# 6

# EXPONENTIALS

We have now managed to unify most of the universal mapping properties that we have seen so far with the notion of limits (or colimits). Of course, the free algebras are an exception to this. In fact, it turns out that there is a common source of such universal mapping properties (UMPs), but it lies somewhat deeper, in the notion of *adjoints*, which unify free algebras, limits, and other universals of various kinds.

Next we are going to look at one more elementary universal structure, which is also an example of a universal that is not a limit. This important structure is called an "exponential," and it can be thought of as a categorical notion of a "function space." As we shall see it subsumes much more than just that, however.

## 6.1 Exponential in a category

Let us start by considering a function of sets,

$$f(x, y) : A \times B \to C$$

written using variables $x$ over $A$ and $y$ over $B$. If we now hold $a \in A$ fixed, we have a function

$$f(a, y) : B \to C$$

and thus an element

$$f(a, y) \in C^B$$

of the set of all such functions.

Letting $a$ vary over $A$ then gives a map, which I write like this

$$\tilde{f} : A \to C^B$$

defined by $a \mapsto f(a, y)$.

The map $\tilde{f} : A \to C^B$ takes the "parameter" $a$ to the function $f(y) : B \to C$. It is uniquely determined by the equation

$$\tilde{f}(a)(b) = f(a, b).$$

Indeed, *any* map

$$\phi : A \to C^B$$

is uniquely of the form

$$\phi = \tilde{f}$$

for some $f : A \times B \to C$. For we can set

$$f(a, b) := \phi(a)(b).$$

What this means, in sum, is that we have an isomorphism of Hom-sets:

$$\mathrm{Hom}_{\mathbf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{Sets}}(A, C^B)$$

That is, there is a bijective correspondence between functions of the form $f : A \times B \to C$ and those of the form $\tilde{f} : A \to C^B$, which we can display schematically, thus,

$$\frac{f : A \times B \to C}{\tilde{f} : A \to C^B}$$

This bijection is mediated by a certain operation of *evaluation*, which we have indicated in the foregoing by using variables. In order to generalize the indicated bijection to other categories, we are going to need to make this evaluation operation explicit, too.

In **Sets**, it is the function

$$\mathrm{eval} : C^B \times B \to C$$

defined by $(g, b) \mapsto g(b)$, that is,

$$\mathrm{eval}(g, b) = g(b).$$

This evaluation function has the following UMP: given any set $A$ and any function

$$f : A \times B \to C$$

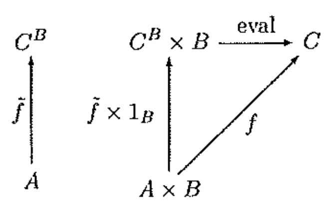
there is a unique function

$$\tilde{f} : A \to C^B$$

such that $\mathrm{eval} \circ (\tilde{f} \times 1_B) = f$. That is,

$$\mathrm{eval}(\tilde{f}(a), b) = f(a, b). \tag{6.1}$$

Here is the diagram:



You can read the equation (6.1) off from this diagram by taking a pair of elements $(a, b) \in A \times B$ and chasing them around both ways, using the fact that $(\tilde{f} \times 1_B)(a, b) = (\tilde{f}(a), b)$.

Now, the property just stated of the set $C^B$ and the evaluation function $\text{eval} : C^B \times B \to C$ is one that makes sense in any category having binary products. It says that evaluation is "the universal map into $C$ from a product with $B$." Precisely, ~~it states~~ we have the following:

**Definition 6.1.** Let the category **C** have binary products. An *exponential* of objects $B$ and $C$ consists of an object

$$C^B$$

and an arrow

$$\epsilon : C^B \times B \to C$$

such that, for any object $A$ and arrow

$$f : A \times B \to C$$

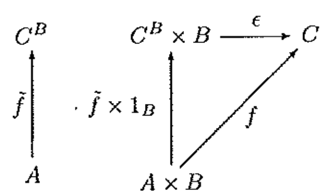
there is a unique arrow

$$\tilde{f} : A \to C^B$$

such that

$$\epsilon \circ (\tilde{f} \times 1_B) = f$$

all as in the diagram




Here is some terminology:

- $\epsilon : C^B \times B \to C$ is called *evaluation.*
- $\tilde{f} : A \to C^B$ is called the (exponential) *transpose* of $f$.
- Given any arrow

  $$g : A \to C^B$$

  we write

  $$\bar{g} := \epsilon \circ (g \times 1_B) : A \times B \to C$$

and also call $\bar{g}$ the *transpose* of $g$. By the uniqueness clause of the definition, we then have

$$\tilde{\bar{g}} = g$$

and for any $f : A \times B \to C$,

$$\bar{\tilde{f}} = f.$$

Briefly, transposition of transposition is the identity.

Thus in sum, the transposition operation

$$(f : A \times B \to C) \longmapsto (\tilde{f} : A \to C^B)$$

provides an inverse to the induced operation

$$(g : A \to C^B) \longmapsto (\bar{g} = \epsilon \circ (g \times 1_B) : A \times B \to C),$$

yielding the desired isomorphism,

$$\mathrm{Hom}_{\mathbf{C}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{C}}(A, C^B).$$

Au: Please confirm "Cartesian closed categories" and "CCC" are one and the same.

yes

## 6.2 Cartesian closed categories

**Definition 6.2.** A category is called *cartesian closed*, if it has all finite products and exponentials.

*Example* 6.3. We already have **Sets** as one example, but note that also $\mathbf{Sets}_{fin}$ is cartesian closed, since for finite sets $M, N$, the set of functions $N^M$ has cardinality

$$|N^M| = |N|^{|M|}$$

and so is also finite.

*Example* 6.4. Recall that the category **Pos** of posets has as arrows $f : P \to Q$ the monotone functions, $p \leq p'$ implies $fp \leq fp'$. Given posets $P$ and $Q$, the poset $P \times Q$ has pairs $(p, q)$ as elements, and is partially ordered by

$$(p, q) \leq (p', q') \quad \text{iff} \quad p \leq p' \text{ and } q \leq q'.$$

Thus, the evident projections

$$P \xleftarrow[\pi_1]{} P \times Q \xrightarrow[\pi_2]{} Q$$

are monotone, as is the pairing

$$\langle f, g \rangle : X \to P \times Q$$

if $f : X \to P$ and $g : X \to Q$ are monotone.

For the exponential $Q^P$, we take the set of monotone functions,

$$Q^P = \{f : P \to Q \mid f \text{ monotone }\}$$

ordered *pointwise*, that is,

$$f \leq g \quad \text{iff} \quad fp \leq gp \text{ for all } p \in P.$$

The evaluation

$$\epsilon : Q^P \times P \to Q$$

and transposition

$$\tilde{f} : X \to Q^P$$

of a given arrow

$$f : X \times P \to Q$$

are the usual ones of the underlying functions. Thus, we need only show that these are monotone.

To that end, given $(f, p) \leq (f', p')$ in $Q^P \times P$, we have

$$\begin{aligned} \epsilon(f, p) &= f(p) \\ &\leq f(p') \\ &\leq f'(p') \\ &= \epsilon(f', p') \end{aligned}$$

so $\epsilon$ is monotone. Now take $f : X \times P \to Q$ monotone and let $x \leq x'$. We need to show

$$\tilde{f}(x) \leq \tilde{f}(x') \quad \text{in } Q^P$$

which means

$$\tilde{f}(x)(p) \leq \tilde{f}(x')(p) \quad \text{for all } p \in P.$$

But $\tilde{f}(x)(p) = f(x, p) \leq f(x', p) = \tilde{f}(x')(p)$.

*Example* 6.5. Now let us consider what happens if we restrict to the category of $\omega$CPOs (see example 5.33). Given two $\omega$CPOs $P$ and $Q$, we take as an exponential the subset,

$$Q^P = \{f : P \to Q \mid f \text{ monotone and } \omega\text{-continuous}\}.$$

Then take evaluation $\epsilon : Q^P \times P \to Q$ and transposition as before, for functions. Then, since we know that the required equations are satisfied, we just need to check the following:

- $Q^P$ is an $\omega$CPO

- $\epsilon$ is $\omega$-continuous
- $\tilde{f}$ is $\omega$-continuous if $f$ is

We leave this as an exercise!

Au: Please check the sentence for completeness.

*Example* 6.6. An example of a somewhat different sort is provided by the category **Graphs** of graphs and their homomorphisms. Recall that a graph $G$ consists of a pair of sets $G_e$ and $G_v$—the edges and vertices—and a pair of functions,

$$\begin{array}{c} G_e \\ s_G \downarrow\;\downarrow t_G \\ G_v \end{array}$$

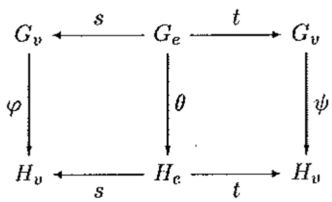
called the source and target maps. A homomorphism of graphs $h : G \to H$ is a mapping of edges to edges and vertices to vertices, preserving sources and targets, that is, is a pair of maps $h_v : G_v \to H_v$ and $h_e : G_e \to H_e$, making the two obvious squares commute.

$$\begin{array}{ccc} G_e & \xrightarrow{h_e} & H_e \\ s_G \downarrow\;\downarrow t_G & & s_H \downarrow\;\downarrow t_H \\ G_v & \xrightarrow{h_v} & H_e \end{array}$$

The product $G \times H$ of two graphs $G$ and $H$, like the product of categories, has as vertices the pairs $(g, h)$ of vertices $g \in G$ and $h \in H$, and similarly the edges are pairs of edges $(u, v)$ with $u$ an edge in $G$ and $v$ and edge in $H$. The source and target operations are, then, "pointwise": $s(u, v) = (s(u), s(v))$, etc.

$$\begin{array}{c} G_e \times H_e \\ s_G \times s_H \downarrow\;\downarrow t_G \times t_H \\ G_v \times H_v \end{array}$$

Now, the exponential graph $H^G$ has as vertices the (arbitrary!) maps of vertices $\varphi : G_v \to H_v$. An edge $\theta$ from $\varphi$ to another vertex $\psi : G_v \to H_v$ is a family of edges $(\theta_e)$ in $H$, one for each edge $e \in G$, such that $s(\theta_e) = \varphi(s(e))$ and $t(\theta_e) = \psi(t(e))$. In other words, $\theta$ is a map $\theta : G_e \to H_e$ making the following commute:




Imagining $G$ as a certain configuration of edges and vertices, and the maps $\varphi$ and $\psi$ as two different "pictures" or "images" of the vertices of $G$ in $H$, the edge $\theta : \varphi \to \psi$ appears as a family of edges in $H$, labeled by the edges of $G$, each connecting the source vertex in $\varphi$ to the corresponding target one in $\psi$. (The reader should draw a diagram at this point.) The evaluation homomorphism $\epsilon : H^G \times G \to H$ takes a vertex $(\varphi, g)$ to the vertex $\varphi(g)$, and an edge $(\theta, e)$ to the edge $\theta_e$. The transpose of a graph homomorphism $f : F \times G \to H$ is the homomorphism $\tilde{f} : F \to H^G$ taking a vertex $a \in F$ to the mapping on vertices $f(a, -) : G_v \to H_v$, and an edge $c : a \to b$ in $F$ to the mapping of edges $f(c, -) : G_e \to H_e$.

We leave the verification of this cartesian closed structure as an exercise for the reader.

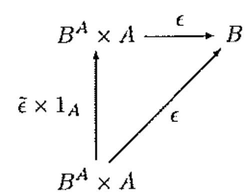
Next, we derive some of the basic facts about exponentials and cartesian closed categories. First, let us ask, what is the transpose of evaluation?

$$\epsilon : B^A \times A \to B$$

It must be an arrow $\tilde{\epsilon} : B^A \to B^A$ such that

$$\epsilon(\tilde{\epsilon} \times 1_A) = \epsilon$$

that is, making the following diagram commute:




Since $1_{B^A} \times 1_A = 1_{(B^A \times A)}$ clearly has this property, we must have

$$\tilde{\epsilon} = 1_{B^A}$$

and so we also know that $\epsilon = \overline{(1_{B^A})}$.

Now let us show that the operation $X \mapsto X^A$ on a CCC is *functorial.*

**Proposition 6.7.** *In any cartesian closed category* $\mathbf{C}$*, exponentiation by a fixed object* $A$ *is a functor,*

$$(-)^A : \mathbf{C} \to \mathbf{C}.$$

Toward the proof, consider first the case of sets. Given some function

$$\beta : B \to C,$$

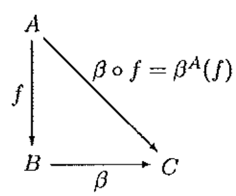
we put

$$\beta^A : B^A \to C^A$$

defined by

$$f \mapsto \beta \circ f.$$

That is,




This assignment is functorial, because for any $\alpha : C \to D$

$$\begin{aligned}(\alpha \circ \beta)^A(f) &= \alpha \circ \beta \circ f \\ &= \alpha \circ \beta^A(f) \\ &= \alpha^A \circ \beta^A(f).\end{aligned}$$

Whence $(\alpha \circ \beta)^A = \alpha^A \circ \beta^A$. Also,

$$\begin{aligned}(1_B)^A(f) &= 1_B \circ f \\ &= f \\ &= 1_{B^A}(f).\end{aligned}$$

So $(1_B)^A = 1_{B^A}$. Thus, $(-)^A$ is indeed a functor; of course, it is just the representable functor $\mathrm{Hom}(A, -)$ that we have already considered.

In a general CCC then, given $\beta : B \to C$, we define

$$\beta^A : B^A \to C^A$$

by

$$\beta^A := \widetilde{(\beta \circ \epsilon)}.$$

That is, we take the transpose of the composite

$$B^A \times A \xrightarrow{\epsilon} B \xrightarrow{\beta} C$$

giving

$$\beta^A : B^A \to C^A.$$

It is easier to see in the form

$$\begin{array}{ccccc} C^A & & C^A \times A & \xrightarrow{\epsilon} & C \\ {\scriptstyle \beta^A}\uparrow & & {\scriptstyle \beta^A \times 1_A}\uparrow & & \uparrow{\scriptstyle \beta} \\ B^A & & B^A \times A & \xrightarrow[\epsilon]{} & B \end{array}$$

Now, clearly,

$$(1_B)^A = 1_{B^A} : B^A \to B^A$$

by examining

$$\begin{array}{ccc} B^A \times A & \xrightarrow{\epsilon} & B \\ {\scriptstyle 1_{(B^A \times A)} = 1_{B^A} \times 1_A}\uparrow & & \uparrow{\scriptstyle 1_B} \\ B^A \times A & \xrightarrow[\epsilon]{} & B \end{array}$$

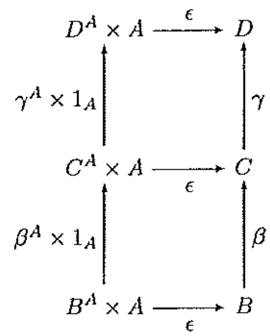
Quite similarly, given

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} D$$

we have

$$\gamma^A \circ \beta^A = (\gamma \circ \beta)^A.$$

This follows from considering the commutative diagram:




We use the fact that

$$(\gamma^A \times 1_A) \circ (\beta^A \times 1_A) = ((\gamma^A \circ \beta^A) \times 1_A).$$

The result follows by the uniqueness of transposes.

There is also another distinguished "universal" arrow; rather than transposing $1_{B^A} : B^A \to B^A$, we can transpose the identity $1_{A\times B} : A \times B \to A \times B$, to get

$$\tilde{1}_{A\times B} : A \to (A \times B)^B.$$

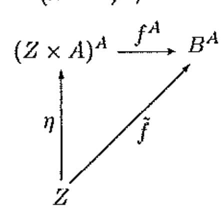
In **Sets**, it has the values $\tilde{1}_{A\times B}(a)(b) = (a, b)$. Let us denote this map by $\eta = \tilde{1}_{A\times B}$, so that

$$\eta(a)(b) = (a, b).$$

The map $\eta$ lets us compute $\tilde{f}$ from the functor $-^A$. Indeed, given $f : Z \times A \to B$, take

$$f^A : (Z \times A)^A \to B^A$$

and precompose with $\eta : Z \to (Z \times A)^A$, as indicated in




This gives the useful equation

$$\tilde{f} = f^A \circ \eta$$

which the reader should prove.

### 6.3 Heyting algebras

Any Boolean algebra $B$, regarded as a poset category, has finite products 1 and $a \wedge b$. We can also define the exponential in $B$ by

$$b^a = (\neg a \vee b)$$

which we also write $a \Rightarrow b$. The evaluation arrow is

$$(a \Rightarrow b) \wedge a \leq b.$$

This always holds since

$$(\neg a \vee b) \wedge a = (\neg a \wedge a) \vee (b \wedge a) = 0 \vee (b \wedge a) = b \wedge a \leq b.$$

To show that $a \Rightarrow b$ is indeed an exponential in $B$, we just need to verify that if $a \wedge b \leq c$ then $a \leq b \Rightarrow c$, that is, transposition. But if $a \wedge b \leq c$, then

$$\neg b \vee (a \wedge b) \leq \neg b \vee c = b \Rightarrow c.$$

But we also have

$$a \leq \neg b \vee a \leq (\neg b \vee a) \wedge (\neg b \vee b) = \neg b \vee (a \wedge b).$$

This example suggests generalizing the notion of a Boolean algebra to that of a cartesian closed poset. Indeed, consider first the following stronger notion.

