10

MONADS AND ALGEBRAS

In Chapter 9, the adjoint functor theorem was seen to imply that the category of algebras for an equational theory T always has a "free T-algebra" functor, left adjoint to the forgetful functor into **Sets**. This adjunction describes the notion of a T-algebra in a way that is independent of the specific syntactic description given by the theory T, the operations and equations of which are rather like a particular *presentation* of that notion. In a certain sense that we are about to make precise, it turns out that *every* adjunction describes, in a "syntax invariant" way, a notion of an "algebra" for an abstract "equational theory."

Toward this end, we begin with yet a third characterization of adjunctions. This one has the virtue of being entirely equational.

10.1 The triangle identities

Suppose we are given an adjunction,

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U.$$

with unit and counit,

$$\eta : 1_{\mathbf{C}} \to UF$$
$$\epsilon : FU \to 1_{\mathbf{D}}.$$

We can take any $f : FC \to D$ to

$$\phi(f) = U(f) \circ \eta_C : C \to UD,$$

and for any $g : C \to UD$, we have

$$\phi^{-1}(g) = \epsilon_D \circ F(g) : FC \to D.$$

This we know gives the isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(FC, D) \cong_{\phi} \mathrm{Hom}_{\mathbf{C}}(C, UD).$$

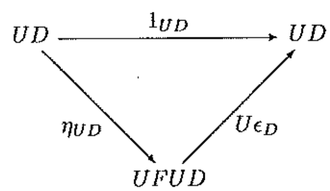
Now put $1_{UD} : UD \to UD$ in place of $g : C \to UD$ in the foregoing set. We know that $\phi^{-1}(1_{UD}) = \epsilon_D$, and so

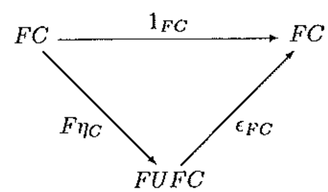
$$1_{UD} = \phi(\epsilon_D)$$
$$= U(\epsilon_D) \circ \eta_{UD}.$$

And similarly, $\phi(1_{FC}) = \eta_C$, so

$$\begin{aligned} 1_{FC} &= \phi^{-1}(\eta_C) \\ &= \epsilon_{FC} \circ F(\eta_C). \end{aligned}$$

Thus, we have shown that the following two diagrams commute:





Indeed, one has the following equations of natural transformations:

$$U\epsilon \circ \eta_U = 1_U \tag{10.1}$$

$$\epsilon_F \circ F\eta = 1_F \tag{10.2}$$

These are called the "triangle identities."

Proposition 10.1. *Given categories, functors, and natural transformations*

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U$$

$$\begin{aligned} \eta &: 1_{\mathbf{C}} \to U \circ F \\ \epsilon &: F \circ U \to 1_{\mathbf{D}} \end{aligned}$$

one has $F \dashv U$ with unit η and counit ϵ iff the triangle identities (10.1) *and* (10.2) *hold.*

Proof. We have already shown one direction. For the other, we just need a natural isomorphism,

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD).$$

As earlier, we put

$$\begin{aligned} \phi(f : FC \to D) &= U(f) \circ \eta_C \\ \vartheta(g : C \to UD) &= \epsilon_D \circ F(g). \end{aligned}$$

Then we check that these are mutually inverse:

$$\begin{aligned}
\phi(\vartheta(g)) &= \phi(\epsilon_D \circ F(g)) \\
&= U(\epsilon_D) \circ UF(g) \circ \eta_C \\
&= U(\epsilon_D) \circ \eta_{UD} \circ g && \eta \text{ natural} \\
&= g && (10.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\vartheta(\phi(f)) &= \vartheta(U(f) \circ \eta_C) \\
&= \epsilon_D \circ FU(f) \circ F\eta_C \\
&= f \circ \epsilon_{FC} \circ F\eta_C && \epsilon \text{ natural} \\
&= f && (10.2)
\end{aligned}$$

Moreover, this isomorphism is easily seen to be natural. □

The triangle identities have the virtue of being entirely "algebraic"—no quantifiers, limits, Hom-sets, infinite conditions, etc. Thus, anything defined by adjoints such as free groups, product spaces, quantifiers, ... can be defined *equationally*. This is not only a matter of conceptual simplification; it also has important consequences for the existence and properties of the structures that are so determined.

10.2 Monads and adjoints

Next consider an adjunction $F \dashv U$ and the composite functor

$$U \circ F : \mathbf{C} \to \mathbf{D} \to \mathbf{C}.$$

Given *any* category $\mathbf{C}$ and endofunctor

$$T : \mathbf{C} \to \mathbf{C}$$

one can ask the following ~~question.~~ :

Question: When is $T = U \circ F$ for some adjoint functors $F \dashv U$ to and from another category $\mathbf{D}$?

