. For right-to-left, assume that $\{\Diamond A\colon A \in \Delta\} \subseteq \Gamma$ and that $\Box A$ is in Γ. Since Γ contains Df$\Box$, $\neg\Diamond\neg A$ is in Γ, and so $\Diamond\neg A$ is not. Hence $\neg A$ is not a member of Δ, which means that A is, which is what we wished to show.

Note that when $k = 0$ the preceding theorem means that for Σ-maximal sets Γ and Δ, $\Gamma \subseteq \Delta$ if and only if $\Delta \subseteq \Gamma$, and so $\Gamma = \Delta$ just in case $\Gamma \subseteq \Delta$.

The next theorem may be regarded as an extension, for normal modal logics, of theorem 2.18 (particularly parts (3)–(9)).

THEOREM 4.30. *Let* Γ *be a maximal set of sentences in a normal system* Σ. *Then:*

(1) $\Box A \in \Gamma$ *iff for every* $\mathrm{Max}_\Sigma \Delta$ *such that* $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$, $A \in \Delta$.

(2) $\Diamond A \in \Gamma$ *iff for some* $\mathrm{Max}_\Sigma \Delta$ *such that* $\{\Diamond A\colon A \in \Delta\} \subseteq \Gamma$, $A \in \Delta$.

Proof. Suppose Σ is a normal system and that Γ is a Σ-maximal set of sentences.

For (1). From left to right the theorem is trivial, for if $\Box A \in \Gamma$ and $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$, then $A \in \Delta$.

The reverse is thus the interesting direction. Suppose that $A \in \Delta$, for every Σ-maximal set Δ such that $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$, i.e. that A belongs

to every Σ-maximal extension of the set $\{A\colon \Box A \in \Gamma\}$. By a corollary to Lindenbaum's lemma, theorem 2.20 (1), this means that A is Σ-deducible from this set of sentences; i.e.

$$\{A\colon \Box A \in \Gamma\} \vdash_\Sigma A.$$

This in turn means that there are sentences $A_1, \ldots, A_n$ ($n \geqslant 0$) in the set $\{A\colon \Box A \in \Gamma\}$ that are such that

$$\vdash_\Sigma (A_1 \wedge \ldots \wedge A_n) \to A.$$

Because Σ is normal, we may infer by RK that

$$\vdash_\Sigma (\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A.$$

But Γ contains each of $\Box A_1, \ldots, \Box A_n$, so $\Box A$ is Σ-deducible from Γ; i.e.

$$\Gamma \vdash_\Sigma \Box A.$$

By theorem 2.18 (1) this means that

$$\Box A \in \Gamma,$$

which was to be proved.

For (2):

$\Diamond A \in \Gamma$ iff $\neg\Box\neg A \in \Gamma$
– Df$\Diamond$ and theorem 2.18 (9);
iff $\Box\neg A \notin \Gamma$
– theorem 2.18 (5);
iff for some $\mathrm{Max}_\Sigma \Delta$ such that $\{A\colon \Box A \in \Gamma\} \subseteq \Delta$, $\neg A \notin \Delta$
– part (1);
iff for some $\mathrm{Max}_\Sigma \Delta$ such that $\{\Diamond A\colon A \in \Delta\} \subseteq \Gamma$, $\neg A \notin \Delta$
– theorem 4.29;
iff for some $\mathrm{Max}_\Sigma \Delta$ such that $\{\Diamond A\colon A \in \Delta\} \subseteq \Gamma$, $A \in \Delta$
– theorem 2.18 (5).

THEOREM 4.31. *Let* Γ *and* Δ *be maximal sets of sentences in a normal system* Σ. *Then for every* $k \geqslant 0$:

$\{A\colon \Box^{k+1} A \in \Gamma\} \subseteq \Delta$ *iff for some* $\mathrm{Max}_\Sigma E$,
$\{A\colon \Box A \in \Gamma\} \subseteq E$ *and* $\{A\colon \Box^k A \in E\} \subseteq \Delta$.