**Definition 6.8.** A *Heyting algebra* is a poset with

1. Finite meets: 1 and $p \wedge q$,
2. Finite joins: 0 and $p \vee q$,
3. Exponentials: for each $a, b$, an element $a \Rightarrow b$ such that

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \Rightarrow c.$$

The stated condition on exponentials $a \Rightarrow b$ is equivalent to the UMP in the case of posets. Indeed, given the condition, the transpose of $a \wedge b \leq c$ is $a \leq b \Rightarrow c$ and the evaluation $(a \Rightarrow b) \wedge a \leq b$ follows immediately from $a \Rightarrow b \leq a \Rightarrow b$ (the converse is just as simple).

First, observe that every Heyting algebra is a *distributive lattice*, that is, for any $a, b, c$, one has

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

Indeed, we have

$$\begin{aligned}(a \vee b) \wedge c \leq z \ &\text{iff } a \vee b \leq c \Rightarrow z \\ &\text{iff } a \leq c \Rightarrow z \text{ and } b \leq c \Rightarrow z \\ &\text{iff } a \wedge c \leq z \text{ and } b \wedge c \leq z \\ &\text{iff } (a \wedge c) \vee (b \wedge c) \leq z.\end{aligned}$$

Now pick $z = (a \vee b) \wedge c$ and read the equivalences downward to get one direction, then do the same with $z = (a \wedge c) \vee (b \wedge c)$ and reading the equivalences upward to get the other direction.

*Remark* 6.9. The foregoing distributivity is actually a special case of the more general fact that in a cartesian closed category with coproducts, the products necessarily distribute over the coproducts,

$$(A + B) \times C \cong (A \times C) + (B \times C).$$

Although we could prove this now directly, a much more elegant proof (generalizing the one above for the poset case) will be available to us once we have access to the Yoneda Lemma. For this reason, we defer the proof of distributivity to 8.6.

One may well wonder whether all distributive lattices are Heyting algebras. The answer is in general, no; but certain ones always are.

**Definition 6.10.** A poset is (co) *complete* if it is so as a category, thus if it has all set-indexed meets $\bigwedge_{i \in I} a_i$ (resp. joins $\bigvee_{i \in I} a_i$). For posets, completeness and cocompleteness are equivalent (exercise!). A lattice, Heyting algebra, Boolean algebra, etc. is called *complete* if it is so as a poset.

**Proposition 6.11.** *A complete lattice is a Heyting algebra iff it satisfies the infinite distributive law*

$$a \wedge \left( \bigvee_i b_i \right) = \bigvee_i (a \wedge b_i).$$

*Proof.* One shows that Heyting algebra implies distributivity just as in the finite case. To show that the infinite distributive law implies Heyting algebra, set

$$a \Rightarrow b = \bigvee_{x \wedge a \leq b} x.$$

Then, if

$$y \wedge a \leq b$$

then $y \leq \bigvee_{x \wedge a \leq b} x = a \Rightarrow b$. And conversely, if $y \leq a \Rightarrow b$, then $y \wedge a \leq (\bigvee_{x \wedge a \leq b} x) \wedge a = \bigvee_{x \wedge a \leq b} (x \wedge a) \leq \bigvee b = b$. □

*Example* 6.12. For any set $A$, the powerset $P(A)$ is a complete Heyting algebra with unions and intersections as joins and meets, since it satisfies the infinite distributive law. More generally, the lattice of open sets of a topological space is also a Heyting algebra, since the open sets are closed under finite intersections and arbitrary unions.

Of course, every Boolean algebra is a Heyting algebra with $a \Rightarrow b = \neg a \vee b$, as we already showed. But in general, a Heyting algebra is not Boolean. Indeed, we can define a proposed negation by

$$\neg a = a \Rightarrow 0$$

as must be the case, since in a Boolean algebra $\neg a = \neg a \vee 0 = a \Rightarrow 0$. Then $a \leq \neg\neg a$ since $a \wedge (a \Rightarrow 0) \leq 0$. But, conversely, $\neg\neg a \leq a$ need not hold in a Heyting algebra. Indeed, in a topological space $X$, the negation $\neg U$ of an open subset $U$ is the *interior* of the complement $X - U$. Thus, for example, in the real interval $[0, 1]$, we have $\neg\neg(0, 1) = [0, 1]$.

Moreover, the law,

$$1 \leq a \vee \neg a$$

also need not hold in general. In fact, the concept of a Heyting algebra is the algebraic equivalent of the *intuitionistic* propositional calculus (IPC), in the same sense that Boolean algebras are an algebraic formulation of the *classical* propositional calculus.

## 6.4 Propositional calculus

In order to make the connection between Heyting algebras and propositional calculus more rigorous, let us first give a specific system of rules for the IPC. This we do in terms of entailments $p \vdash q$ between formulas $p$ and $q$:

1. $\vdash$ is reflexive and transitive
2. $p \vdash \top$
3. $\bot \vdash p$
4. $p \vdash q$ and $p \vdash r$ iff $p \vdash q \wedge r$
5. $p \vdash r$ and $q \vdash r$ iff $p \vee q \vdash r$
6. $p \wedge q \vdash r$ iff $p \vdash q \Rightarrow r$

This is a complete system for IPC, equivalent to the more standard presentations the reader may have seen. To compare with one perhaps more familiar presentation, note first that we have an "evaluation" entailment by reflexivity and (6):

$$\frac{p \Rightarrow q \vdash p \Rightarrow q}{(p \Rightarrow q) \wedge p \vdash q}$$

We therefore have the rule of "modus ponens" by (4) and transitivity:

$$\frac{\dfrac{\top \vdash p \Rightarrow q \quad \text{and} \quad \top \vdash p}{\top \vdash (p \Rightarrow q) \wedge p}}{\top \vdash q}$$

Moreover, by (4) there are "projections":

$$p \wedge q \vdash p \wedge q$$
$$p \wedge q \vdash p \quad (\text{resp. } q)$$

from which it follows that $p \dashv\vdash \top \wedge p$. Thus, we get one of the usual axioms for products:

$$\begin{aligned} p \wedge q &\vdash p \\ \top \wedge (p \wedge q) &\vdash p \\ \top &\vdash (p \wedge q) \Rightarrow p \end{aligned}$$

Now let us derive the usual axioms for $\Rightarrow$, namely,

1. $p \Rightarrow p$,
2. $p \Rightarrow (q \Rightarrow p)$,
3. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

The first two are almost immediate:

$$\begin{aligned} p &\vdash p \\ \top \wedge p &\vdash p \\ \top &\vdash p \Rightarrow p \end{aligned}$$

$$\begin{aligned} p \wedge q &\vdash p \\ p &\vdash q \Rightarrow p \\ \top \wedge p &\vdash (q \Rightarrow p) \\ \top &\vdash p \Rightarrow (q \Rightarrow p) \end{aligned}$$

For the third one, we use the fact that $\Rightarrow$ distributes over $\wedge$ on the right:

$$a \Rightarrow (b \wedge c) \dashv\vdash (a \Rightarrow b) \wedge (a \Rightarrow c)$$

This is a special case of the exercise:

$$(B \times C)^A \cong B^A \times C^A$$

We also use the following simple fact, which will be recognized as a special case of proposition 6.7:

$$a \vdash b \quad \text{implies} \quad p \Rightarrow a \vdash p \Rightarrow b \tag{6.2}$$

Then we have

$$
\begin{aligned}
(q \Rightarrow r) \wedge q &\vdash r \\
p \Rightarrow ((q \Rightarrow r) \wedge q) &\vdash p \Rightarrow r \\
(p \Rightarrow (q \Rightarrow r)) \wedge (p \Rightarrow q) &\vdash p \Rightarrow r \qquad \text{by (6.3)} \\
(p \Rightarrow (q \Rightarrow r)) &\vdash (p \Rightarrow q) \Rightarrow (p \Rightarrow r) \\
\top &\vdash (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \wedge (p \Rightarrow r)).
\end{aligned}
$$

The "positive" fragment of IPC, involving only the logical operations

$$\top,\ \wedge,\ \Rightarrow$$

corresponds to the notion of a cartesian closed poset. We then add $\bot$ and disjunction $p \vee q$ on the logical side and finite joins on the algebraic side to arrive at a correspondence between IPC and Heyting algebras. The exact correspondence is given by mutually inverse constructions between Heyting algebras and IPCs. We briefly indicate one direction of this correspondence, leaving the other one to the reader's ingenuity.

Given any IPSs $\mathcal{L}$, consisting of propositional formulas $p, q, r, \dots$ over some set of variables $x, y, z, \dots$ together with the rules of inference stated above, and perhaps some distinguished formulas $a, b, c, \dots$ as axioms, one constructs from $\mathcal{L}$ a Heyting algebra (HA)($\mathcal{L}$), called the *Lindenbaum–Tarski algebra*, consisting of equivalence classes $[p]$ of formulas $p$, where

$$[p] = [q] \quad \text{iff} \quad p \dashv\vdash q \tag{6.3}$$

The ordering in $\mathrm{HA}(\mathcal{L})$ is given by

$$[p] \leq [q] \quad \text{iff} \quad p \vdash q \tag{6.4}$$

This is clearly well defined on equivalence classes, in the sense that if $p \vdash q$ and $[p] = [p']$ then $p' \vdash q$, and similarly for $q$. The operations in $\mathrm{HA}(\mathcal{L})$ are then induced in the expected way by the logical operations:

$$
\begin{aligned}
1 &= [\top] \\
0 &= [\bot] \\
[p] \wedge [q] &= [p \wedge q] \\
[p] \vee [q] &= [p \vee q] \\
[p] \Rightarrow [q] &= [p \Rightarrow q]
\end{aligned}
$$

Again, these operations are easily seen to be well defined on equivalence classes, and they satisfy the laws for a Heyting algebra because the logical rules evidently imply them.

**Lemma 6.13.** *Observe that, by* (6.3), $\mathrm{HA}(\mathcal{L})$ *has the property that a formula $p$ is provable $\top \vdash p$ if and only if $[p] = 1$.*

Now define an *interpretation* $M$ of $\mathcal{L}$ in a Heyting algebra $H$ to be an assignment of the basic propositional variables $x, y, z, \dots$ to elements of $H$, which we shall write as $[\![x]\!], [\![y]\!], [\![z]\!], \dots$. An interpretation then extends to all formulas by recursion in the evident way, that is, $[\![p \wedge q]\!] = [\![p]\!] \wedge [\![q]\!]$, etc. An interpretation is called a *model* of $\mathcal{L}$ if for every theorem $\top \vdash p$, one has $[\![p]\!] = 1$. Observe that there is a canonical interpretation of $\mathcal{L}$ in $\mathrm{HA}(\mathcal{L})$ given by $[\![x]\!] = [x]$. One shows easily by induction that, for any formula $p$, moreover, $[\![p]\!] = [p]$. Now lemma 6.13 tells us that this interpretation is in fact a model of $\mathcal{L}$ and that, moreover, it is "generic," in the sense that it validates *only* the provable formulas. We therefore have the following logical *completeness theorem for IPC.*

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**Proposition 6.14.** *The ~~IPC~~ is complete with respect to models in Heyting algebras.*

*Proof.* Suppose a formula $p$ is true in all models in all Heyting algebras. Then in particular, it is so in $\mathrm{HA}(\mathcal{L})$. Thus, $1 = [\![p]\!] = [p]$ in $\mathrm{HA}(\mathcal{L})$, and so $\top \vdash p$. $\square$

In sum, then, a particular instance $\mathcal{L}$ of IPC can be regarded as a way of specifying (and reasoning about) a particular Heyting algebra $\mathrm{HA}(\mathcal{L})$. Indeed, it is essentially a presentation by generators and relations, in just the way that we have already seen for other algebraic objects like monoids. The Heyting algebra $\mathrm{HA}(\mathcal{L})$ even has a UMP with respect to $\mathcal{L}$ that is entirely analogous to the UMP of a finitely presented monoid given by generators and relations. Specifically, if, for instance, $\mathcal{L}$ is generated by the two elements $x, y$ subject to the single "axiom" $x \vee y \Rightarrow x \wedge y$, then in $\mathrm{HA}(\mathcal{L})$ the elements $[x]$ and $[y]$ satisfy $[x] \vee [y] \leq [x] \wedge [y]$ (which is of course equivalent to $([x] \vee [y] \Rightarrow [x] \wedge [y]) = 1$), and given any Heyting algebra $A$ with two elements $a$ and $b$ satisfying $a \vee b \leq a \wedge b$, there is a unique Heyting homomorphism $h : \mathrm{HA}(\mathcal{L}) \to A$ with $h([x]) = a$ and $h([y]) = b$. In this sense, the Lindenbaum–Tarski Heyting algebra $\mathrm{HA}(\mathcal{L})$, being finitely presented by the generators and axioms of $\mathcal{L}$, can be said to contain a "universal model" of the theory determined by $\mathcal{L}$.

## 6.5 Equational definition of CCC

The following description of CCCs in terms of operations and equations on a category is often useful. The proof is entirely routine and left to the reader.

**Proposition 6.15.** *A category* $\mathbf{C}$ *is a CCC iff it has the following structure:*

- *A distinguished object* 1*, and for each object* $C$ *there is given an arrow*

$$!_C : C \to 1$$

*such that for each arrow* $f : C \to 1$,

$$f = !_C.$$

- *For each pair of objects $A, B$, there is given an object $A \times B$ and arrows,*

$$p_1 : A \times B \to A \quad \text{and} \quad p_2 : A \times B \to B$$

*and for each pair of arrows $f : Z \to A$ and $g : Z \to B$, there is given an arrow,*

$$\langle f, g \rangle : Z \to A \times B$$

*such that*

$$p_1 \langle f, g \rangle = f$$
$$p_2 \langle f, g \rangle = g$$
$$\langle p_1 h, p_2 h \rangle = h \quad \text{for all } h : Z \to A \times B.$$

- *For each pair of objects $A, B$, there is given an object $B^A$ and an arrow,*

$$\epsilon : B^A \times A \to B$$

*and for each arrow $f : Z \times A \to B$, there is given an arrow*

$$\tilde{f} : Z \to B^A$$

*such that*

$$\epsilon \circ (\tilde{f} \times 1_A) = f$$

*and*

$$\widetilde{(\epsilon \circ (g \times 1_A))} = g$$

*for all $g : Z \to B^A$. Here, and generally, for any $a : X \to A$ and $b : Y \to B$, we write*

$$a \times b = \langle a \circ p_1, b \circ p_2 \rangle : X \times Y \to A \times B.$$

It is sometimes easier to check these equational conditions than to verify the corresponding UMPs. Section 6.6 provides an example of this sort.

## 6.6 $\lambda$-calculus

We have seen that the notions of a cartesian closed poset with finite joins (i.e., a Heyting algebra) and IPC are essentially the same:

$$\text{HA} \sim \text{IPC}.$$

intuitionistic propositional calculus

These are two different ways of describing one and the same structure; whereby, to be sure, the logical description contains some superfluous data in the choice of a particular presentation.

We now want to consider another, very similar, correspondence between systems of logic and categories, involving more general CCCs. Indeed, the foregoing correspondence was the poset case of the following general one between CCCs and $\lambda$-calculus:

$$\text{CCC} \sim \lambda\text{-calculus.}$$

These notions are also essentially equivalent, in a sense that we now sketch (a more detailed treatment can be found in the book by Lambek and Scott). These are two different ways of representing the same idea, namely that of a collection of objects and functions, with operations of *pairing*, *projection*, *application*, and *transposition* (or "*currying*").

First, recall the notion of a (typed) $\lambda$-calculus from Chapter 2. It consists of

- Types: $A \times B,\ A \to B, \ldots$ (and some basic types)
- Terms: $x, y, z, \ldots : A$ (variables for each type $A$)
  $a : A,\ b : B, \ldots$ (possibly some typed constants)

$$\begin{aligned} \langle a, b\rangle : A \times B &\quad (a : A,\ b : B)\\ \text{fst}(c) : A &\quad (c : A \times B)\\ \text{snd}(c) : B &\quad (c : A \times B)\\ ca : B &\quad (c : A \to B,\ a : A)\\ \lambda x.b : A \to B &\quad (x : A,\ b : B)\end{aligned}$$

- Equations, including at least all instances of the following:

$$\begin{aligned} \text{fst}(\langle a, b\rangle) &= a\\ \text{snd}(\langle a, b\rangle) &= b\\ \langle \text{fst}(c), \text{snd}(c)\rangle &= c\\ (\lambda x.b)a &= b[a/x]\\ \lambda x.cx &= c \quad (\text{no } x \text{ in } c)\end{aligned}$$

Given a particular such $\lambda$-calculus $\mathcal{L}$, the associated *category of types* $\mathbf{C}(\mathcal{L})$ was then defined as follows:

- Objects: the types,
- Arrows $A \to B$: equivalence classes of closed terms $[c] : A \to B$, identified according to (renaming of bound variables and),

$$[a] = [b] \quad \text{iff } \mathcal{L} \vdash a = b \tag{6.5}$$

- Identities: $1_A = [\lambda x.x]$ (where $x : A$),
- Composition: $[c] \circ [b] = [\lambda x.c(bx)]$.