Thus, we seek necessary and sufficient conditions on the given endofunctor $T : \mathbf{C} \to \mathbf{C}$ for recovering a category $\mathbf{D}$ and adjunction $F \dashv U$. Of course, not every T arises so, and we see that even if $T = U \circ F$ for *some* $\mathbf{D}$ and $F \dashv U$, we cannot always recover *that* adjunction. Thus, a better way to ask the question would be, given an adjunction what sort of "trace" does it leave on a category and can we recover the adjunction from this?

First, suppose we have $\mathbf{D}$ and $F \dashv U$ and T is the composite functor $T = U \circ F$. We then have a natural transformation,

$$\eta : 1 \to T.$$

And from the counit ϵ at FC,

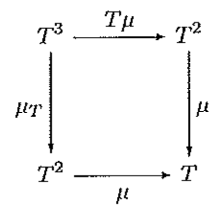
$$\epsilon_{FC} : FUFC \to FC$$

we have $U\epsilon_{FC} : UFUFC \to UFC$, which we call,

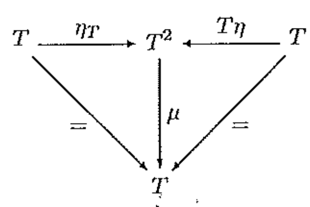
$$\mu : T^2 \to T.$$

In general, then, as a first step toward answering our question, if T arises from an adjunction, then it should have such a structure $\eta : 1 \to T$ and $\mu : T^2 \to T$.

Now, what can be said about the structure (T, η, μ)? Actually, quite a bit! Indeed, the triangle equalities give us the following commutative diagrams:

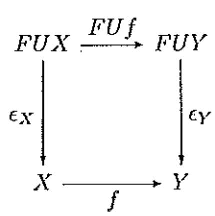


$$\mu \circ \mu_T = \mu \circ T\mu \tag{10.3}$$

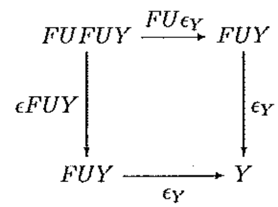


$$\mu \circ \eta_T = 1_T = \mu \circ T\eta \tag{10.4}$$

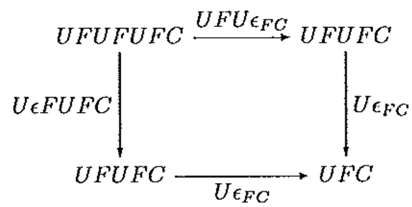
To prove the first one, for any $f : X \to Y$ in $\mathbf{D}$, the following square in $\mathbf{C}$ commutes, just since ϵ is natural:



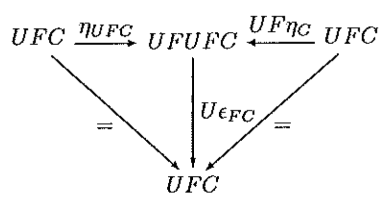
Now take $X = FUY$ and $f = \epsilon_Y$ to get the following:



Putting FC for Y and applying U, therefore, gives this



which has the required form (10.3). The equations (10.4) in the form



are simply the triangle identities, once taken at FC, and once under U. We record this data in the following definition.

Definition 10.2. A *monad* on a category $\mathbf{C}$ consists of an endofunctor $T : \mathbf{C} \to \mathbf{C}$, and natural transformations $\eta : 1_{\mathbf{C}} \to T$, and $\mu : T^2 \to T$ satisfying the two commutative diagrams above, that is,

$$\mu \circ \mu_T = \mu \circ T\mu \qquad (10.5)$$

$$\mu \circ \eta_T = 1 = \mu \circ T\eta. \qquad (10.6)$$

Note the formal analogy to the definition of a monoid. In fact, a monad is exactly the same thing a *monoidal* monoid in the monoidal category $\mathbf{C}^{\mathbf{C}}$ with composition as the monoidal product, $G \otimes F = G \circ F$ (cf. section 7.8). For this reason, the equations (10.5) and (10.6) above are called the *associativity* and *unit* laws, respectively.

We have now shown the following proposition.