Proof. We assume that Γ and Δ are maximal sets of sentences in a normal modal logic Σ. From right to left the proof is straightforward, and we leave it as an exercise. For left-to-right we suppose that $\{A: \Box^{k+1}A \in \Gamma\} \subseteq \Delta$, to show that there exists a Σ-maximal set E such that

$$\{A : \Box A \in \Gamma\} \subseteq E \text{ and } \{A : \Box^k A \in E\} \subseteq \Delta,$$

i.e. by theorem 4.29, such that

$$\{A : \Box A \in \Gamma\} \subseteq E \text{ and } \{\Diamond^k A : A \in \Delta\} \subseteq E.$$

In other words, we wish to show that there is a Σ-maximal set of sentences E that includes the set

$$\{A : \Box A \in \Gamma\} \cup \{\Diamond^k A : A \in \Delta\}.$$

By Lindenbaum's lemma (theorem 2.19) this is equivalent to showing that this union is Σ-consistent.

Let us suppose otherwise, and argue to a contradiction. If the set is Σ-inconsistent, then $\perp$ is Σ-deducible from it, and this in turn means that for some $m, n \geqslant 0$ there are sentences $B_1, \ldots, B_m$ in $\{A : \Box A \in \Gamma\}$ and sentences $\Diamond^k C_1, \ldots, \Diamond^k C_n$ in $\{\Diamond^k A : A \in \Delta\}$ such that

$$\vdash_\Sigma (B_1 \wedge \ldots \wedge B_m \wedge \Diamond^k C_1 \wedge \ldots \wedge \Diamond^k C_n) \rightarrow \perp.$$

By a rule of inference present in every normal modal logic (see exercise 4.11) we infer that

$$\vdash_\Sigma (\Box B_1 \wedge \ldots \wedge \Box B_m) \rightarrow \Box^{k+1} \neg (C_1 \wedge \ldots \wedge C_n).$$

Because each of $\Box B_1, \ldots, \Box B_m$ is in Γ, the consequent $\Box^{k+1}\neg(C_1 \wedge \ldots \wedge C_n)$ is Σ-deducible from Γ, and so belongs to Γ. By our original assumption, then, Δ contains $\neg(C_1 \wedge \ldots \wedge C_n)$. But Δ contains $C_1 \wedge \ldots \wedge C_n$ too, since this is the consequent of the theorem $(C_1 \wedge \ldots \wedge C_n) \rightarrow (C_1 \wedge \ldots \wedge C_n)$, for which Δ contains each conjunct of the antecedent. So Δ is Σ-inconsistent, which is a contradiction, and we may consider the proof complete.

EXERCISES

4.74. Prove part (2) of theorem 4.29.

4.75. Give the proof for right-to-left in theorem 4.31.

4.76. Prove the following generalizations of theorem 4.30, for any $k > 0$, where Γ is a maximal set of sentences in a normal system Σ.

(*a*) $\Box^k A \in \Gamma$ iff for every $\text{Max}_\Sigma \Delta$ such that $\{A : \Box^k A \in \Gamma\} \subseteq \Delta$, $A \in \Delta$.

(*b*) $\Diamond^k A \in \Gamma$ iff for some $\text{Max}_\Sigma \Delta$ such that $\{\Diamond^k A : A \in \Delta\} \subseteq \Gamma$, $A \in \Delta$.

4.77. Let Σ be a normal system, and define the relation R on the set of Σ-maximal sets of sentences by:

$$\Gamma R \Delta \text{ iff } \{A : \Box A \in \Gamma\} \subseteq \Delta.$$

(Thus, by theorem 4.29, $\Gamma R \Delta$ if and only if $\{\Diamond A : A \in \Delta\} \subseteq \Gamma$.) Prove:

(*a*) R is serial if Σ contains D.

(*b*) R is reflexive if Σ contains T.

(*c*) R is symmetric if Σ contains B.

(*d*) R is transitive if Σ contains 4.

(*e*) R is euclidean if Σ contains 5.

(See section 3.2 for these properties of R.)