We have already seen that this is a well-defined category, and that it has binary products. It is a simple matter to add a terminal object. Now let us use the equational characterization of CCCs to show that it is cartesian closed. Given any objects $A, B$, we set $B^A = A \to B$, and as the evaluation arrow, we take (the equivalence class of),

$$\epsilon = \lambda z.\mathrm{fst}(z)\mathrm{snd}(z) : B^A \times A \to B \quad (z : Z).$$

Then for any arrow $f : Z \times A \to B$, we take as the transpose,

$$\tilde{f} = \lambda z \lambda x. f\langle z, x\rangle : Z \to B^A \quad (z : Z,\ x : A).$$

It is now a straightforward $\lambda$-calculus calculation to verify the two required equations, namely,

$$\epsilon \circ (\tilde{f} \times 1_A) = f,$$
$$(\epsilon \circ \widetilde{(g \times 1_A)}) = g.$$

In detail, for the first one recall that

$$\alpha \times \beta = \lambda w.\langle \alpha\mathrm{fst}(w), \beta\mathrm{snd}(w)\rangle.$$

So, we have

$$\begin{aligned}
\epsilon \circ (\tilde{f} \times 1_A) &= (\lambda z.\mathrm{fst}(z)\mathrm{snd}(z)) \circ [(\lambda y \lambda x. f\langle y, x\rangle) \times \lambda u.u] \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[(\lambda y \lambda x. f\langle y, x\rangle) \times \lambda u.u]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\lambda w.\langle(\lambda y \lambda x. f\langle y, x\rangle)\mathrm{fst}(w), (\lambda u.u)\mathrm{snd}(w)\rangle]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\lambda w.\langle(\lambda x. f\langle \mathrm{fst}(w), x\rangle), \mathrm{snd}(w)\rangle]v \\
&= \lambda v.(\lambda z.\mathrm{fst}(z)\mathrm{snd}(z))[\langle(\lambda x. f\langle \mathrm{fst}(v), x\rangle), \mathrm{snd}(v)\rangle] \\
&= \lambda v.(\lambda x. f\langle \mathrm{fst}(v), x\rangle)\mathrm{snd}(v) \\
&= \lambda v. f\langle \mathrm{fst}(v), \mathrm{snd}(v)\rangle \\
&= \lambda v. f v \\
&= f.
\end{aligned}$$

The second equation is proved similarly.

Let us call a set of basic types and terms, together with a set of equations between terms, a *theory* in the $\lambda$-calculus. Given such a theory $\mathcal{L}$, the cartesian closed category $\mathbf{C}(\mathcal{L})$ built from the $\lambda$-calculus over $\mathcal{L}$ is the CCC presented by the generators and relations stated by $\mathcal{L}$. Just as in the poset case of IPC and Heyting algebras, there is a logical completeness theorem that follows from this fact. To state it, we require the notion of a model of a theory $\mathcal{L}$ in the $\lambda$-calculus in an arbitrary cartesian closed category $\mathbf{C}$. We give only a brief sketch to give the reader the general idea.

**Definition 6.16.** A *model* of $\mathcal{L}$ in $\mathbf{C}$ is an assignment of the types and terms of $\mathcal{L}$ to objects and arrows of $\mathbf{C}$:

$$X \text{ basic type} \quad \rightsquigarrow \quad [\![X]\!] \text{ object}$$
$$b : A \to B \text{ basic term} \quad \rightsquigarrow \quad [\![b]\!] : [\![A]\!] \to [\![B]\!] \text{ arrow}$$

This assignment is then extended to all types and terms in such a way that the $\lambda$-calculus operations are taken to the corresponding CCC ones:

$$[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$$
$$[\![\langle f, g\rangle]\!] = \langle [\![f]\!], [\![g]\!]\rangle$$
$$\text{etc.}$$

Finally, it is required that all the equations of $\mathcal{L}$ are satisfied, in the sense that

$$\mathcal{L} \vdash [a] = [b] : A \to B \quad \text{implies} \quad [\![a]\!] = [\![b]\!] : [\![A]\!] \to [\![B]\!]. \tag{6.6}$$

This is what is sometimes called "denotational semantics" for the $\lambda$-calculus. It is essentially the conventional, set-theoretic semantics for first-order logic, but extended to higher types, restricted to equational theories, and generalized to CCCs.

For example, let $\mathcal{L}$ be the theory with one basic type $X$, two basic terms,

$$u : X$$
$$m : X \times X \to X$$

and the usual equations for associativity and units,

$$m\langle u, x\rangle = x$$
$$m\langle x, u\rangle = x$$
$$m\langle x, m\langle y, z\rangle\rangle = m\langle m\langle x, y\rangle, z\rangle.$$

Thus, $\mathcal{L}$ is just the usual equational theory of monoids. Then a model of $\mathcal{L}$ in a cartesian closed category $\mathbf{C}$ is nothing but a monoid in $\mathbf{C}$, that is, an object $M = [\![X]\!]$ equipped with a distinguished point

$$[\![u]\!] : 1 \to M$$

and a binary operation

$$[\![m]\!] : M \times M \to M$$

satisfying the unit and associativity laws.

Note that by (6.5) and (6.6), there is a model of $\mathcal{L}$ in $\mathbf{C}(\mathcal{L})$ with the property that $[\![a]\!] = [\![b]\!] : X \to Y$ if and only if $a = b$ is provable in $\mathcal{L}$. In this way, one can prove the following *CCC completeness theorem for $\lambda$-calculus.*

**Proposition 6.17.** *For any theory $\mathcal{L}$ in the λ-calculus, one has the following:*

1. *For any terms $a, b$, $\mathcal{L} \vdash a = b$ iff for all models $M$ in CCCs, $[\![a]\!]_M = [\![b]\!]_M$.*
2. *Moreover, for any type $A$, there is a closed $t : A$ iff for all models $M$ in CCCs, there is an arrow $1 \to [\![A]\!]_M$.*

This proposition says that the λ-calculus is *deductively sound and complete* for models in CCCs. It is worth emphasizing that completeness is not true if one restricts attention to models in the single category **Sets**; indeed, there are many examples of theories in λ-calculus in which equations holding for all models in **Sets** are still not provable (see the exercises for an example).

Soundness (i.e., the "only if" direction of the above statements) follows from the following UMP of the cartesian closed category $\mathbf{C}(\mathcal{L})$, analogous to the one for any algebra presented by generators and relations. Given any model $M$ of $\mathcal{L}$ in any cartesian closed category $\mathbf{C}$, there is a unique functor,

$$[\![-]\!]_M : \mathbf{C}(\mathcal{L}) \to \mathbf{C}$$

preserving the CCC structure, given by

$$[\![X]\!]_M = M$$

for the basic type $X$, and similarly for the other basic types and terms of $\mathcal{L}$. In this precise sense, the theory $\mathcal{L}$ is a presentation of the cartesian closed category $\mathbf{C}(\mathcal{L})$ by generators and relations.

Finally, let us note that the notions of λ-calculus and CCC are essentially "equivalent," in the sense that any cartesian closed category $\mathbf{C}$ also gives rise to a λ-calculus $\mathcal{L}(\mathbf{C})$, and this construction is essentially inverse to the one just sketched.

Briefly, given $\mathbf{C}$, we define $\mathcal{L}(\mathbf{C})$ by

- Basic types: the objects of $\mathbf{C}$
- Basic terms: $a : A \to B$ for each $a : A \to B$ in $\mathbf{C}$
- Equations: many equations identifying the λ-calculus operations with the corresponding category and CCC structure on $\mathbf{C}$, for example,

$$\lambda x.\text{fst}(x) = p_1$$
$$\lambda x.\text{snd}(x) = p_2$$
$$\lambda y.f(x, y) = \tilde{f}(x)$$
$$g(f(x)) = (g \circ f)(x)$$
$$\lambda y.y = 1_A$$

This suffices to ensure that there is an isomorphism of categories,

$$\mathbf{C}(\mathcal{L}(\mathbf{C})) \cong \mathbf{C}.$$

Moreover, the theories $\mathcal{L}$ and $\mathcal{L}(\mathrm{C}(\mathcal{L}))$ are also "equivalent" in a suitable sense, involving the kinds of considerations typical of comparing different presentations of algebras. We refer the reader to the excellent book by Lambek and Scott (1986), for further details.

## 6.7 Variable sets

We conclude with a special kind of CCC related to the so-called Kripke models of logic, namely categories of *variable sets.* These categories provide specific examples of the "algebraic" semantics of IPC and $\lambda$-calculus just given.

### *6.7.1 IPC*

Let us begin by very briefly reviewing the notion of a Kripke model of IPC from our algebraic point of view; we focus on the positive fragment involving only $\top, p \wedge q, p \Rightarrow q$, and variables.

A *Kripke model* of this language $\mathcal{L}$ consists of a poset $I$ of "possible worlds," which we write $i \leq j$, together with a relation between worlds $i$ and propositions $p$,

$$i \Vdash p,$$

read "$p$ holds at i." This relation is assumed to satisfy the following conditions:

(1) $i \Vdash p$ and $i \leq j$ implies $j \Vdash p$
(2) $i \Vdash \top$
(3) $i \Vdash p \wedge q$ iff $i \Vdash p$ and $i \Vdash q$
(4) $i \Vdash p \Rightarrow q$ iff $j \Vdash p$ implies $j \Vdash q$ for all $j \geq i$.

One then sets

$$I \Vdash p \quad \text{iff} \quad i \Vdash p \quad \text{for all } i \in I.$$

And finally, we have the well-known theorem,

**Theorem 6.18 (Kripke completeness for IPC).** *A propositional formula $p$ is provable from the rules for* IPC *iff it holds in all Kripke models, that is, iff $I \Vdash p$ for all relations $\Vdash$ over all posets $I$,*

$$\mathrm{IPC} \vdash p \quad \textit{iff} \quad I \Vdash p \quad \textit{for all } I.$$

Now let us see how to relate this result to our formulation of the semantics of IPC in Heyting algebras. First, the relation $\Vdash \subseteq I \times \mathrm{Prop}(\mathcal{L})$ between worlds $I$ and propositional formulas $\mathrm{Prop}(\mathcal{L})$ can be equivalently formulated as a mapping,

$$[\![-]\!] : \mathrm{Prop}(\mathcal{L}) \longrightarrow \mathbf{2}^I, \tag{6.7}$$

where we write $\mathbf{2}^I = \mathrm{Hom}_{\mathbf{Pos}}(I, \mathbf{2})$ for the exponential poset of monotone maps from $I$ into the poset $\mathbf{2} = \{\bot \leq \top\}$. This poset is a CCC, and indeed a Heyting algebra, the proof of which we leave as an exercise for the reader. The mapping (6.7) is determined by the condition

$$[\![p]\!](i) = \top \quad \text{iff} \quad i \Vdash p.$$

Now, in terms of the Heyting algebra semantics of IPC developed in Section 6.4 (adapted in the evident way to the current setting without the coproducts $\bot, p \vee q$, and writing $\mathrm{HA}^-$ for Heyting algebras without coproducts, i.e., poset CCCs), the poset $\mathrm{HA}^-(\mathcal{L})$ is a quotient of $\mathrm{Prop}(\mathcal{L})$ by the equivalence relation of mutual derivability $p \dashv\vdash q$, which clearly makes it a CCC, and the map (6.7) therefore determines a model (with the same name),

$$[\![-]\!] : \mathrm{HA}^-(\mathcal{L}) \longrightarrow \mathbf{2}^I.$$

Indeed, condition (1) above ensures that $[\![p]\!] : I \to \mathbf{2}$ is monotone, and (2)–(4) ensure that $[\![-]\!]$ is a homomorphism of poset CCCs, that is, that it is monotone and preserves the CCC structure (exercise!). Thus, a Kripke model is just an "algebraic" model in a Heyting algebra of the special form $\mathbf{2}^I$. The Kripke completeness theorem for positive IPC above then follows from Heyting-valued completeness theorem proposition 6.14 together with the following, purely algebraic, embedding theorem for poset CCCs.

**Proposition 6.19.** *For every poset CCC* $\mathbf{A}$, *there is a poset* $I$ *and an injective, monotone map,*

$$y : \mathbf{A} \rightarrowtail \mathbf{2}^I,$$

*preserving CCC structure.*

*Proof.* We can take $I = \mathbf{A}^{\mathrm{op}}$ and $y(a) : \mathbf{A}^{\mathrm{op}} \to \mathbf{2}$, the "truth-value" of $x \leq a$, that is, $y(a)$ is determined by

$$y(a)(x) = \top \quad \text{iff} \quad x \leq a.$$

Clearly, $y(a)$ is monotone and contravariant, while $y$ itself is monotone and covariant. We leave it as an exercise to verify that $y$ is injective and preserves the CCC structure, but note that $\mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ can be identified with the collection of *lower sets* $S \subseteq \mathbf{A}$ in $\mathbf{A}$, that is, subsets that are closed downward: $x \leq y \in S$ implies $x \in S$. Under this identification, we then have $y(a) = \downarrow(a) = \{x \mid x \leq a\}$.

A proof is also given in Chapter 8 as a consequence of the Yoneda Lemma. □

The result can be extended from poset CCCs to Heyting algebras, thus recovering the usual Kripke completeness theorem for full IPC, by the same argument using a more delicate embedding theorem that also preserves the coproducts $\bot$ and $p \vee q$.

### *6.7.2 λ-calculus*

We now want to generalize the foregoing from propositional logic to the λ-calculus, motivated by the insight that the latter is the proof theory of the former (according to the Curry–Howard–correspondence). Categorically speaking, we are generalizing from the poset case to the general case of a CCC. According to the "propositions-as-types" conception behind the C–H correspondence, we therefore should replace the poset CCC of idealized propositions **2** with the general CCC of idealized types **Sets**. We therefore model the $\lambda$-calculus in categories of the form $\mathbf{Sets}^I$ for posets $I$, which can be regarded as comprised of "$I$-indexed," or "variable sets," as we now indicate.

Given a poset $I$, an *$I$-indexed set* is a family of sets $(A_i)_{i\in I}$ together with transition functions $\alpha_{ij} : A_i \to A_j$ for each $i \leq j$, satisfying the compatibility conditions:

- $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ whenever $i \leq j \leq k$,
- $\alpha_{ii} = 1_{A_i}$ for all $i$.

In other words, it is simply a functor,

$$A : I \longrightarrow \mathbf{Sets}.$$

We can think of such $I$-indexed sets as "sets varying in a parameter" from the poset $I$. For instance, if $I = \mathbb{R}$ thought of as time, then an $\mathbb{R}$-indexed set $A$ may be thought of as a set varying through time: some elements $a, b \in A_t$ may become identified over time (the $\alpha$s need not be injective), and new elements may appear over time (the $\alpha$s need not be surjective), but once an element is in the set ($a \in A_t$), it stays in forever ($\alpha_{tt'}(a) \in A_{t'}$). For a more general poset $I$, the variation is parameterized accordingly.

A product of two variable sets $A$ and $B$ can be constructed by taking the pointwise products $(A \times B)(i) = A(i) \times B(i)$ with the evident transition maps,

$$\alpha_{ij} \times \beta_{ij} : A(i) \times B(i) \to A(j) \times B(j) \qquad i \leq j$$

where $\beta_{ij} : B_i \to B_j$ is the transition map for $B$. This plainly gives an $I$-indexed set, but to check that it really is a product we need to make $\mathbf{Sets}^I$ into a category and verify the UMP (respectively, the operations and equations of Section 6.5). What is a map of $I$-indexed sets $f : A \to B$? One natural proposal is this: it is an $I$-indexed family of functions $(f_i : A_i \to B_i)_{i\in I}$ that are compatible with the transition maps, in the sense that whenever $i \leq j$, then the following commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle \alpha_{ij}}\downarrow & & \downarrow{\scriptstyle \beta_{ij}} \\ A_j & \xrightarrow[f_j]{} & B_j \end{array}$$

We can think of this condition as saying that $f$ takes elements $a \in A$ to elements $f(a) \in B$ without regard to when the transition is made, since given $a \in A_i$ it does not matter if we first wait until $j \geq i$ and then take $f_j(\alpha_{ij}(a))$, or go right away to $f_i(a)$ and then wait until $\beta_{ij}(f_i(a))$.

Indeed, in Chapter 7 we see that this type of map is exactly what is called a "natural transformation" of the functors $A$ and $B$. These maps $f : A \to B$ compose in the evident way:

$$(g \circ f)_i = g_i \circ f_i : A_i \longrightarrow B_i$$

to make $\mathit{Sets}^I$ into a category, the *category of I-indexed sets.* It is now an easy exercise to confirm that the specification of the product $A \times B$ just given really is a product in the resulting category $\mathbf{Sets}^I$, and the terminal object is obviously the constant index set 1, so $\mathbf{Sets}^I$ has all finite products.