Proposition 10.3. *Every adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$, unit $\eta : UF \to 1_{\mathbf{C}}$ and counit $\epsilon : 1_{\mathbf{D}} \to FU$ gives rise to a monad (T, η, μ) on $\mathbf{C}$ with*

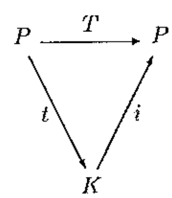
$$T = U \circ F : \mathbf{C} \to \mathbf{C}$$

$$\eta : 1 \to T \quad \textit{the unit}$$

$$\mu = U\epsilon_F : T^2 \to T.$$

Example 10.4. Let P be a poset. A monad on P is a monotone function $T : P \to P$ with $x \leq Tx$ and $T^2x \leq Tx$. But then $T^2 = T$, that is, T is *idempotent.* Such a T, that is both inflationary and idempotent, is sometimes called a *closure operation* and written $Tp = \bar{p}$, since it acts like the closure operation on the subsets of a topological space. The "possibility operator" $\diamond p$ in modal logic is another example.

In the poset case, we can easily recover an adjunction from the monad. First, let $K = \mathrm{im}(T)(P)$ (the fixed points of T), and let $i : K \to P$ be the inclusion. Then let t be the factorization of T through K, as indicated in



Observe that since $TTp = Tp$, for any element $k \in K$ we then have, for some $p \in P$, the equation $itik = ititp = itp = ik$, whence $tik = k$ since i is monic. We therefore have

$$p \leq ik \quad \text{implies} \quad tp \leq tik = k$$

$$tp \leq k \quad \text{implies} \quad p \leq itp \leq ik$$

So indeed $t \dashv i$.

Example 10.5. Consider the covariant powerset functor

$$\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$$

which takes each function $f : X \to Y$ to the image mapping $\mathrm{im}(f) : P(X) \to P(Y)$. Let $\eta_X : X \to \mathcal{P}(X)$ be the singleton operation

$$\eta_X(x) = \{x\}$$

and let $\mu_X : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$ be the union operation

$$\mu_X(\alpha) = \bigcup \alpha.$$

The reader should verify as an exercise that these operations are in fact natural in X and that this defines a monad $(\mathcal{P}, \{-\}, \bigcup)$ on $\mathbf{Sets}$.

As we see in these examples, monads can, and often do, arise without coming from evident adjunctions. In fact, the notion of a monad originally did occur independently of adjunctions! Monads were originally also known by the names "triples" and sometimes "standard constructions." Despite their independent origin, however, our question "when does an endofunctor T arise from an adjunction?" has the simple answer: just if it is the functor part of a monad.

10.3 Algebras for a monad

Proposition 10.6. *Every monad arises from an adjunction. More precisely, given a monad (T, η, μ) on the category $\mathbf{C}$, there exists a category $\mathbf{D}$ and an adjunction $F \dashv U$, $\eta : 1 \to UF$, $\epsilon : FU \to 1$ with $U : \mathbf{D} \to \mathbf{C}$ such that*

$$T = U \circ F$$
$$\eta = \eta \quad \textit{(the unit)}$$
$$\mu = U\epsilon_F.$$

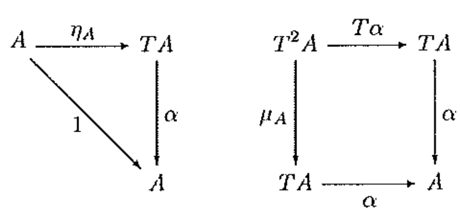
Proof. We first define the important category $\mathbf{C}^T$ called the *Eilenberg–Moore category of T*. This will be our "$\mathbf{D}$." Then we need suitable functors

$$F : \mathbf{C} \rightleftarrows \mathbf{C}^T : U.$$

And, finally, we need natural transformations $\eta : 1 \to UF$ and $\epsilon : FU \to 1$ satisfying the triangle identities.

To begin, $\mathbf{C}^T$ has as *objects* the "T-algebras," which are pairs (A, α) of the form $\alpha : TA \to A$ in $\mathbf{C}$, such that

$$1_A = \alpha \circ \eta_A \quad \text{and} \quad \alpha \circ \mu_A = \alpha \circ T\alpha. \tag{10.7}$$



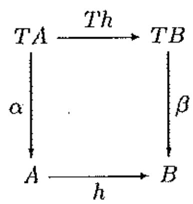
A *morphism* of T-algebras,

$$h : (A, \alpha) \to (B, \beta)$$

is simply an arrow $h : A \to B$ in $\mathbf{C}$, such that,

$$h \circ \alpha = \beta \circ T(h)$$

as indicated in the following diagram:



It is obvious that $\mathbf{C}^T$ is a category with the expected composites and identities coming from $\mathbf{C}$, and that T is a functor.

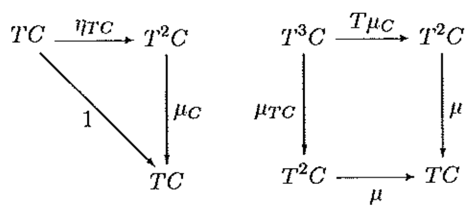
Now define the functors,

$$U : \mathbf{C}^T \to \mathbf{C}$$
$$U(A, \alpha) = A$$

and

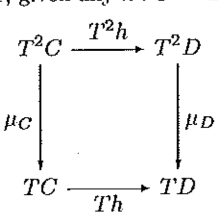
$$F : \mathbf{C} \to \mathbf{C}^T$$
$$FC = (TC, \mu_C).$$

We need to check that (TC, μ_C) is a T-algebra. The equations (10.7) for T-algebras in this case become



But these come directly from the definition of a monad.

To see that F is a functor, given any $h : C \to D$ in $\mathbf{C}$, we have



since μ is natural. But this is a T-algebra homomorphism $FC \to FD$, so we can put

$$Fh = Th : TC \to TD$$

to get an arrow in $\mathbf{C}^T$.

Now we have defined the category $\mathbf{C}^T$ and the functors

$$\mathbf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{C}^{\mathbf{T}}$$

and we want to show that $F \dashv U$. Next, we need the unit and counit:

$$\begin{aligned} \bar{\eta} &: 1_{\mathbf{C}} \to U \circ F \\ \epsilon &: F \circ U \to 1_{\mathbf{C}^T} \end{aligned}$$

Given $C \in \mathbf{C}$, we have

$$UF(C) = U(TC, \mu_C) = TC.$$

So we can take $\bar{\eta} = \eta : 1_{\mathbf{C}} \to U \circ F$, as required.

Given $(A, \alpha) \in \mathbf{C}^T$,

$$FU(A, \alpha) = (TA, \mu_A)$$

and the definition of a T-algebra makes the following diagram commute:

$$\begin{array}{ccc} T^2A & \xrightarrow{T\alpha} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow[\alpha]{} & A \end{array}$$

But this is a morphism $\epsilon_{(A,\alpha)} : (TA, \mu_A) \to (A, \alpha)$ in $\mathbf{C}^T$. Thus we are setting

$$\epsilon_{(A,\alpha)} = \alpha.$$

And ϵ is *natural* by the definition of a morphism of T-algebras, as follows. Given any $h : (A, \alpha) \to (B, \beta)$, we need to show

$$h \circ \epsilon_{(A,\alpha)} = \epsilon_{(B,\beta)} \circ Th.$$

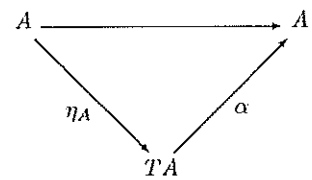
But by the definition of ϵ, that is, $h \circ \alpha = \beta \circ Th$, which holds since h is a T-algebra homomorphism.

Finally, the triangle identities now read as follows:

1. For (A, α), a T-algebra

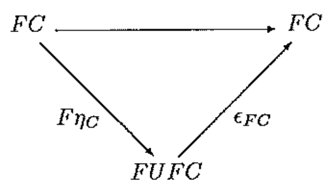
$$\begin{array}{ccc} U(A,\alpha) & \xrightarrow{\qquad\qquad} & U(A,\alpha) \\ & {\scriptstyle \eta_{U(A,\alpha)}}\searrow \quad \nearrow{\scriptstyle U\epsilon_{(A,\alpha)}} & \\ & UFU(A,\alpha) & \end{array}$$

which amounts to

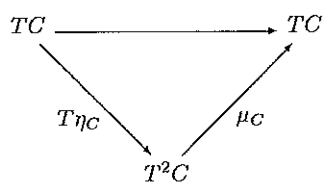


which holds since (A, α) is T-algebra.

2. For any $C \in \mathbf{C}$



which is



which holds by one of the unit laws for T.

Finally, note that we indeed have

$$T = U \circ F$$
$$\eta = \text{ unit of } F \dashv U.$$

And for the multiplication,

$$\bar{\mu} = U\epsilon F$$

we have, for any $C \in \mathbf{C}$,

$$\bar{\mu}_C = U\epsilon_{FC} = U\epsilon_{(TC,\mu_C)} = U\mu_C = \mu_C.$$

So $\bar{\mu} = \mu$ and we are done; the adjunction $F \dashv U$ via η and ϵ gives rise to the monad (T, η, μ). □