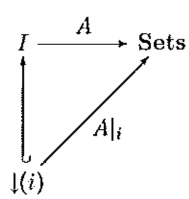
What about exponentials? The first attempt at defining pointwise exponentials,

$$(B^A)_i = B_i^{A_i}$$

fails, because the indexing is covariant in $B$ and *contra*variant in $A$, as the reader should confirm. The idea that maybe $B^A$ is just the collection of all index maps from $A$ to $B$ also fails, because it is not indexed! The solution is a combintion of these two ideas which generalizes the "Kripke" exponential as follows. For each $i \in I$, let

$$\downarrow(i) \subseteq I$$

be the lower set below $i$, regarded as a subposet. Then for any $A : I \to \mathbf{Sets}$, let $A|_i$ be the restriction,




This determines an indexed set over $\downarrow(i)$. Given any $f : A \to B$ and $i \in I$, there is an evident restriction $f|_i : A|_i \to B|_i$ which is defined to be simply $(f|_i)_j = f_j$ for any $j \leq i$. Now we can define

$$(B^A)_i \;=\; \{f : A|_i \to B|_i \mid f \text{ is } \downarrow(i)\text{-indexed}\}$$

with the transition maps given by

$$f \mapsto f|_j \qquad j \leq i$$

It is immediate that this determines an $I$-indexed set $B^A$. That it is actually the exponential of $A$ and $B$ in $\mathbf{Sets}^I$ is shown later, as an easy consequence of the Yoneda Lemma. For the record, we therefore have the following (proof deferred).

**Proposition 6.20.** *For any poset $I$, the category $\mathit{Sets}^I$ of $I$-indexed sets and functions is cartesian closed.*

**Definition 6.21.** A *Kripke model* of a theory $\mathcal{L}$ in the $\lambda$-calculus is a model (in the sense of definition 6.16) in a cartesian closed category of the form $\mathbf{Sets}^I$ for a poset $I$.

For instance, it can be seen that a Kripke model over a poset $I$ of a conventional algebraic theory such as the theory of groups is just an $I$-indexed group, that is, a functor $I \to \mathbf{Group}$. In particular, if $I = \mathcal{O}(X)^{\mathrm{op}}$ for a topological space $X$, then this is just what the topologist calls a "presheaf of groups." On the other hand, it also agrees with (or generalizes) the logician's notion of a Kripke model of a first-order language, in that it consists of a varying domain of "individuals" equipped with varying structure.

Finally, in order to generalize the Kripke completeness theorem for IPC to $\lambda$-calculus, it clearly suffices to sharpen our general CCC completeness theorem, proposition 6.17, to the special models in CCCs of the form $\mathbf{Sets}^I$ by means of an embedding theorem analogous to proposition 6.19. Indeed, one can prove ~~this.~~ the following.

**Proposition 6.22.** *For every CCC* $\mathbf{C}$, *there is a poset* $I$ *and a functor,*

$$y : \mathbf{C} \rightarrowtail \mathbf{Sets}^I,$$

*that is injective on both objects and arrows and preserves CCC structure. Moreover, every map between objects in the image of* $y$ *is itself in the image of* $y$ *(*$y$ *is "full").*

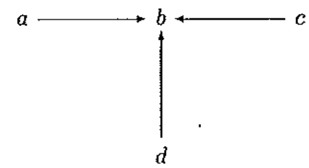
The full proof of this result involves methods from topos theory that are beyond the scope of this book. But a significant part of it, to be given below, is entirely analogous to the proof of the poset case, and will again be a consequence of the Yoneda Lemma.

### 6.8 Exercises

1. Show that for all finite sets $M$ and $N$,
$$|N^M| = |N|^{|M|},$$
where $|K|$ is the number of elements in the set $K$, while $N^M$ is the exponential in the category of sets (the set of all functions $f : M \to N$), and $n^m$ is the usual exponentiation operation of arithmetic.
2. Show that for any three objects $A, B, C$ in a cartesian closed category, there are isomorphisms:
   (a) $(A \times B)^C \cong A^C \times B^C$
   (b) $(A^B)^C \cong A^{B \times C}$
3. Determine the exponential transpose $\tilde{\varepsilon}$ of evaluation $\varepsilon : B^A \times A \to B$ (for any objects in any CCC). In $\mathbf{Sets}$, determine the transpose $\tilde{1}$ of the identity

$1 : A \times B \to A \times B$. Also determine the transpose of $\varepsilon \circ \tau : A \times B^A \to B$, where $\tau : A \times B^A \to B^A \times A$ is the "twist" arrow $\tau = \langle p_2, p_1 \rangle$.

4. Is the category of monoids cartesian closed?
5. Verify the description given in the text of the exponential graph $H^G$ for two graphs $G$ and $H$. Determine the exponential $\mathbf{2}^G$, where $\mathbf{2}$ is the graph $v_1 \to v_2$ with two vertices and one edge, and $G$ is an arbitrary graph. Determine $\mathbf{2}^G$ explicitely for $G$ the graph pictured below.




6. Consider the category of sets equipped with a (binary) relation, $(A, R \subseteq A \times A)$, with maps $f : (A, R) \to (B, S)$ being those functions $f : A \to B$ such that $aRa'$ implies $f(a)Sf(a')$. Show this category is cartesian closed by describing it as a subcategory of graphs.
7. Consider the category of sets equipped with a distinguished subset, $(A, P \subseteq A)$, with maps $f : (A, P) \to (B, Q)$ being those functions $f : A \to B$ such that $a \in P$ iff $f(a) \in Q$. Show this category is cartesian closed by describing it as a category of pairs of sets.
8. Consider the category of "pointed sets," that is, sets equipped with a distinguished element, $(A, a \in A)$, with maps $f : (A, a) \to (B, b)$ being those functions $f : A \to B$ such that $f(a) = b$. Is this category cartesian closed?
9. Show that for any objects $A, B$ in a cartesian closed category, there is a bijective correspondence between points of the exponential $1 \to B^A$ and arrows $A \to B$.
10. Show that the category of $\omega$CPOs is cartesian closed, but that the category of *strict* $\omega$CPOs is not (the strict $\omega$CPOs are the ones with initial object $\bot$, and the continuous maps between them are supposed to preserve $\bot$).
11. (a) Show that in any cartesian closed poset with joins $p \vee q$, the following "distributive" law of IPC holds:

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

    (b) Generalize the foregoing problem to an arbitrary category (not necessarily a poset), by showing that there is always an arrow of the corresponding form.

    (c) If you are brave, show that the previous two arrows are isomorphisms.

12. Prove that in a CCC $\mathbf{C}$, exponentiation with a fixed base object $C$ is a contravariant functor $C^{(-)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, where $C^{(-)}(A) = C^A$.

Text is mismatch i followed the hard copy plz confirm.

13. Show that in a cartesian closed category with coproducts, the products necessarily distribute over the coproducts,

$$(A \times C) + (B \times C) \cong (A + B) \times C.$$

14. In the $\lambda$-calculus, consider the theory (due to Dana Scott) of a *reflexive domain*: there is one basic type $D$, two constants $s$ and $r$ of types

$$s : (D \to D) \to D$$
$$r : D \to (D \to D),$$

and two equations,

$$srx = x \quad (x : D)$$
$$rsy = y \quad (y : D \to D).$$

Prove that, up to isomorphism, this theory has only one model $M$ in **Sets**, and that *every* equation holds in $M$.

15. Complete the proof from the text of Kripke completeness for the positive fragement of IPC as follows:

    (a) Show that for any poset $I$, the exponential poset $\mathbf{2}^I$ is a Heyting algebra. (Hint: the limits and colimits are "pointwise," and the Heyting implication $p \Rightarrow q$ is defined at $i \in I$ by $(p \Rightarrow q)(i) = \top$ iff for all $j \geq i, p(j) \leq q(j)$.)
    (b) Show that for any poset CCC $\mathbf{A}$, the map $y : \mathbf{A} \to \mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ defined in the text is indeed (i) monotone, (ii) injective, and (iii) preserves CCC structure.

16. Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of $I$-indexed sets ($I$ a poset) can be computed "pointwise." Show, moreover, that the same is true for all limits and colimits.

# 7

# NATURALITY

We now want to start considering categories and functors more systematically, developing the "category theory" of category theory itself, rather than of other mathematical objects, like groups, or formulas in a logical system. Let me emphasize that, while some of this may look a bit like "abstract nonsense," the idea behind it is that when one has a particular application at hand, the theory can then be specialized to that concrete case. The notion of a functor is a case in point; developing its general theory makes it a clarifying, simplifying, and powerful tool in its many instances.

## 7.1 Category of categories

We begin by reviewing what we know about the category **Cat** of categories and functors and tying up some loose ends.

We have already seen that **Cat** has finite coproducts $\mathbf{0}$, $\mathbf{C} + \mathbf{D}$; and finite products $\mathbf{1}$, $\mathbf{C} \times \mathbf{D}$. It is very easy to see that there are also all small coproducts and products, constructed analogously. We can therefore show that **Cat** has all limits by constructing equalizers. Thus, let categories $\mathbf{C}$ and $\mathbf{D}$ and parallel functors $F$ and $G$ be given, and define the category $\mathbf{E}$ and functor $E$,

$$\mathbf{E} \xrightarrow{\ E\ } \mathbf{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{D}$$

as follows (recall that for a category $\mathbf{C}$, we write $\mathbf{C}_0$ and $\mathbf{C}_1$ for the collections of objects and arrows, respectively):

$$\mathbf{E}_0 = \{C \in \mathbf{C}_0 \mid F(C) = G(C)\}$$
$$\mathbf{E}_1 = \{f \in \mathbf{C}_1 \mid F(f) = G(f)\}$$

and let $E : \mathbf{E} \to \mathbf{C}$ be the evident inclusion. This is then an equalizer, as the reader can easily check.

The category $\mathbf{E}$ is an example of a *subcategory*, that is, a monomorphism in **Cat** (recall that equalizers are monic). Often, by a subcategory of a category $\mathbf{C}$ one means specifically a collection $\mathbf{U}$ of some of the objects and arrows, $\mathbf{U}_0 \subseteq \mathbf{C}_0$ and $\mathbf{U}_1 \subseteq \mathbf{C}_1$), that is closed under the operations dom, cod, id, and $\circ$. There is then an evident inclusion functor

$$i : \mathbf{U} \to \mathbf{C}$$

which is clearly monic.

In general, coequalizers of categories are more complicated to describe—indeed, even for posets, determining the coequalizer of a pair of monotone maps can be quite involved, as the reader should consider.

There are various properties of functors other than being monic and epic that turn out to be quite useful in **Cat**. A few of these are given by the following:

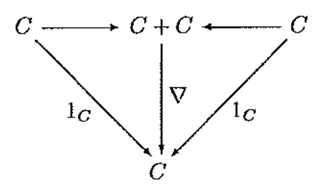
**Definition 7.1.** A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be

- *injective on objects* if the object part $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ is injective, it is *surjective on objects* if $F_0$ is surjective.
- Similarly, $F$ is *injective* (resp. *surjective*) *on arrows* if the arrow part $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ is injective (resp. surjective).
- $F$ is *faithful* if for all $A, B \in \mathbf{C}_0$, the map

  $$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(FA, FB)$$

  defined by $f \mapsto F(f)$ is injective.
- Similarly, $F$ is *full* if $F_{A,B}$ is always surjective.

What is the difference between being faithful and being injective on arrows? Consider, for example, the "codiagonal functor" $\nabla : \mathbf{C} + \mathbf{C} \to \mathbf{C}$, as indicated in the following:




$\nabla$ is faithful, but not injective on arrows.

A *full subcategory*

$$\mathbf{U} \rightarrowtail \mathbf{C}$$

consists of some objects of $\mathbf{C}$ and *all* of the arrows between them (thus satisfying the closure conditions for a subcategory). For example, the inclusion functor $\mathbf{Sets}_{fin} \rightarrowtail \mathbf{Sets}$ is full and faithful, but the forgetful functor $\mathbf{Groups} \rightarrowtail \mathbf{Sets}$ is faithful but not full.

*Example* 7.2. There is another "forgetful" functor for groups, namely to the category **Cat** of categories,

$$G : \mathbf{Groups} \to \mathbf{Cat}.$$

Observe that this functor is full and faithful, since a functor between groups $F : G(A) \to G(B)$ is exactly the same thing as a group homomorphism.

And exactly the same situation holds for monoids.

For posets, too, there is a full and faithful, forgetful functor

$$P : \mathbf{Pos} \to \mathbf{Cat}$$

again because a functor between posets $F : P(A) \to P(B)$ is exactly a monotone map. And the same thing holds for the "discrete category" functor $S : \mathbf{Sets} \to \mathbf{Cat}$.

Thus, **Cat** provides a setting for *comparing structures of many different kinds.* For instance, one can have a functor $R : G \to \mathbf{C}$ from a group $G$ to a category $\mathbf{C}$ that is not a group. If $\mathbf{C}$ is a poset, then any such functor must be trivial (why?). But if $\mathbf{C}$ is, say, the category of finite dimensional, real vector spaces and linear maps, then a functor $R$ is exactly a *linear representation* of the group $G$, representing every element of $G$ as an invertible matrix of real numbers and the group multiplication as matrix multiplication.

What is a functor $g : P \to G$ from a poset to a group? Since $G$ has only one object $*$, it has $g(p) = * = g(q)$ for all $p, q \in P$. For each $p \leq q$, it picks an element $g_{p,q}$ in such a way that

$$g_{p,p} = u \qquad \text{(the unit of } G\text{)}$$
$$g_{q,r} \cdot g_{p,q} = g_{p,r}.$$

For example, take $P = (\mathbb{R}, \leq)$ to be the ordered real numbers and $G = (\mathbb{R}, +)$ the additive group of reals, then *subtraction* is a functor,

$$g : (\mathbb{R}, \leq) \to (\mathbb{R}, +)$$

defined by

$$g_{x,y} = (y - x).$$

Indeed, we have

$$g_{x,x} = (x - x) = 0$$
$$g_{y,z} \cdot g_{x,y} = (z - y) + (y - x) = (z - x) = g_{x,z}.$$

## 7.2 Representable structure

Let $\mathbf{C}$ be a *locally small category*, so that we have the *representable functors*,

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

for all objects $C \in \mathbf{C}$. This functor is evidently faithful if the object $C$ has the property that for any objects $X$ and $Y$ and arrows $f, g : X \rightrightarrows Y$, if $f \neq g$ there is an arrow $x : C \to X$ such that $fx \neq gx$. That is, the arrows in the category are distinguished by their effect on generalized elements based at $C$. Such an object $C$ is called a *generator* for $\mathbf{C}$.

In the category of sets, for example, the terminal object 1 is a generator. In groups, as we have already discussed, the free group $F(1)$ on one element is a generator. Indeed, the functor represented by $F(1)$ is isomorphic to the forgetful functor $U : \mathbf{Grp} \to \mathbf{Sets}$,

$$\mathrm{Hom}(F(1), G) \cong U(G). \tag{7.1}$$

This isomorphism not only holds for each group $G$, but also respects group homomorphisms, in the sense that for any such $h : G \to H$, there is a commutative square,

$$\begin{array}{ccccc} G & \quad & \mathrm{Hom}(F(1), G) & \xrightarrow{\cong} & U(G) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h_*}\big\downarrow & & \big\downarrow{\scriptstyle U(h)} \\ H & & \mathrm{Hom}(F(1), H) & \xrightarrow[\cong]{} & U(H) \end{array}$$

One says that the isomorphism (7.1) is "natural in $G$." In a certain sense, this also "explains" why the forgetful functor $U$ preserves all limits, since representable functors necessarily do. The related fact that the forgetful functor is faithful is a precise way to capture the vague idea, which we initially used for motivation, that the category of groups is "concrete."

Recall that there are also *contravariant* representable functors

$$\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

taking $f : A \to B$ to $f^* : \mathrm{Hom}_{\mathbf{C}}(B, C) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$ by $f^*(h) = h \circ f$ for $h : B \to C$.

*Example* 7.3. Given a group $G$ in a (locally small) category $\mathbf{C}$, the contravariant representable functor $\mathrm{Hom}_{\mathbf{C}}(-, G)$ actually has a group structure, giving a functor

$$\mathrm{Hom}_{\mathbf{C}}(-, G) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grp}.$$

In $\mathbf{Sets}$, for example, for each set $X$, we can define the operations on the group $\mathrm{Hom}(X, G)$ pointwise,

$$\begin{aligned} u(x) &= u \quad \text{(the unit of } G) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ f^{-1}(x) &= f(x)^{-1}. \end{aligned}$$

In this case, we have an isomorphism

$$\mathrm{Hom}(X, G) \cong \Pi_{x \in X} G$$

with the product group. Functoriality in $X$ is given simply by precomposition; thus, for any function $h : Y \to X$, one has

$$\begin{aligned} h^*(f \cdot g)(y) &= (f \cdot g)(h(y)) \\ &= f(h(y)) \cdot g(h(y)) \\ &= h^*(f)(y) \cdot h^*(g)(y) \\ &= (h^*(f) \cdot h^*(g))(y) \end{aligned}$$

and similarly for inverses and the unit. Indeed, it is easy to see that this construction works just as well for any other algebraic structure defined by operations and equations. Nor is there anything special about the category **Sets** here; we can do the same thing in any category with an internal algebraic structure.