Example 10.7. Take the free monoid adjunction,

$$F : \mathbf{Sets} \rightleftarrows \mathbf{Mon} : \mathbf{U}.$$

The monad on **Sets** is then $T : \mathbf{Sets} \to \mathbf{Sets}$, where for any set X, $T(X) = UF(X) =$ "strings over X." The unit $\eta : X \to TX$ is the usual "string of length one" operation, but what is the multiplication?

$$\mu : T^2X \to TX$$

Here T^2X is the set of strings of strings,

$$[[x_{11}, \ldots, x_{1n}], [x_{21}, \ldots, x_{2n}], \ldots, [x_{m1}, \ldots, x_{mn}]].$$

And μ of such a string of strings is the string of their elements,

$$\mu([[x_{11}, \ldots, x_{1n}], [x_{21}, \ldots, x_{2n}], \ldots, [x_{m1}, \ldots, x_{mn}]]) = [x_{11}, \ldots, x_{mn}].$$

Now, what is a T-algebra in this case? By the equations for a T-algebra, it is a map,

$$\alpha : TA \to A$$

from strings over A to elements of A, such that

$$\alpha[a] = a$$

and

$$\alpha(\mu([[\ldots], [\ldots], \ldots, [\ldots]])) = \alpha(\alpha[\ldots], \alpha[\ldots], \ldots, \alpha[\ldots]).$$

If we start with a monoid, then we can get a T-algebra $\alpha : TM \to M$ by

$$\alpha[m_1, \ldots, m_n] = m_1 \cdot \ldots \cdot m_n.$$

This clearly satisfies the required conditions. Observe that we can even recover the monoid structure from m by $u = m(-)$ for the unit and $x \cdot y = m(x, y)$ for the multiplication. Indeed, *every* T-algebra is of this form for a *unique* monoid (exercise!).

We have now given constructions back and forth between adjunctions and monads. And we know that if we start with a monad $T : \mathbf{C} \to \mathbf{C}$, and then take the adjunction,

$$F^T : \mathbf{C} \rightleftarrows \mathbf{C}^T : U^T$$

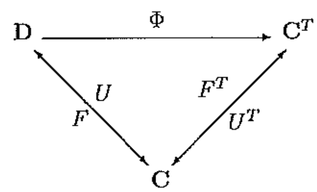
then we can get the monad back by $T = U^T \circ F^T$. Thus, in particular, every monoid arises from *some* adjunction. But are $\mathbf{C}^T, U^T, F^T$ unique with this property?

In general, the answer is *no*. There may be many different categories $\mathbf{D}$ and adjunctions $F \dashv U : \mathbf{D} \to \mathbf{C}$, all giving the same monad on $\mathbf{C}$. We have used the Eilenberg–Moore category $\mathbf{C}^T$, but there is also something called the "Kleisli category," which is in general different from $\mathbf{C}^T$, but also has an adjoint pair to $\mathbf{C}$ giving rise to the same monad (see the exercises).

If we start with an adjunction $F \dashv U$ and construct $\mathbf{C}^T$ for $T = U \circ F$, we then get a comparison functor $\Phi : \mathbf{D} \to \mathbf{C}^T$, with

$$U^T \circ \Phi \cong U$$

$$\Phi \circ F = F^T$$



In fact, Φ is *unique* with this property. A functor $U : \mathbf{D} \to \mathbf{C}$ is called *monadic* if it has a left adjoint $F \dashv U$, such that this comparison functor is an equivalence of categories,

$$\mathbf{D} \xrightarrow[\cong]{\Phi} \mathbf{C}^T$$

for $T = UF$.

Typical examples of monadic forgetful functors $U : \mathbf{C} \to \mathbf{Sets}$ are those from the "algebraic" categories arising as models for equational theories, like monoids, groups, rings, etc. Indeed, one can reasonably take monadicity as the *definition* of being "algebraic."

An example of a right adjoint that is *not* monadic is the forgetful functor from posets,

$$U : \mathbf{Pos} \to \mathbf{Sets}.$$

Its left adjoint F is the discrete poset functor. For any set X, therefore, one has as the unit the identity function $X = UF(X)$. The reader can easily show that the Eilenberg–Moore category for $T = 1_{\mathbf{Sets}}$ is then just $\mathbf{Sets}$ itself.