For instance, in topological spaces, one has the ring $\mathbb{R}$ of real numbers and, for any space $X$, the ring

$$\mathcal{C}(X) = \mathrm{Hom}_{\mathbf{Top}}(X, \mathbb{R})$$

of real-valued, continuous functions on $X$. Just as in the previous case, if

$$h : Y \to X$$

is any continuous function, we then get a ring homomorphism

$$h^* : \mathcal{C}(X) \to \mathcal{C}(Y)$$

by precomposing with $h$. The recognition of $\mathcal{C}(X)$ as representable ensures that this "ring of real-valued functions" construction is functorial,

$$\mathcal{C} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Rings}.$$

Note that in passing from $\mathbb{R}$ to $\mathrm{Hom}_{\mathbf{Top}}(X, \mathbb{R})$, all the algebraic structure of $\mathbb{R}$ is retained, but properties determined by conditions that are not strictly equational are not necessarily preserved. For instance, $\mathbb{R}$ is not only a ring, but also a *field*, meaning that every nonzero real number $r$ has a multiplicative inverse $r^{-1}$; formally,

$$\forall x (x = 0 \vee \exists y.\ y \cdot x = 1).$$

To see that this condition fails in, for example, $\mathcal{C}(\mathbb{R})$, consider the continuous function $f(x) = x^2$. For any argument $y \neq 0$, the multiplicative inverse must be $g(y) = 1/y^2$. But if this function were to be continuous, at 0 it would have to be $\lim_{y \to 0} 1/y^2$ that does not exist in $\mathbb{R}$.

*Example* 7.4. A very similar situation occurs in the category **BA** of Boolean algebras. Given the Boolean algebra **2** with the usual (truth-table) operations $\wedge, \vee, \neg, 0, 1$, for any set $X$, we make the set

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2})$$

into a Boolean algebra with the pointwise operations:

$$
\begin{aligned}
0(x) &= 0 \\
1(x) &= 1 \\
(f \wedge g)(x) &= f(x) \wedge g(x) \\
&\text{etc.}
\end{aligned}
$$

When we define the operations in this way in terms of those on $\mathbf{2}$, we see immediately that $\mathrm{Hom}(X, \mathbf{2})$ is a Boolean algebra too, and that precomposition is a contravariant functor,

$$\mathrm{Hom}(-, \mathbf{2}) : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$$

into the category $\mathbf{BA}$ of Boolean algebras and their homomorphisms.

Now observe that for any set $X$, the familiar isomorphism

$$\mathrm{Hom}(X, \mathbf{2}) \cong \mathcal{P}(X)$$

between characteristic functions $\phi : X \to \mathbf{2}$ and subsets $V_\phi = \phi^{-1}(1) \subseteq X$, relates the pointwise Boolean operations in $\mathrm{Hom}(X, \mathbf{2})$ to the subset operations of intersection, union, etc. in $\mathcal{P}(X)$:

$$
\begin{aligned}
V_{\phi \wedge \psi} &= V_\phi \cap V_\psi \\
V_{\phi \vee \psi} &= V_\phi \cup V_\psi \\
V_{\neg \phi} &= X - V_\phi \\
V_1 &= X \\
V_0 &= \emptyset
\end{aligned}
$$

In this sense, the set-theoretic Boolean operations on $\mathcal{P}(X)$ are induced by those on $\mathbf{2}$, and the powerset $\mathcal{P}$ is seen to be a contravariant functor to the category of Boolean algebras,

$$\mathcal{P}^{\mathrm{BA}} : \mathbf{Sets}^{\mathrm{op}} \to \mathrm{BA}.$$

As was the case for the covariant representable functor $\mathrm{Hom}_{\mathbf{Grp}}(F(1), -)$ and the forgetful functor $U$ from groups to sets, here the contravariant functors $\mathrm{Hom}_{\mathbf{Sets}}(-, \mathbf{2})$ and $\mathcal{P}^{\mathrm{BA}}$ from sets to Boolean algebras can also be seen to be *naturally* isomorphic, in the sense that for any function $f : Y \to X$, the following square of Boolean algebras and homomorphisms commutes:

$$
\begin{array}{ccccc}
X & \quad & \mathrm{Hom}(X, \mathbf{2}) & \xrightarrow{\cong} & P(X) \\
{\scriptstyle f}\uparrow & & {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\
Y & & \mathrm{Hom}(Y, \mathbf{2}) & \xrightarrow[\cong]{} & P(Y)
\end{array}
$$

### 7.3 Stone duality

Before considering the topic of naturality more systematically, let us take a closer look at the foregoing example of powersets and Boolean algebras.

Recall that an *ultrafilter* in a Boolean algebra $B$ is a proper subset $U \subset B$ such that

- $1 \in U$
- $x, y \in U$ implies $x \wedge y \in U$
- $x \in U$ and $x \leq y$ implies $y \in U$
- if $U \subset U'$ and $U'$ is a filter, then $U' = B$

The maximality condition on $U$ is equivalent to the condition that for every $x \in B$, either $x \in U$ or $\neg x \in U$ but not both (exercise!).

We already know that there is an isomorphism between the set $\mathrm{Ult}(B)$ of ultrafilters on $B$ and the Boolean homomorphisms $B \to \mathbf{2}$,

$$\mathrm{Ult}(B) \cong \mathrm{Hom}_{\mathbf{BA}}(B, \mathbf{2}).$$

This assignment $\mathrm{Ult}(B)$ is functorial and contravariant, and the displayed isomorphism above is natural in $B$. Indeed, given a Boolean homomorphism $h : B' \to B$, let

$$\mathrm{Ult}(h) = h^{-1} : \mathrm{Ult}(B) \to \mathrm{Ult}(B').$$

Of course, we have to show that the inverse image $h^{-1}(U) \subset B$ of an ultrafilter $U \subset B'$ is an ultrafilter in $B$. But since we know that $U = \chi_U^{-1}(1)$ for some $\chi_U : B' \to 2$, we have

$$\begin{aligned} \mathrm{Ult}(h)(U) &= h^{-1}(\chi_U^{-1}(1)) \\ &= (\chi_U \circ h)^{-1}(1). \end{aligned}$$

Therefore, $\mathrm{Ult}(h)(U)$ is also an ultrafilter. Thus, we have a contravariant functor of ultrafilters

$$\mathrm{Ult} : \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets},$$

as well as the contravariant powerset functor coming back

$$\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}.$$

The constructions,

$$\mathbf{BA}^{\mathrm{op}} \underset{\mathrm{Ult}}{\overset{(\mathcal{P}^{\mathbf{BA}})^{\mathrm{op}}}{\rightleftarrows}} \mathbf{Sets}$$

are *not* mutually inverse, however. For in general, $\mathrm{Ult}(\mathcal{P}(X))$ is much larger than $X$, since there are many ultrafilters in $\mathcal{P}(X)$ that are not "principal," that is, of the form $\{U \subseteq X \mid x \in U\}$ for some $x \in X$. (But what if $X$ is finite?) Instead, there is a more subtle relation between these functors that we consider in more detail later; namely, these are an example of adjoint functors.

For now, consider the following observations. Let

$$\mathcal{U} = \mathrm{Ult} \circ (\mathcal{P}^{\mathbf{BA}})^{\mathrm{op}} : \mathbf{Sets} \to \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets}$$

so that

$$\mathcal{U}(X) = \{U \subseteq \mathcal{P}(X) \mid U \text{ is an ultrafilter}\}$$

is a *covariant* functor on **Sets**. Now, observe that for any set $X$, there is a function

$$\eta : X \to \mathcal{U}(X)$$

taking each element $x \in X$ to the principal ultrafilter

$$\eta(x) = \{U \subseteq X \mid x \in U\}.$$

This map is "natural" in $X$, that is, for any function $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \mathcal{U}(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(f)} \\
Y & \xrightarrow[\eta_Y]{} & \mathcal{U}(Y)
\end{array}$$

This is so because, for any ultrafilter $\mathcal{V}$ in $\mathcal{P}(X)$,

$$\mathcal{U}(f)(\mathcal{V}) = \{U \subseteq Y \mid f^{-1}(U) \in \mathcal{V}\}.$$

So in the case of the principal ultrafilters $\eta(x)$, we have

$$\begin{aligned}
(\mathcal{U}(f) \circ \eta_X)(x) &= \mathcal{U}(f)(\eta_X(x)) \\
&= \{V \subseteq Y \mid f^{-1}(V) \in \eta_X(x)\} \\
&= \{V \subseteq Y \mid x \in f^{-1}(V)\} \\
&= \{V \subseteq Y \mid fx \in V\} \\
&= \eta_Y(fx) \\
&= (\eta_Y \circ f)(x).
\end{aligned}$$

Finally, observe that there is an analogous natural map at the "other side" of this situation, in the category of Boolean algebras. Specifically, for every Boolean algebra $B$, there is a homomorphism similar to the function $\eta$,

$$\phi_B : B \to \mathcal{P}(\mathrm{Ult}(B))$$

given by

$$\phi_B(b) = \{\mathcal{V} \in \mathrm{Ult}(B) \mid b \in \mathcal{V}\}.$$

It is not hard to see that $\phi_B$ is always injective. For, given any distinct elements $b, b' \in B$, the Boolean prime ideal theorem implies that there is an ultrafilter $\mathcal{V}$ containing one but not the other. The Boolean algebra $\mathcal{P}(\mathrm{Ult}(B))$, together with the homomorphism $\phi_B$, is called the *Stone representation* of $B$. It presents the arbitrary Boolean algebra $B$ as an algebra of subsets. For the record, we thus have the following step toward a special case of the far-reaching *Stone duality theorem.*

**Proposition 7.5.** *Every Boolean algebra $B$ is isomorphic to one consisting of subsets of some set $X$, equipped with the set-theoretical Boolean operations.*

## 7.4 Naturality

A natural transformation is a morphism of functors. That is right: for fixed categories $\mathbf{C}$ and $\mathbf{D}$, we can regard the functors $\mathbf{C} \to \mathbf{D}$ as the *objects* of a new category, and the arrows between these objects are what we are going to call natural transformations. They are to be thought of as different ways of "relating" functors to each other, in a sense that we now explain.

Let us begin by considering a certain kind of situation that often arises: we have some "construction" on a category $\mathbf{C}$ and some other "construction," and we observe that these two "constructions" are related to each other in a way that is independent of the specific objects and arrows involved. That is, the relation is really between the constructions themselves. To give a simple example, suppose $\mathbf{C}$ has products and consider, for objects $A, B, C \in \mathbf{C}$,

$$(A \times B) \times C \quad \text{and} \quad A \times (B \times C).$$

Regardless of what objects $A, B$, and $C$ are, we have an isomorphism

$$h : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C).$$

What does it mean that this isomorphism does not really depend on the particular objects $A, B, C$? One way to explain it is this:

Given any $f : A \to A'$, we get a commutative square

$$\begin{array}{ccc}
(A \times B) \times C & \xrightarrow{h_A} & A \times (B \times C) \\
\downarrow & & \downarrow \\
(A' \times B) \times C & \xrightarrow[h_{A'}]{} & A' \times (B \times C)
\end{array}$$

So what we really have is an isomorphism between the "constructions"

$$(- \times B) \times C \quad \text{and} \quad - \times (B \times C)$$

without regard to what is in the argument-place of these.

Now, by a "construction," we of course just mean a functor, and by a "relation between constructors" we mean a *morphism of functors* (which is what we are about to define). In the example, it is an isomorphism

$$(- \times B) \times C \;\cong\; - \times (B \times C)$$

of functors $\mathbf{C} \to \mathbf{C}$. In fact, we can of course consider the functors of three arguments:

$$F = (-_1 \times -_2) \times -_3 : \mathbf{C}^3 \to \mathbf{C}$$

and

$$G = -_1 \times (-_2 \times -_3) : \mathbf{C}^3 \to \mathbf{C}$$

and there is an analogous isomorphism

$$F \cong G.$$

But an *isomorphism* is a special morphism, so let us define the general notion first.

**Definition 7.6.** For categories $\mathbf{C}, \mathbf{D}$ and functors

$$F, G : \mathbf{C} \to \mathbf{D}$$

a *natural transformation* $\vartheta : F \to G$ is a family of arrows in $\mathbf{D}$

$$(\vartheta_C : FC \to GC)_{C \in \mathbf{C}_0}$$

such that, for any $f : C \to C'$ in $\mathbf{C}$, one has $\vartheta_{C'} \circ F(f) = G(f) \circ \vartheta_C$, that is, the following commutes:

$$\begin{array}{ccc} FC & \xrightarrow{\vartheta_C} & GC \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FC' & \xrightarrow[\vartheta_{C'}]{} & GC' \end{array}$$

Given such a natural transformation $\vartheta : F \to G$, the $\mathbf{D}$-arrow $\vartheta_C : FC \to GC$ is called the *component of* $\vartheta$ at $C$.

If you think of a functor $F : \mathbf{C} \to \mathbf{D}$ as a "picture" of $\mathbf{C}$ in $\mathbf{D}$, then you can think of a natural transformation $\vartheta_C : FC \to GC$ as a "cylinder" with such a picture at each end.

## 7.5 Examples of natural transformations

We have already seen several examples of natural transformations in previous sections, namely the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Grp}}(F(1), G) &\cong U(G) \\ \mathrm{Hom}_{\mathbf{Sets}}(X, 2) &\cong \mathcal{P}(X) \\ \mathrm{Hom}_{\mathbf{BA}}(B, 2) &\cong \mathrm{Ult}(B). \end{aligned}$$

There were also the maps from Stone duality,

$$\eta_X : X \to \mathrm{Ult}(\mathcal{P}(X))$$
$$\phi_B : B \to \mathcal{P}(\mathrm{Ult}(B)).$$

We now consider some further examples.

*Example* 7.7. Consider the free monoid $M(X)$ on a set $X$ and define a natural transformation $\eta : 1_{\mathbf{Sets}} \to UM$, such that each component $\eta_X : X \to UM(X)$ is given by the "insertion of generators" taking every element $x$ to itself, considered as a word.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UM(X) \\ {\scriptstyle f}\downarrow & & \vdots\, {\scriptstyle UM(f)} \\ Y & \xrightarrow[\eta_Y]{} & UM(Y) \end{array}$$

This is natural, because the homomorphism $M(f)$ on the free monoid $M(X)$ is completely determined by what $f$ does to the generators.

*Example* 7.8. Let $\mathbf{C}$ be a category with products, and $A \in \mathbf{C}$ fixed. A natural transformation from the functor $A \times - : \mathbf{C} \to \mathbf{C}$ to $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is given by taking the component at $C$ to be the second projection

$$\pi_2 : A \times C \to C.$$

From this, together with the pairing operation $\langle -, - \rangle$, one can build up the isomorphism,

$$h : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C).$$

For another such example in more detail, consider the functors

$$\times : \mathbf{C}^2 \to \mathbf{C}$$
$$\tilde{\times} : \mathbf{C}^2 \to \mathbf{C}$$

where $\bar{\times}$ is defined on objects by

$$A \mathbin{\bar{\times}} B = B \times A$$

and on arrows by

$$\alpha \mathbin{\bar{\times}} \beta = \beta \times \alpha.$$

Define a "twist" natural transformation $t : \times \to \bar{\times}$ by

$$t_{(A,B)}\langle a, b\rangle = \langle b, a\rangle.$$

To check that the following commutes,

$$\begin{array}{ccc} A \times B & \xrightarrow{t_{(A,B)}} & B \times A \\ {\scriptstyle \alpha \times \beta}\downarrow & & \downarrow{\scriptstyle \beta \times \alpha} \\ A' \times B' & \xrightarrow[t_{(A',B')}]{} & B' \times A' \end{array}$$

observe that for any generalized elements $a : Z \to A$ and $b : Z \to B$,

$$\begin{aligned} (\beta \times \alpha) t_{(A,B)}\langle a, b\rangle &= (\beta \times \alpha)\langle b, a\rangle \\ &= \langle \beta b, \alpha a\rangle \\ &= t_{(A',B')}\langle \alpha a, \beta b\rangle \\ &= t_{(A',B')} \circ (\alpha \times \beta)\langle a, b\rangle. \end{aligned}$$

Thus, $t : \times \to \bar{\times}$ is natural. In fact, each component $t_{(A,B)}$ is an isomorphism with inverse $t_{(B,A)}$. This is a simple case of an isomorphism of functors.

**Definition 7.9.** The *functor category* $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ has

**Objects:** functors $F : \mathbf{C} \to \mathbf{D}$,

**Arrows:** natural transformations $\vartheta : F \to G$.

For each object $F$, the natural transformation $1_F$ has components

$$(1_F)_C = 1_{FC} : FC \to FC$$

and the composite natural transformation of $F \xrightarrow{\vartheta} G \xrightarrow{\phi} H$ has components

$$(\phi \circ \vartheta)_C = \phi_C \circ \vartheta_C.$$

**Definition 7.10.** A *natural isomorphism* is a natural transformation

$$\vartheta : F \to G$$

which is an isomorphism in the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

**Lemma 7.11.** *A natural transformation $\vartheta : F \to G$ is a natural isomorphism iff each component $\vartheta_C : FC \to GC$ is an isomorphism.*

*Proof.* Exercise! □

In our first example, we can therefore say that the isomorphism

$$\vartheta_A : (A \times B) \times C \cong A \times (B \times C)$$

is *natural in* $A$, meaning that the functors

$$F(A) = (A \times B) \times C$$
$$G(A) = A \times (B \times C)$$

are naturally isomorphic.

Here is a classical example of a natural isomorphism.

*Example* 7.12. Consider the category

$$\mathrm{Vect}(\mathbb{R})$$

of real vector spaces and linear transformations

$$f : V \to W.$$

Every vector space $V$ has a *dual space*

$$V^* = \mathrm{Vect}(V, \mathbb{R})$$

of linear transformations. And every linear transformation

$$f : V \to W$$

gives rise to a *dual linear transformation*

$$f^* : W^* \to V^*$$

defined by precomposition, $f^*(A) = A \circ f$ for $A : W \to \mathbb{R}$. In brief, $(-)^* = \mathrm{Vect}(-, \mathbb{R}) : \mathbf{Vect}^{\mathrm{op}} \to \mathbf{Vect}$ is the *contravariant representable* functor endowed with vector space structure, just like the examples already considered in Section 7.2.

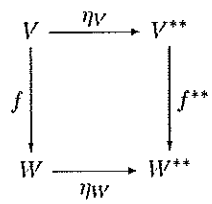
As in those examples, there is a canonical linear transformation from each vector space to its double dual,

$$\eta_V : V \to V^{**}$$
$$x \mapsto (\mathrm{ev}_x : V^* \to \mathbb{R})$$

where $\mathrm{ev}_x(A) = A(x)$ for every $A : V \to \mathbb{R}$. This map is the component of a natural transformation,

$$\eta : 1_{\mathbf{Vect}} \to {**}$$

since the following always commutes:




in **Vect**. Indeed, given any $v \in V$ and $A : W \to \mathbb{R}$ in $W^*$, we have

$$\begin{aligned}(f^{**} \circ \eta_V)(v)(A) &= f^{**}(\mathrm{ev}_v)(A)\\ &= \mathrm{ev}_v(f^*(A))\\ &= \mathrm{ev}_v(A \circ f)\\ &= (A \circ f)(v)\\ &= A(fv)\\ &= \mathrm{ev}_{fv}(A)\\ &= (\eta_W \circ f)(v)(A).\end{aligned}$$

Now, it is a well-known fact in linear algebra that every *finite dimensional* vector space $V$ is isomorphic to its dual space $V \cong V^*$ just for reasons of dimension. However, there is no "natural" way to choose such an isomorphism. On the other hand, the natural transformation,

$$\eta_V : V \to V^{**}$$

*is* a natural isomorphism when $V$ is finite dimensional.

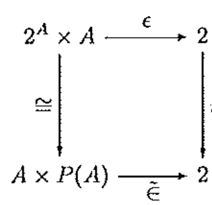
Thus, the formal notion of naturality captures the informal fact that $V \cong V^{**}$ "naturally," unlike $V \cong V^*$.

A similar situation occurs in **Sets**. Here we take 2 instead of $\mathbb{R}$, and the dual $A^*$ of a set $A$ then becomes

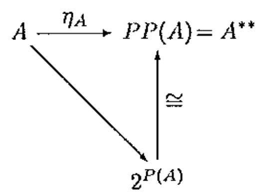
$$A^* = \mathcal{P}(A) \cong \mathbf{Sets}(A, 2)$$

while the dual of a map $f : A \to B$ is the inverse image $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$.

Note that the exponential evaluation corresponds to (the characteristic function of) the membership relation on $A \times \mathcal{P}(A)$.



Transposing again gives a map




which is described by

$$\eta_A(a) = \{U \subseteq A \mid a \in U\}.$$

In **Sets**, one always has $A$ strictly smaller than $\mathcal{P}(A)$, so $\eta_A : A \to A^{**}$ is *never* an isomorphism. Nonetheless, $\eta : 1_{\mathbf{Sets}} \to **$ is a natural transformation, which the reader should prove.

### 7.6 Exponentials of categories

We now want to show that the category **Cat** of (small) categories and functors is cartesian closed, by showing that any two categories $\mathbf{C}, \mathbf{D}$ have an exponential $\mathbf{D}^{\mathbf{C}}$. Of course, we take $\mathbf{D}^{\mathbf{C}} = \mathrm{Fun}(\mathbf{C}, \mathbf{D})$, the category of functors and natural transformations, for which we need to prove the required universal mapping property (UMP).

**Proposition 7.13.** **Cat** *is cartesian closed, with the exponentials*

$$\mathbf{D}^{\mathbf{C}} = \mathrm{Fun}(\mathbf{C}, \mathbf{D}).$$

Before giving the proof, let us note the following. Since exponentials are unique up to isomorphism, this gives us a way to verify that we have found the *"right"* definition of a morphism of functors. For the notion of a natural transformation is completely determined by the requirement that it makes the set $\mathrm{Hom}(\mathbf{C}, \mathbf{D})$ into an exponential category. This is an example of how category theory can serve as a conceptual tool for discovering new concepts. Before giving the proof, we need the following.

**Lemma 7.14** (bifunctor lemma). *Given categories* $\mathbf{A}, \mathbf{B}$, *and* $\mathbf{C}$, *a map of arrows and objects,*

$$F_0 : \mathbf{A}_0 \times \mathbf{B}_0 \to \mathbf{C}_0$$
$$F_1 : \mathbf{A}_1 \times \mathbf{B}_1 \to \mathbf{C}_1$$

*is a functor* $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ *iff*

1. *F is functorial in each argument:* $F(A, -) : \mathbf{B} \to \mathbf{C}$ *and* $F(-, B) : \mathbf{A} \to \mathbf{C}$ *are functors for all* $A \in \mathbf{A}_0$ *and* $B \in \mathbf{B}_0$.
2. *F satisfies the following "interchange law." Given* $\alpha : A \to A' \in \mathbf{A}$ *and* $\beta : B \to B' \in \mathbf{B}$, *the following commutes:*

$$
\begin{array}{ccc}
F(A, B) & \xrightarrow{F(A,\beta)} & F(A, B') \\
{\scriptstyle F(\alpha, B)}\downarrow & & \downarrow{\scriptstyle F(\alpha, B')} \\
F(A', B) & \xrightarrow[F(A',\beta)]{} & F(A', B')
\end{array}
$$

*that is,* $F(A', \beta) \circ F(\alpha, B) = F(\alpha, B') \circ F(A, \beta)$ *in* $\mathbf{C}$.

*Proof.* ~~(of lemma)~~ (Lemma) In $\mathbf{A} \times \mathbf{B}$, any arrow

$$\langle \alpha, \beta \rangle : \langle A, B \rangle \to \langle A', B' \rangle$$

factors as

$$
\begin{array}{ccc}
\langle A, B \rangle & \xrightarrow{\langle 1_A,\beta \rangle} & \langle A, B' \rangle \\
{\scriptstyle \langle \alpha, 1_B \rangle}\downarrow & & \downarrow{\scriptstyle \langle \alpha, 1_{B'} \rangle} \\
\langle A', B \rangle & \xrightarrow[\langle 1_{A'},\beta \rangle]{} & \langle A', B' \rangle
\end{array}
$$

So (1) and (2) are clearly necessary. To show that they are also sufficient, we can define the (proposed) functor:

$$
\begin{aligned}
F(\langle A, B \rangle) &= F(A, B) \\
F(\langle \alpha, \beta \rangle) &= F(A', \beta) \circ F(\alpha, B)
\end{aligned}
$$

The interchange law, together with functoriality in each argument, then ensures that

$$F(\alpha', \beta') \circ F(\alpha, \beta) = F(\langle \alpha', \beta' \rangle \circ \langle \alpha, \beta \rangle)$$

as can be read off from the following diagram:

$$\begin{array}{ccccc}
F(A,B) & & & & \\
{\scriptstyle F(\alpha,B)}\downarrow & \searrow^{F(\alpha,\beta)} & & & \\
F(A',B) & \xrightarrow[F(A',\beta)]{} & F(A',B') & & \\
{\scriptstyle F(\alpha',B)}\vdots & & {\scriptstyle F(\alpha,B')}\downarrow & \searrow^{F(\alpha',\beta')} & \\
F(A',B) & \xrightarrow[F(A'',\beta)]{\cdots\cdots} & F(A'',B') & \xrightarrow[F(A'',\beta')]{} & F(A'',B'')
\end{array}$$

□

*Proof.* (of proposition) We need to show:

1. $\epsilon = \text{eval} : \text{Fun}(\mathbf{C},\mathbf{D}) \times \mathbf{C} \to \mathbf{D}$ is functorial.
2. For any category $\mathbf{X}$ and functor

$$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

   there is a functor

$$\tilde{F} : \mathbf{X} \to \text{Fun}(\mathbf{C},\mathbf{D})$$

   such that $\epsilon \circ (\tilde{F} \times 1_{\mathbf{C}}) = F$.
3. Given any functor

$$G : \mathbf{X} \to \text{Fun}(\mathbf{C},\mathbf{D}),$$

   one has $(\epsilon \circ \widetilde{(G \times 1_{\mathbf{C}})}) = G$.

(1) Using the bifunctor lemma, we show that $\epsilon$ is functorial. First, fix $F : \mathbf{C} \to \mathbf{D}$ and consider $\epsilon(F,-) = F : \mathbf{C} \to \mathbf{D}$. This is clearly functorial! Next, fix $C \in \mathbf{C}_0$ and consider $\epsilon(-,C) : \text{Fun}(\mathbf{C},\mathbf{D}) \to \mathbf{D}$ defined by

$$(\vartheta : F \to G) \mapsto (\vartheta_C : FC \to GC).$$

This is also clearly functorial.

For the interchange law, consider any $\vartheta : F \to G \in \text{Fun}(\mathbf{C},\mathbf{D})$ and $(f : C \to C') \in \mathbf{C}$, then we need the following to commute:

$$\begin{array}{ccc}
\epsilon(F,C) & \xrightarrow{\vartheta_C} & \epsilon(G,C) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
\epsilon(F,C') & \xrightarrow[\vartheta_{C'}]{} & \epsilon(G,C')
\end{array}$$

But this holds because $\epsilon(F, C) = F(C)$ and $\vartheta$ is a natural transformation.

The conditions (2) and (3) are now routine. For example, for (2), given

$$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

let

$$\tilde{F} : \mathbf{X} \to \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

be defined by

$$\tilde{F}(X)(C) = F(X, C).$$

□

## 7.7 Functor categories

Let us consider some particular functor categories.

*Example* 7.15. First, clearly $\mathbf{C}^{\mathbf{1}} = \mathbf{C}$ for the terminal category $\mathbf{1}$. Next, what about $\mathbf{C}^{\mathbf{2}}$, where $\mathbf{2} = \cdot \to \cdot$ is the single arrow category? This is just the arrow category of $\mathbf{C}$ that we already know,

$$\mathbf{C}^{\mathbf{2}} = \mathbf{C}^{\to}.$$

Consider instead the discrete category, $2 = \{0, 1\}$. Then clearly,

$$\mathbf{C}^{2} \cong \mathbf{C} \times \mathbf{C}.$$

Similarly, for any set $I$ (regarded as a discrete category), we have

$$\mathbf{C}^{I} \cong \prod_{i \in I} \mathbf{C}.$$

*Example* 7.16. "Transcendental deduction of natural transformations"
Given the *possibility* of functor categories $\mathbf{D}^{\mathbf{C}}$, we can determine what the objects and arrows therein *must* be as follows:

**Objects:** these correspond uniquely to functors of the form

$$\mathbf{1} \to \mathbf{D}^{\mathbf{C}}$$

and hence to functors

$$\mathbf{C} \to \mathbf{D}.$$

**Arrows:** by the foregoing example, arrows in the functor category correspond uniquely to functors of the form

$$\mathbf{1} \to (\mathbf{D}^{\mathbf{C}})^{\mathbf{2}}$$

thus to functors of the form

$$\mathbf{2} \to \mathbf{D}^{\mathbf{C}}$$

and hence to functors

$$\mathbf{C} \times \mathbf{2} \to \mathbf{D}$$

respectively

$$\mathbf{C} \to \mathbf{D}^{\mathbf{2}}.$$

But a functor from $\mathbf{C}$ into the arrow category $\mathbf{D}^{\mathbf{2}}$ (respectively a functor into $\mathbf{D}$ from the cylinder category $\mathbf{C} \times \mathbf{2}$) is exactly a natural transformation between two functors from $\mathbf{C}$ into $\mathbf{D}$, as the reader can see by drawing a picture of the functor's image in $\mathbf{D}$.

*Example* 7.17. Recall that a (directed) graph can be regarded as a pair of sets and a pair of functions,

$$G_1 \underset{s}{\overset{t}{\rightrightarrows}} G_0$$

where $G_1$ is the set of edges, $G_0$ is the set of vertices, and $s$ and $t$ are the source and target operations.

A homomorphism of graphs $h : G \to H$ is a map that preserves sources and targets. In detail, this is a pair of functions $h_1 : G_1 \to H_1$ and $h_0 : G_0 \to H_0$ such that for all edges $e \in G$, we have $sh_1(e) = h_0 s(e)$ and similarly for $t$ as well. But this amounts exactly to saying that the following two diagrams commute:

$$\begin{array}{ccc} G_1 & \xrightarrow{h_1} & H_1 \\ {\scriptstyle s_G}\downarrow & & \downarrow{\scriptstyle s_H} \\ G_0 & \xrightarrow[h_0]{} & H_0 \end{array} \qquad\qquad \begin{array}{ccc} G_1 & \xrightarrow{h_1} & H_1 \\ {\scriptstyle t_G}\downarrow & & \downarrow{\scriptstyle t_H} \\ G_0 & \xrightarrow[h_0]{} & H_0 \end{array}$$

Now consider the category $\Gamma$, pictured as follows:

$$\cdot \rightrightarrows \cdot$$

It has exactly two objects and two distinct, parallel, nonidentity arrows. A graph $G$ is then exactly a functor,

$$G : \Gamma \to \mathbf{Sets}$$

and a homomorphism of graphs $h : G \to H$ is exactly a natural transformation between these functors. Thus, the category of graphs is a functor category,

$$\mathbf{Graphs} = \mathbf{Sets}^{\Gamma}.$$

As we see later, it follows from this fact that **Graphs** is cartesian closed.

*Example* 7.18. Given a product $\mathbf{C} \times \mathbf{D}$ of categories, take the first product projection

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C}$$

and transpose it to get a functor

$$\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}.$$

For $C \in \mathbf{C}$, the functor $\Delta(C)$ is the "constant $C$-valued functor,"

- $\Delta(C)(X) = C$ for all $X \in \mathbf{D}_0$
- $\Delta(x) = 1_C$ for all $x \in \mathbf{D}_1$.

Moreover, $\Delta(f) : \Delta(C) \to \Delta(C')$ is the natural transformation, each component of which is $f$.

Now suppose we have any functor

$$F : \mathbf{D} \to \mathbf{C}$$

and a natural transformation

$$\vartheta : \Delta(C) \to F.$$

Then, the components of $\vartheta$ all look like

$$\vartheta_D : C \to F(D)$$

since $\Delta(C)(D) = C$. Moreover, for any $d : D \to D'$ in $\mathbf{D}$, the usual naturality square becomes a triangle, since $\Delta(C)(d) = 1_C$ for all $d : D \to D'$.

$$\begin{array}{ccc} C & \xrightarrow{\vartheta_D} & FD \\ {\scriptstyle 1_C}\downarrow & & \downarrow{\scriptstyle Fd} \\ C & \xrightarrow[\vartheta_{D'}]{} & FD' \end{array}$$

Thus, such a natural transformation $\vartheta : \Delta(C) \to F$ is exactly a cone to the base $F$ (with vertex $C$). Similarly, a map of cones $\vartheta \to \varphi$ is a constant natural transformation, that is, one of the forms $\Delta(h)$ for some $h : C \to D$, making a commutative triangle

$$\begin{array}{ccc} \Delta(C) & \xrightarrow{\Delta(h)} & \Delta(D) \\ & {\scriptstyle \vartheta}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & F & \end{array}$$

*Example* 7.19. Take posets $P, Q$ and consider the functor category,

$$Q^P.$$

The functors $Q \to P$, as we know, are just monotone maps, but what is a natural transformation?

$$\vartheta : f \to g$$

For each $p \in P$, we must have

$$\vartheta_p : fp \leq gp$$

and if $p \leq q$, then there must be a commutative square involving $fp \leq fq$ and $gp \leq gq$, which, however, is automatic. Thus, the only condition is that $fp \leq gp$ for all $p$, that is, $f \leq g$ *pointwise*. Since this is just the usual ordering of the poset $Q^P$, the exponential poset agrees with the functor category. Thus, we have the following.

**Proposition 7.20.** *The inclusion functor,*

$$\mathbf{Pos} \to \mathbf{Cat}$$

*preserves CCC structure.*

*Example* 7.21. What happens if we take the functor *category* of two groups $G$ and $H$?

$$H^G$$

Do we get an exponential of groups? Let us first ask, what is a natural transformation between two group homomorphisms $f, g : G \to H$ ? Such a map $\vartheta : f \to g$ would be an element $h \in H$ such that for every $x \in G$, we have

$$g(x) \cdot h = h \cdot f(x)$$

or, equivalently,

$$g(x) = h \cdot f(x) \cdot h^{-1}.$$

Therefore, a natural transformation $\vartheta : f \to g$ is an *inner automorphism* $y \mapsto h{\cdot}y{\cdot}h^{-1}$ of $H$ (called *conjugation by* $h$) that takes $f$ to $g$. Clearly, every such arrow $\vartheta : f \to g$ has an inverse $\vartheta^{-1} : g \to f$ (conjugation by $h^{-1}$). But $H^G$ is still not usually a group, simply because there may be many *different* homomorphisms $G \to H$, so the functor category $H^G$ has more than one object.