10.4 Comonads and coalgebras

By definition, a *comonad* on a category $\mathbf{C}$ is a monad on $\mathbf{C}^{\mathrm{op}}$. Explicitly, this consists of an endofunctor $G : \mathbf{C} \to \mathbf{C}$ and natural transformations,

$$\epsilon : G \to 1 \qquad \text{the counit}$$

$$\delta : G \to G^2 \quad \text{comultiplication}$$

satisfying the duals of the equations for a monad, namely

$$\delta_G \circ \delta = G\delta \circ \delta$$

$$\epsilon_G \circ \delta = 1_G = G\epsilon \circ \delta.$$

We leave it as an exercise in duality to verify that an adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$ and $F : \mathbf{C} \to \mathbf{D}$ and $\eta : 1_{\mathbf{C}} \to UF$ and $\epsilon : FU \to 1_{\mathbf{D}}$ gives rise to a comonad (G, ϵ, δ) on $\mathbf{D}$, where

$$G = F \circ U : \mathbf{D} \to \mathbf{D}$$
$$\epsilon : G \to 1$$
$$\delta = F\eta_U : G \to G^2.$$

The notions of coalgebra for a comonad, and of a comonadic functor, are of course also precisely dual to the corresponding ones for monads. Why do we even bother to study these notions separately, rather than just considering their duals? As in other examples of duality, there are actually two distinct reasons:

1. We may be interested in a particular category with special properties not had by its dual. A comonad on $\mathbf{Sets}^{\mathbf{C}}$ is of course a monad on $(\mathbf{Sets}^{\mathbf{C}})^{\mathrm{op}}$, but as we now know, $\mathbf{Sets}^{\mathbf{C}}$ has many special properties that its dual does not have (e.g., it is a topos!). So we can profitably consider the notion of a comonad on such a category.

   A simple example of this kind is the comonad $G = \Delta \circ \varprojlim$ resulting from composing the "constant functor" functor $\Delta : \mathbf{Sets} \to \mathbf{Sets}^{\mathbf{C}}$ with the "limit" functor $\varprojlim : \mathbf{Sets}^{\mathbf{C}} \to \mathbf{Sets}$. It can be shown in general that the coalgebras for this comonad again form a topos. In fact, they are just the constant functors $\Delta(S)$ for sets S, and the category $\mathbf{Sets}$ is thus comonadic over $\mathbf{Sets}^{\mathbf{C}}$.

2. It may happen that both structures—monad and comonad—occur together, and interact. Taking the opposite category will not alter this situation! This happens for instance when a system of *three* adjoint functors are composed:

$$L \dashv U \dashv R \qquad \mathbf{C} \; \begin{array}{c} \xrightarrow{\;R\;} \\ \xleftarrow{\;U\;} \\ \xrightarrow[\;L\;]{} \end{array} \; \mathbf{D}$$

   resulting in a monad $T = U \circ L$ and a comonad $G = U \circ R$, both on $\mathbf{C}$. In such a case, T and G are then of course also adjoint $T \dashv G$.

   This arises, for instance, in the foregoing example with $R = \varprojlim$, and $U = \Delta$, and $L = \varinjlim$ the "colimit" functor. It also occurs in propositional modal logic, with $T = \Diamond$ "possibility" and $G = \Box$ "necessity," where the adjointness $\Diamond \dashv \Box$ is equivalent to the law known to modal logicians as "S5."

   A related example is given by the open and closed subsets of a topological space: the topological interior operation on arbitrary subsets is a comonad and closure is a monad. We leave the details as an exercise.

10.5 Algebras for endofunctors

Some very basic kinds of algebraic structures have a more simple description than as algebras for a monad, and this description generalizes to structures that are not algebras for any monad, but still have some algebra-like properties.

As a familiar example, consider first the underlying structure of the notion of a group. We have a set G equipped with operations as indicated in the following:

$$\begin{array}{ccccc} G \times G & \xrightarrow{m} & G & \xleftarrow{i} & G \\ & & \uparrow{\scriptstyle u} & & \\ & & 1 & & \end{array}$$

We do not assume, however, that these operations satisfy the group equations of associativity, etc. Observe that this description of what we call a "group structure" can plainly be compressed into a single arrow of the form

$$1 + G + G \times G \xrightarrow{[u,i,m]} G$$

Now let us define the functor $F : \mathbf{Sets} \to \mathbf{Sets}$ by

$$F(X) = 1 + X + X \times X$$

Then a group structure is simply an arrow,

$$\gamma : F(G) \to G.$$

Moreover, a homomorphism of group structures in the conventional sense

$$h : G \to H,$$

$$\begin{aligned} h(u_G) &= u_H \\ h(i(x)) &= i(h(x)) \\ h(m(x,y)) &= m(h(x), h(y)) \end{aligned}$$

is then exactly a function $h : G \to H$ such that the following diagram commutes:

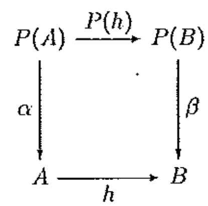
$$\begin{array}{ccc} F(G) & \xrightarrow{F(h)} & F(H) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \vartheta} \\ G & \xrightarrow[h]{} & H \end{array}$$

where $\vartheta : F(H) \to H$ is the group structure on H. This observation motivates the following definition.