This suggests enlarging the category of groups to include also categories with more than one object, but still having inverses for all arrows. Such categories are called *groupoids*, and have been studied by topologists (they occur as the collection of paths between different points in a topological space). A groupoid can thus be regarded as a generalized group, in which the domains and codomains of elements $x$ and $y$ must match up, as in any category, for the multiplication $x \cdot y$ to be defined.

It is clear that if $G$ and $H$ are any groupoids, then the functor category $H^G$ is also a groupoid. Thus, we have the following proposition, the detailed proof of which is left as an exercise.

**Proposition 7.22.** *The category* Grpd *of groupoids is cartesian closed and the inclusion functor*

$$\mathbf{Grpd} \to \mathbf{Cat}$$

*preserves the CCC structure.*

## 7.8 Monoidal categories

Au: Please specify section number.

Section 4.1

As a further application of natural transformations, we can finally give the general notion of a monoidal category, as opposed to the special case of a *strict* one. Recall from Section ?? that a strict monoidal category is by definition a monoid in Cat, that is, a category C equipped with an associative multiplication functor,

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and a distinguished object $I$ that acts as a unit for $\otimes$. A monoidal category with a discrete category C is just a monoid in the usual sense, and every set $X$ gives rise to one of these, with C the set of endomorphisms End($X$) under composition. Another example, not discrete, is now had by considering the category End(D) of endo*functors* of an *arbitrary* category D, with their natural transformations as arrows; that is, let,

$$\mathbf{C} = \mathrm{End}(\mathbf{D}), \quad G \otimes F = G \circ F, \quad I = 1_{\mathbf{D}}.$$

This can also be seen to be a strict monoidal category. Indeed, the multiplication is clearly associative and has $1_{\mathbf{D}}$ as unit, so we just need to check that composition is a bifunctor $\mathrm{End}(\mathbf{D}) \times \mathrm{End}(\mathbf{D}) \longrightarrow \mathrm{End}(\mathbf{D})$. Of course, for this we can use the bifunctor lemma. Fixing $F$ and taking any natural transformation $\alpha : G \to G'$, we have, for any object $D$,

$$\alpha_{FD} : G(FD) \to G'(FD)$$

which is clearly functorial as an operation $\mathrm{End}(\mathbf{D}) \longrightarrow \mathrm{End}(\mathbf{D})$. Fixing $G$ and taking $\beta : F \to F'$ gives

$$G(\beta_D) : G(FD) \to G(F'D)$$

which is also easily seen to be functorial. So it just remains to check the exchange law. This comes down to seeing that the square below commutes, which it plainly does just because $\alpha$ is natural.

$$\begin{array}{ccc} GFD & \xrightarrow{\alpha_{FD}} & G'FD \\ {\scriptstyle G\beta_D}\big\downarrow & & \big\downarrow{\scriptstyle G'\beta_D} \\ GF'D & \xrightarrow[\alpha_{F'D}]{} & G'F'D \end{array}$$

Some of the other examples of strict monoidal categories that we have seen involved "product-like" operations such as meets $a \wedge b$ and joins $a \vee b$ in posets. We would like to also capture general products $A \times B$ and coproducts $A + B$ in categories having these; however, these operations are not generally associative on the nose, but only up to isomorphism. Specifically, given any three objects $A, B, C$ in a category with all finite products, we do not have $A \times (B \times C) = (A \times B) \times C$, but instead an isomorphism,

$$A \times (B \times C) \cong (A \times B) \times C.$$

Note, however, that there is exactly one such isomorphism that commutes with all three projections, and it is natural in all three arguments. Similarly, taking a terminal object 1, rather than $1 \times A = A = A \times 1$, we have natural isomorphisms,

$$1 \times A \cong A \cong A \times 1$$

which, again, are uniquely determined by the condition that they commute with the projections. This leads us to the following definition.

**Definition 7.23.** A *monoidal category* consists of a category $\mathbf{C}$ equipped with a functor

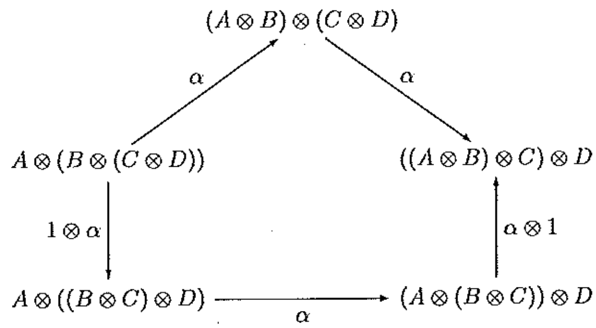
$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and a distinguished object $I$, together with natural isomorphisms,

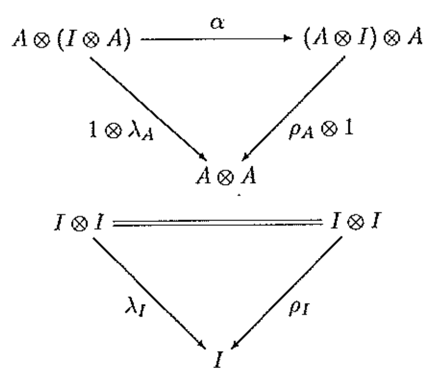
$$\alpha_{ABC} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \times C,$$

$$\lambda_A : I \otimes A \xrightarrow{\sim} A, \quad \rho_A : A \otimes I \xrightarrow{\sim} A.$$

Moreover, these are required to always make the following diagrams commute:




("Mac Lane's pentagon")




In this precise sense, a monoidal category is thus a category that is strict monoidal "up to natural isomorphism"—where the natural isomorphisms are specified and compatible. An example is, of course, a category with all finite products, where the required equations above are ensured by the UMP of products and the selection of the maps $\alpha, \lambda, \rho$ as the unique ones commuting with projections. We leave the verification as an exercise. The reader familiar with tensor products of vector spaces, modules, rings, etc., will have no trouble verifying that these, too, give examples of monoidal categories.

A further example comes from an unexpected source: linear logic. The logical operations of linear conjunction and disjunction, sometimes written $P \otimes Q$ and $P \oplus Q$, can be modeled in a monoidal category, usually with extra structure $\sigma_{AB} : A \otimes B \xrightarrow{\sim} B \otimes A$ making these operations "symmetric" (up to isomorphism). Here, too, we leave the verification to the reader familiar with this logical system.

The basic theorem regarding monoidal categories is Mac Lane's coherence theorem, which says that "all diagrams commute." Somewhat more precisely, it says that any diagram in a monoidal category constructed, like those above, just from identities, the functor $\otimes$, and the maps $\alpha, \lambda, \rho$ will necessarily commute. We shall not state the theorem more precisely than this, nor will we give its somewhat technical proof which, surprisingly, uses ideas from proof theory related to Gentzen's cut-elimination theorem! The details can be found in Mac Lane's book, *Categories Work*.

### 7.9 Equivalence of categories

Before examining some particular functor categories in more detail, we consider one very special application of the concept of natural isomorphism. Consider first the following situation.

*Example* 7.24. Let $\mathbf{Ord}_{\text{fin}}$ be the category of finite ordinal numbers. Thus, the *objects* are the sets $0, 1, 2, \ldots$, where $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$, while the *arrows* are *all* functions between these sets. Now suppose that for each finite set $A$ we select an ordinal $|A|$ that is its cardinal and an isomorphism,

$$A \cong |A|.$$

Then for each function $f : A \to B$ of finite sets, we have a function $|f|$ by completing the square

$$\begin{array}{ccc} A & \xrightarrow{\cong} & |A| \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle |f|} \\ B & \xrightarrow[\cong]{} & |B| \end{array} \qquad (7.2)$$

This clearly gives us a functor

$$|-| : \mathbf{Sets}_{\text{fin}} \to \mathbf{Ord}_{\text{fin}}.$$

Actually, all the maps in the above square are in $\mathbf{Sets}_{\text{fin}}$; so we should also make the inclusion functor

$$i : \mathbf{Ord}_{\text{fin}} \to \mathbf{Sets}_{\text{fin}}$$

explicit. Then we have the selected isos,

$$\vartheta_A : A \xrightarrow{\sim} i|A|$$

and we know by (7.2) that

$$i(|f|) \circ \vartheta_A = \vartheta_B \circ f.$$

This, of course, says that we have a natural isomorphism

$$\vartheta : 1_{\mathbf{Sets}_{\mathrm{fin}}} \to i \circ |-|$$

between two functors of the form

$$\mathbf{Sets}_{\mathrm{fin}} \to \mathbf{Sets}_{\mathrm{fin}}.$$

On the other hand, if we take an ordinal and take its ordinal, we get nothing new,

$$|i(-)| = 1_{\mathbf{Ord}_{\mathrm{fin}}} : \mathbf{Ord}_{\mathrm{fin}} \to \mathbf{Ord}_{\mathrm{fin}}.$$

This is so because, for any finite ordinal $n$,

$$|i(n)| = n$$

and we can assume that we take $\vartheta_n = 1_n : n \to |i(n)|$, so that also,

$$|i(f)| = f : n \to m.$$

In sum, then, we have a situation where two categories are very similar; but they are *not* the same and they are *not even isomorphic* (why?). This kind of correspondence is what is captured by the notion of equivalence of categories.

**Definition 7.25.** An *equivalence of categories* consists of a pair of functors

$$E : \mathbf{C} \to \mathbf{D}$$
$$F : \mathbf{D} \to \mathbf{C}$$

and a pair of natural isomorphisms

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} F \circ E \qquad \text{in } \mathbf{C}^{\mathbf{C}}$$
$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} E \circ F \qquad \text{in } \mathbf{D}^{\mathbf{D}}.$$

In this situation, the functor $F$ is called a *pseudo-inverse* of $E$. The categories $\mathbf{C}$ and $\mathbf{D}$ are then said to be *equivalent*, written $\mathbf{C} \simeq \mathbf{D}$.

Observe that equivalence of categories is a generalization of isomorphism. Indeed, two categories $\mathbf{C}, \mathbf{D}$ are isomorphic if there are functors.

$$E : \mathbf{C} \to \mathbf{D}$$
$$F : \mathbf{D} \to \mathbf{C}$$

such that

$$1_{\mathbf{C}} = F \circ E$$
$$1_{\mathbf{D}} = E \circ F.$$

In the case of equivalence $\mathbf{C} \simeq \mathbf{D}$, we replace the identity natural transformations by natural isomorphisms. In that sense, equivalence of categories as "isomorphism up to isomorphism."

Experience has shown that the mathematically significant properties of objects are those that are invariant under isomorphisms, and in category theory, identity of objects is a much less important relation than isomorphism. So it is really equivalence of categories that is the more important notion of "similarity" for categories.

In the foregoing example $\mathbf{Sets}_{\text{fin}} \simeq \mathbf{Ord}_{\text{fin}}$, we see that every set is isomorphic to an ordinal, and the maps between ordinals are just the maps between them *as sets*. Thus, we have

1. for every set $A$, there is some ordinal $n$ with $A \cong i(n)$,
2. for any ordinals $n, m$, there is an isomorphism,

$$\mathrm{Hom}_{\mathbf{Ord}_{\text{fin}}}(n, m) \cong \mathrm{Hom}_{\mathbf{Sets}_{\text{fin}}}(i(n), i(m))$$

   where $i : \mathbf{Ord}_{\text{fin}} \to \mathbf{Sets}_{\text{fin}}$ is the inclusion functor.

In fact, these conditions are characteristic of equivalences, as the following proposition shows.

**Proposition 7.26.** *The following conditions on a functor $F : \mathbf{C} \to \mathbf{D}$ are equivalent:*

1. *$F$ is (part of) an equivalence of categories.*
2. *$F$ is full and faithful and "essentially surjective" on objects: for every $D \in \mathbf{D}$ there is some $C \in \mathbf{C}$ such that $FC \cong D$.*

*Proof.* (1 implies 2) Take $E : \mathbf{D} \to \mathbf{C}$, and

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} EF$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE.$$

In $\mathbf{C}$, for any $C$, we then have $\alpha_C : C \xrightarrow{\sim} EF(C)$, and

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & EF(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle EF(f)} \\ C' & \xrightarrow[\alpha_{C'}]{} & EF(C') \end{array}$$

commutes for any $f : C \to C'$.

Thus, if $F(f) = F(f')$, then $EF(f) = EF(f')$, so $f = f'$. So $F$ is faithful. Note that by symmetry, $E$ is also faithful.

Now take any arrow

$$h : F(C) \to F(C') \text{ in } \mathbf{D},$$

and consider

$$\begin{array}{ccc} C & \xrightarrow{\cong} & EF(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle E(h)} \\ C' & \xrightarrow[\cong]{} & EF(C') \end{array}$$

where $f = (\alpha_{C'})^{-1} \circ E(h) \circ \alpha_C$. Then, we have also $F(f) : F(C) \to F(C')$ and

$$EF(f) = E(h) : EF(C) \to EF(C')$$

by the naturality square

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & EF(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle EF(f)} \\ C' & \xrightarrow[\alpha_{C'}]{} & EF(C') \end{array}$$

Since $E$ is faithful, $F(f) = h$. So $F$ is also *full.*

Finally, for any object $D \in \mathbf{D}$, we have

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$$

so

$$\beta_D : D \cong F(ED), \quad \text{for } ED \in \mathbf{C}_0.$$

(2 implies 1) We need to define $E : \mathbf{D} \to \mathbf{C}$ and natural transformations,

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} EF$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE.$$

Since $F$ is essentially surjective, for each $D \in \mathbf{D}_0$, we can *choose* some $E(D) \in \mathbf{C}_0$ along with some $\beta_D : D \xrightarrow{\sim} FE(D)$. That gives $E$ on objects and the proposed components of $\beta : 1_{\mathbf{D}} \to FE$.

Given $h : D \to D'$ in $\mathbf{D}$, consider

$$\begin{array}{ccc} D & \xrightarrow{\beta_D} & FE(D) \\ {\scriptstyle h}\downarrow & & \vdots\, {\scriptstyle \beta_{D'} \circ h \circ \beta_D^{-1}} \\ D' & \xrightarrow[\beta_{D'}]{} & FE(D') \end{array}$$

Since $F : \mathbf{C} \to \mathbf{D}$ is full and faithful, there is a unique arrow

$$E(h) : E(D) \to E(D')$$

with $FE(h) = \beta_{D'} \circ h \circ \beta_D^{-1}$. It is easy to see that then $E : \mathbf{D} \to \mathbf{C}$ is a functor and $\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$ is clearly a natural isomorphism.

To find $\alpha : 1_{\mathbf{C}} \to EF$, apply $F$ to any $C$ and consider $\beta_{FC} : F(C) \to FEF(C)$. Since $F$ is full and faithful, the preimage of $\beta_{FC}$ is an isomorphism,

$$\alpha_C = F^{-1}(\beta_{FC}) : C \xrightarrow{\sim} EF(C)$$

which is easily seen to be natural, since $\beta$ is. □

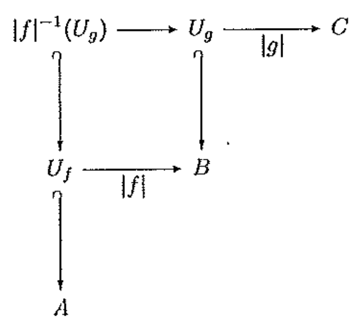
## 7.10 Examples of equivalence

*Example* 7.27. *Pointed sets and partial maps*

Let **Par** be the category of sets and partial functions. An arrow

$$f : A \rightharpoonup B$$

is a function $|f| : U_f \to B$ for some $U_f \subseteq A$. Identities in **Par** are the same as those in **Sets**, that is, $1_A$ is the *total* identity function on $A$. The composite of $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is given as follows: Let $U_{(g \circ f)} := f^{-1}(U_g) \subseteq A$, and $|g \circ f| : U_{(g \circ f)} \to C$ is the horizontal composite indicated in the following diagram, in which the square is a pullback:




It is easy to see that composition is associative and that the identities are units, so we have a category **Par**.

The category of *pointed sets*,

$$\mathbf{Sets}_*$$

has as objects, sets $A$ equipped with a distinguished "point" $a \in A$, that is, pairs,

$$(A, a) \text{ with } a \in A.$$

Arrows are functions that preserve the point, that is, an arrow $f : (A, a) \to (B, b)$ is a function $f : A \to B$ such that $f(a) = b$.

Now we show

**Proposition 7.28.** $\mathbf{Par} \simeq \mathbf{Sets}_*$

The functors establishing the equivalence are as follows:

$$F : \mathbf{Par} \to \mathbf{Sets}_*$$

is defined on an object $A$ by $F(A) = (A \cup \{*\}, *)$, where $*$ is a new element that we add to $A$. We also write $A_* = A \cup \{*\}$. For arrows, given $f : A \rightharpoonup B$, $F(f) : A_* \to B_*$ is defined by

$$f_*(x) = \begin{cases} f(x) & \text{if } x \in U_f \\ * & \text{otherwise.} \end{cases}$$

Then clearly $f_*(*_A) = *_B$, so in fact $f_* : A_* \to B_*$ is "pointed," as required.

Coming back, the functor

$$G : \mathbf{Sets}_* \to \mathbf{Par}$$

is defined on an object $(A, a)$ by $G(A, a) = A - \{a\}$ and for an arrow $f : (A, a) \to (B, b)$

$$G(f) : A - \{a\} \rightharpoonup B - \{b\}$$

is the function with domain

$$U_{G(f)} = A - f^{-1}(b)$$

defined by $G(f)(x) = f(x)$ for every $f(x) \neq b$.

Now $G \circ F$ is the identity on $\mathbf{Par}$, because we are just adding a new point and then throwing it away. But $F \circ G$ is only naturally isomorphic to $1_{\mathbf{Sets}_*}$, since we have

$$(A, a) \cong ((A - \{a\}) \cup \{*\}, *).$$

These sets are not equal, since $a \neq *$. It still needs to be checked, of course, that $F$ and $G$ are functorial, and that the comparison $(A, a) \cong ((A - \{a\}) \cup \{*\}, *)$ is natural, but we leave these easy verifications to the reader.

Observe that this equivalence implies that **Par** has all limits, since it is equivalent to a category of "algebras" of a very simple type, namely sets equipped with a single, nullary operation, that is, a "constant." We already know that limits of algebras can always be constructed as limits of the underlying sets, and an easy exercise shows that a category equivalent to one with limits of any type also has such limits.

*Example* 7.29. *Slice categories and indexed families*

For any set $I$, the functor category $\mathbf{Sets}^I$ is the category of $I$-indexed sets. The objects are $I$-indexed families of sets

$$(A_i)_{i \in I}$$

and the arrows are $I$-indexed families of functions,

$$(f_i : A_i \to B_i)_{i \in I} : (A_i)_{i \in I} \longrightarrow (B_i)_{i \in I}.$$

This category has an equivalent description that is often quite useful: it is equivalent to the slice category of **Sets** over $I$, consisting of arrows $\alpha : A \to I$ and "commutative triangles" over $I$ (see Section 1.6),

$$\mathbf{Sets}^I \simeq \mathbf{Sets}/I.$$

Indeed, define functors

$$\Phi : \mathbf{Sets}^I \longrightarrow \mathbf{Sets}/I$$
$$\Psi : \mathbf{Sets}/I \longrightarrow \mathbf{Sets}^I$$

on objects as follows:

$$\Phi((A_i)_{i \in I}) = \pi : \coprod_{i \in I} A_i \to I \quad \text{(the indexing projection)},$$

where the coproduct is conveniently taken to be

$$\coprod_{i \in I} A_i = \{(i, a) \mid a \in A_i\}.$$

And coming back, we have

$$\Psi(\alpha : A \to I) = (\alpha^{-1}\{i\})_{i \in I}.$$

The effect on arrows is analogous and easily inferred. We leave it as an exercise to show that these are indeed mutually pseudo-inverse functors. (Why are they *not* inverses?)

The equivalent description of $\mathbf{Sets}^I$ as $\mathbf{Sets}/I$ leads to the idea that, for a general category $\mathcal{E}$, the slice category $\mathcal{E}/X$, for any object $X$, can also be regarded as the category of "$X$-indexed objects of $\mathcal{E}$", although the functor category $\mathcal{E}^X$ usually does not make sense. This provides a precise notion of an "$X$-indexed family of objects $E_x$ of $\mathcal{E}$," namely as a map $E \to X$.

For instance, in topology, there is the notion of a "fiber bundle" as a continuous function $\pi : Y \to X$, thought of as a family of spaces $Y_x = \pi^{-1}(x)$, the "fibers" of $\pi$, varying continuously in a parameter $x \in X$. Similarly, in dependent type theory there are "dependent types" $x : X \vdash A(x)$, thought of as families of types indexed over a type. These can be modeled as objects $[\![A]\!] \to [\![X]\!]$ in the slice category $\mathcal{E}/[\![X]\!]$ over the interpretation of the (closed) type $X$ as an object of a category $\mathcal{E}$.

If $\mathcal{E}$ has pullbacks, reindexing of an "indexed family" along an arrow $f : Y \to X$ in $\mathcal{E}$ is represented by the pullback functor $f^* : \mathcal{E}/X \to \mathcal{E}/Y$. This is motivated by the fact that in **Sets** the following diagram commutes (up to natural isomorphism) for any $f : J \to I$:

$$\begin{array}{ccc} \mathbf{Sets}^I & \xrightarrow{\simeq} & \mathbf{Sets}/I \\ {\scriptstyle \mathbf{Sets}^f}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{Sets}^J & \xrightarrow[\simeq]{} & \mathbf{Sets}/J \end{array}$$

where the functor $\mathbf{Sets}^f$ is the reindexing along $f$:

$$(\mathbf{Sets}^f(A_i))_j = A_{f(j)}.$$

Moreover, there are also functors going in the other direction,

$$\Sigma_f, \Pi_f : \mathbf{Sets}/J \longrightarrow \mathbf{Sets}/I$$

which, in terms of indexed families, are given by taking sums and products of the fibers:

$$(\Sigma_f(A_j))_i = \sum_{f(j)=i} A_j$$

and similarly for $\Pi$. These functors can be characterized in terms of the pullback functor $f^*$ (as adjoints, see Section 9.7), and so also make sense in categories more general than **Sets**, where there are no "indexed families" in the usual sense. For instance, in dependent type theory, these operations are formalized by logical rules of inference similar to those for the existential and universal quantifier, and the resulting category of types has such operations of dependent sums and products.

*Example* 7.30. *Stone duality*
Many examples of equivalences of categories are given by what are called "dualities." Often, classical duality theorems are not of the form $\mathbf{C} \cong \mathbf{D}^{op}$ (much less $\mathbf{C} = \mathbf{D}^{op}$), but rather $\mathbf{C} \simeq \mathbf{D}^{op}$, that is, $\mathbf{C}$ is equivalent to the opposite

(or "dual") category of **D**. This is because the duality is established by a construction that returns the original thing only up to isomorphism, not "on the nose." Here is a simple example, which is a very special case of the far-reaching *Stone duality* theorem.

**Proposition 7.31.** *The category of finite Boolean algebras is equivalent to the opposite of the category of finite sets,*

$$\mathbf{BA}_{\mathrm{fin}} \simeq \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}}.$$

*Proof.* The functors involved here are the contravariant powerset functor

$$\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{BA}_{\mathrm{fin}}$$

on one side (the powerset of a finite set is finite!). Going back, we use the functor,

$$A : \mathbf{BA}_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{Sets}_{\mathrm{fin}}$$

taking the set of *atoms* of a Boolean algebra,

$$A(\mathcal{B}) = \{a \in \mathcal{B} \mid 0 < a \text{ and } (b < a \Rightarrow b = 0)\}.$$

In the finite case, this is isomorphic to the ultrafilter functor that we have already studied (see Section 7.3).

**Lemma 7.32.** *For any finite Boolean algebra $\mathcal{B}$, there is an isomorphism between atoms $a$ in $\mathcal{B}$ and ultrafilters $U \subseteq \mathcal{B}$, given by*

$$U \mapsto \bigwedge_{b \in U} b$$

*and*

$$a \mapsto \uparrow(a).$$

*Proof.* If $a$ is an atom, then $\uparrow(a)$ is an ultrafilter, since for any $b$ either $a \wedge b = a$ and then $b \in \uparrow(a)$ or $a \wedge b = 0$ and so $\neg b \in \uparrow(a)$.

If $U \subseteq \mathcal{B}$ is an ultrafilter then $0 < \bigwedge_{b \in U} b$, because, since $U$ is finite and closed under intersections, we must have $\bigwedge_{b \in U} b \in U$. If $0 \neq b_0 < \bigwedge_{b \in U} b$ then $b_0$ is not in $U$, so $\neg b_0 \in U$. But then $b_0 < \neg b_0$ and so $b_0 = 0$.

Plainly, $U \subseteq \uparrow(\bigwedge_{b \in U} b)$ since $b \in U$ implies $\bigwedge_{b \in U} b \subseteq b$. Now let $\bigwedge_{b \in U} b \leq a$ for some $a$ not in $U$. Then, $\neg a \in U$ implies that also $\bigwedge_{b \in U} b \leq \neg a$, and so $\bigwedge_{b \in U} b \leq a \wedge \neg a = 0$, which is impossible. □

Since we know that the set of ultrafilters $\mathrm{Ult}(\mathcal{B})$ is contravariantly functorial (it is represented by the Boolean algebra **2**, see Section 7.3), we therefore also have a contravariant functor of atoms $A \cong \mathrm{Ult}$. The explicit description of this functor is this: if $h : \mathcal{B} \to \mathcal{B}'$ and $a' \in A(\mathcal{B}')$, then it follows from the lemma that there is a *unique* atom $a \in \mathcal{B}$ such that $a' \leq h(b)$ iff $a \leq b$ for all $b \in \mathcal{B}$. To find this atom $a$, take the intersection over the ultrafilter $h^{-1}(\uparrow(a'))$,

$$A(a') = a = \bigwedge_{a' \leq h(b)} b.$$

Thus, we get a function

$$A(h) : A(\mathcal{B}') \to A(\mathcal{B}).$$

Of course, we must still check that this is a pseudo-inverse for $\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}_{\text{fin}}^{\text{op}} \to \mathbf{BA}_{\text{fin}}$. The required natural isomorphisms,

$$\alpha_X : X \to A(\mathcal{P}(X))$$
$$\beta_{\mathcal{B}} : \mathcal{B} \to \mathcal{P}(A(\mathcal{B}))$$

are explicitly described as follows:

The atoms in a finite powerset $\mathcal{P}(X)$ are just the singletons $\{x\}$ for $x \in X$, thus $\alpha_X(x) = \{x\}$ is clearly an isomorphism.

To define $\beta_{\mathcal{B}}$, let

$$\beta_{\mathcal{B}}(b) = \{a \in A(\mathcal{B}) \mid a \leq b\}.$$

To see that $\beta_{\mathcal{B}}$ is also iso, consider the proposed inverse,

$$(\beta_{\mathcal{B}})^{-1}(B) = \bigvee_{a \in B} a \qquad \text{for } B \subseteq A(\mathcal{B}).$$

The isomorphism then follows from the following lemma, the proof of which is routine.

**Lemma 7.33.** *For any finite Boolean algebra $\mathcal{B}$,*

1. $b = \bigvee\{a \in A(\mathcal{B}) \mid a \leq b\}$.
2. *If $a$ is an atom and $a \leq b \vee b'$, then $a \leq b$ or $a \leq b'$.*

Of course, one must still check that $\alpha$ and $\beta$ really are natural transformations. This is left to the reader. □

Finally, we remark that the duality

$$\mathbf{BA}_{\text{fin}} \simeq \mathbf{Sets}_{\text{fin}}^{\text{op}}$$

extends to one between *all* sets on the one side and the complete, atomic Boolean algebras, on the other,

$$\mathbf{caBA} \simeq \mathbf{Sets}^{\text{op}},$$

where a Boolean algebra $\mathcal{B}$ is *complete* if every subset $U \subseteq \mathcal{B}$ has a join $\bigvee U \in \mathcal{B}$ and a *complete homomorphism* preserves these joins and $\mathcal{B}$ is *atomic* if every nonzero element $0 \neq b \in \mathcal{B}$ has some $a \leq b$ with $a$ an atom.

Moreover, this is just the discrete case of the full Stone duality theorem, which states an equivalence between the category of *all* Boolean algebras and the opposite of a certain category of topological spaces, called "Stone spaces," and all continuous maps between them. For details, see Johnston (1982)

## 7.11 Exercises

1. Consider the (covariant) composite functor,

$$\mathcal{F} = \mathcal{P}^{\mathbf{BA}} \circ \mathrm{Ult}^{\mathrm{op}} : \mathbf{BA} \to \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$$

taking each Boolean algebra $B$ to the powerset algebra of sets of ultrafilters in $B$. Note that

$$\mathcal{F}(B) \cong \mathrm{Hom}_{\mathbf{Sets}}(\mathrm{Hom}_{\mathbf{BA}}(B, 2), 2)$$

is a sort of "double-dual" Boolean algebra. There is always a homomorphism,

$$\phi_B : B \to \mathcal{F}(B)$$

given by $\phi_B(b) = \{\mathcal{V} \in \mathrm{Ult}(B) \mid b \in \mathcal{V}\}$. Show that for any Boolean homomorphism $h : A \to B$, the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi_A} & \mathcal{F}(A) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(h)} \\ B & \xrightarrow[\phi_B]{} & \mathcal{F}(B) \end{array}$$

2. Show that the homomorphism $\phi_B : B \to \mathcal{F}(B)$ in the foregoing problem is always injective (use the Boolean prime ideal theorem). This is the classical "Stone representation theorem," stating that every Boolean algebra is isomorphic to a "field of sets," that is, a sub-Boolean algebra of a powerset. Is the functor $\mathcal{F}$ faithful?
3. Prove that for any *finite* Boolean algebra $B$, the "Stone representation"

$$\phi : B \to \mathcal{P}(\mathrm{Ult}(B))$$

is in fact an isomorphism of Boolean algebras. (Note the similarity to the case of finite dimensional vector spaces.) This concludes the proof that we have an equivalence of categories,

$$\mathbf{BA}_{\mathrm{fin}} \simeq \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}}$$

This is the "finite" case of Stone duality.

4. Consider the forgetful functors

$$\mathbf{Groups} \xrightarrow{U} \mathbf{Monoids} \xrightarrow{V} \mathbf{Sets}$$

Say whether each is faithful, full, injective on arrows, surjective on arrows, injective on objects, and surjective on objects.

5. Make every poset $(X, \leq)$ into a topological space by letting $U \subseteq X$ be open just if $x \in U$ and $x \leq y$ implies $y \in U$ ($U$ is "closed upward"). This is called the *Alexandroff topology* on $X$. Show that it gives a functor

$$A : \mathbf{Pos} \to \mathbf{Top}$$

from posets and monotone maps to spaces and continuous maps by showing that any monotone map of posets $f : P \to Q$ is continuous with respect to this topology on $P$ and $Q$ (the inverse image of an open set must be open).

Is $A$ faithful? Is it full?

6. Prove that every functor $F : \mathbf{C} \to \mathbf{D}$ can be factored as $D \circ E = F$,

$$\mathbf{C} \xrightarrow{E} \mathbf{E} \xrightarrow{D} \mathbf{D}$$

in the following two ways:

(a) $E : \mathbf{C} \to \mathbf{E}$ is bijective on objects and full, and $D : \mathbf{E} \to \mathbf{D}$ is faithful;

(b) $E : \mathbf{C} \to \mathbf{E}$ surjective on objects and $D : \mathbf{E} \to \mathbf{D}$ is injective on objects and full and faithful.

When do the two factorizations agree?

7. Show that a natural transformation is a natural isomorphism just if each of its components is an isomorphism. Is the same true for monomorphisms?

8. Show that a functor category $\mathbf{D}^{\mathbf{C}}$ has binary products if $\mathbf{D}$ does (construct the product of two functors $F$ and $G$ "objectwise": $(F \times G)(C) = F(C) \times G(C)$).

9. Show that the map of sets

$$\begin{aligned} \eta_A : A &\longrightarrow PP(A) \\ a &\longmapsto \{U \subseteq A | a \in U\} \end{aligned}$$

is the component at $A$ of a natural transformation $\eta : 1_{\mathbf{Sets}} \to PP$, where $P : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is the (contravariant) powerset functor.

10. Let $\mathbf{C}$ be a locally small category. Show that there is a functor

$$\mathrm{Hom} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$$

such that for each object $C$ of $\mathbf{C}$,

$$\mathrm{Hom}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

is the covariant representable functor and

$$\mathrm{Hom}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is the contravariant one. (Hint: *use the bifunctor lemma.*)

11. Recall from the text that a *groupoid* is a category in which every arrow is an isomorphism. Prove that the category of groupoids is cartesian closed.
12. Let $\mathbf{C} \cong \mathbf{D}$ be equivalent categories. Show that $\mathbf{C}$ has binary products if and only if $\mathbf{D}$ does.
13. What sorts of properties of categories do *not* respect equivalence? Find one that respects isomorphism, but not equivalence.
14. Complete the proof that $\mathbf{Par} \cong \mathbf{Sets}_*$.
15. Show that equivalence of categories is an equivalence relation.
16. A category is *skeletal* if isomorphic objects are always identical. Show that every category is equivalent to a skeletal subcategory. (Every category has a "skeleton.")
17. Complete the proof that, for any set $I$, the category of $I$-indexed families of sets, regarded as the functor category $\mathbf{Sets}^I$, is equivalent to the slice category $\mathbf{Sets}/I$ of sets over $I$,

$$\mathbf{Sets}^I \simeq \mathbf{Sets}/I.$$

Show that reindexing of families along a function $f : J \to I$, given by precomposition,

$$\mathbf{Sets}^f((A_i)_{i \in I}) = (A_{f(j)})_{j \in J}$$

is represented by pullback, in the sense that the following diagram of categories and functors commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathbf{Sets}^I & \xrightarrow{\simeq} & \mathbf{Sets}/I \\ {\scriptstyle \mathbf{Sets}^f}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{Sets}^J & \xrightarrow[\simeq]{} & \mathbf{Sets}/J \end{array}$$

Here $f^* : \mathbf{Sets}/J \to \mathbf{Sets}/I$ is the pullback functor along $f : J \to I$. Finally, infer that $\mathbf{Sets}/2 \simeq \mathbf{Sets} \times \mathbf{Sets}$, and similarly for any $n$ other than 2.

18. Show that a category with finite products is a monoidal category. Infer that the same is true for any category with finite coproducts.