Definition 10.8. Given an endofunctor $P : \mathcal{S} \to \mathcal{S}$ on any category $\mathcal{S}$, a P-*algebra* consists of an object A of $\mathcal{S}$ and an arrow,

$$\alpha : PA \to A.$$

A *homomorphism* $h : (A, \alpha) \to (B, \beta)$ of P-algebras is an arrow $h : A \to B$ in $\mathcal{S}$ such that $h \circ \alpha = \beta \circ P(h)$, as indicated in the following diagram:



The category of all such P-algebras and their homomorphisms are denoted as

$$P\text{-Alg}(\mathcal{S})$$

We usually write more simply P-Alg when $\mathcal{S}$ is understood. Also, if there is a monad present, we need to be careful to distinguish between algebras for the monad and algebras for the endofunctor (especially if P is the functor part of the monad!).

Example 10.9. 1. For the functor $P(X) = 1 + X + X \times X$ on **Sets**, we have already seen that the category **GrpStr** of group structures is the same thing as the category of P-algebras,

$$P\text{-Alg} = \mathbf{GrpStr}.$$

2. Clearly, for any other algebraic structure of finite "signature," that is, consisting of finitely many, finitary operations, there is an analogous description of the structures of that sort as algebras for an associated endofunctor. For instance, a *ring structure*, with two nullary, one unary, and two binary operations is given by the endofunctor

$$R(X) = 2 + X + 2 \times X^2.$$

In general, a functor of the form

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

with natural number coefficients C_k, is called a (finitary) *polynomial functor*, for obvious reasons. These functors present exactly the *finitary structures*. The same thing holds for finitary structures in any category $\mathcal{S}$

with finite products and coproducts; these can always be represented as algebras for a suitable endofunctor.

3. In a category such as **Sets** that is complete and cocomplete, there is an evident generalization to infinitary signatures by using generalized or "infinitary" polynomial functors, that is, ones with infinite sets C_k as coefficients (representing infinitely many operations of a given arity), infinite sets B_k as the exponents X^{B_k} (representing operations of infinite arity), or infinitely many terms (representing infinitely many different arities of operations), or some combination of these. The algebras for such an endofunctor
$$P(X) = \sum_{i \in I} C_i \times X^{B_i}$$
can then be naturally viewed as generalized "algebraic structures." Using locally cartesian closed categories, one can even present this notion without needing (co)completeness.

4. One can of course also consider algebras for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ that is not polynomial at all, such as the covariant powerset functor $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$. This leads to a proper generalization of the notion of an "algebra," which however still shares some of the formal properties of conventional algebras, as seen below.

Let $P : \mathbf{Sets} \to \mathbf{Sets}$ be a polynomial functor, say
$$P(X) = 1 + X^2$$
(what structure is this?). Then the notion of an *initial* P-algebra gives rise to a recursion property analogous to that of the natural numbers. Specifically, let
$$[o, m] : 1 + I^2 \to I$$
be an initial P-algebra, that is, an initial object in the category of P-algebras. Then, explicitly, we have the structure
$$o \in I, \qquad m : I \times I \to I$$
and for any set X with a distinguished element and a binary operation
$$a \in X, \qquad * : X \times X \to X$$
there is a unique function $u : I \to X$ such that the following diagram commutes:
$$\begin{array}{ccc}
1 + I^2 & \xrightarrow{P(u)} & 1 + X^2 \\
{\scriptstyle [o,m]} \downarrow & & \downarrow {\scriptstyle [a,*]} \\
I & \xrightarrow[u]{} & X
\end{array}$$

This of course says that, for all $i, j \in I$,

$$u(o) = a$$
$$u(m(i,j)) = u(i) * u(j)$$

which is exactly a *definition by structural recursion* of the function $u : I \to X$. Indeed, the usual recursion property of the natural numbers $\mathbb{N}$ with $0 \in \mathbb{N}$ and successor $s : \mathbb{N} \to \mathbb{N}$ says precisely that $(\mathbb{N}, 0, s)$ is the initial algebra for the endofunctor,

$$P(X) = 1 + X : \mathbf{Sets} \to \mathbf{Sets}$$

as the reader should check.

We next briefly investigate the question: When does an endofunctor have an initial algebra? The existence is constrained by the fact that initial algebras, when they exist, must have the following noteworthy property.

Lemma 10.10 (Lambek). *Given any endofunctor $P : \mathcal{S} \to \mathcal{S}$ on an arbitrary category $\mathcal{S}$, if $i : P(I) \to I$ is an initial P-algebra, then i is an isomorphism,*

$$P(I) \cong I.$$

We leave the proof as an easy exercise.

In this sense, the initial algebra for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ is a "least fixed point" for P. Such algebras are often used in computer science to model "recursive datatypes" determined by the so-called fixed point equations $X = P(X)$.

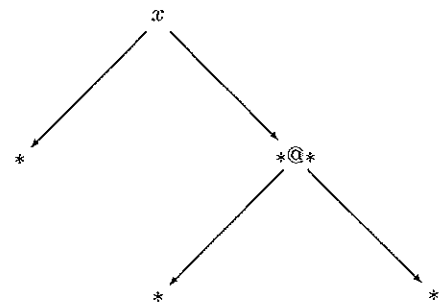
Example 10.11. 1. For the polynomial functor,

$$P(X) = 1 + X^2$$

(monoid structure!), let us "unwind" the initial algebra,

$$[*, @] : 1 + I \times I \cong I.$$

Given any element $x \in I$, it is thus either of the form $*$ or of the form $x_1 @ x_2$ for some elements $x_1, x_2 \in I$. Each of these x_i, in turn, is either of the form $*$ or of the form $x_{i1} @ x_{i2}$, and so on. Continuing in this way, we have a representation of x as a finite, binary tree. For instance, an element of the form $x = *@(*@*)$ looks like



We can present the monoid structure explicitly by letting

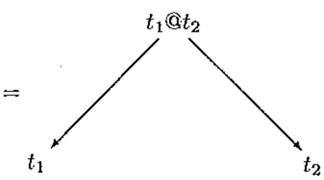
$$I = \{t \mid t \text{ is a finite, binary tree}\}$$

with

$$* = \text{"the empty tree"}$$

$$@(t_1, t_2) = t_1 @ t_2$$

$$=$$



The isomorphism,

$$[*, @] : 1 + I \times I \to I$$

here is plain to see.

2. Similarly, for any other polynomial functor,

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

we can describe the initial algebra (in **Sets**),

$$P(I) \cong I$$

as a set of trees with branching types and labels determined by P. For instance, consider the polynomial

$$P(X) = 1 + A \times X$$

for some set A. What is the initial algebra? Since,

$$[*, @] : 1 + A \times I \cong I$$

we can unwind an element x as

$$x = * \text{ or } a_1 @ x_1$$
$$x_1 = * \text{ or } a_2 @ x_2$$
$$\dots$$

Thus, we essentially have $x = a_1 @ a_2 @ \cdots @ a_n$. So I can be represented as the set A-List of (finite) lists of elements $a_1, a_2, \dots$ of A, with the structure

$$* = \text{"the empty list"}$$
$$@(a, \ell) = a @ \ell$$

The usual procedure of "recursive definition" follows from initiality. For example, the length function for lists length : A-List $\to \mathbb{N}$ is usually defined by

$$\text{length}(*) = 0 \tag{10.8}$$
$$\text{length}(a @ \ell) = 1 + \text{length}(\ell) \tag{10.9}$$

We can do this by equipping $\mathbb{N}$ with a suitable $P(X) = 1 + A \times X$ structure, namely,

$$[0, m] : 1 + A \times \mathbb{N} \to \mathbb{N}$$

where $m(a, n) = 1 + n$ for all $n \in \mathbb{N}$. Then by the universal mapping property (UMP) of the initial algebra, we get a unique function length : A-List $\to \mathbb{N}$ making a commutative square:

$$\begin{array}{ccc} 1 + A \times A\text{-List} & \xrightarrow{1 + A \times \text{length}} & 1 + A \times \mathbb{N} \\ {\scriptstyle [*, @]} \big\downarrow & & \big\downarrow {\scriptstyle [0, m]} \\ A\text{-List} & \xrightarrow[\text{length}]{} & \mathbb{N} \end{array}$$

But this commutativity is, of course, precisely equivalent to the equations (10.8) and (10.9).

In virtue of Lambek's lemma, we at least know that not all endofunctors can have initial algebras. For, consider the covariant powerset functor $\mathcal{P}$: **Sets** $\to$ **Sets**. An initial algebra for this would give us a set I with the property that $\mathcal{P}(I) \cong I$, which is impossible by the well-known theorem of Cantor!

The following proposition gives a useful sufficient condition for the existence of an initial algebra.

Proposition 10.12. *If the category $\mathcal{S}$ has an initial object* 0 *and colimits of diagrams of type ω (call them "ω-colimits"), and the functor*

$$P : \mathcal{S} \to \mathcal{S}$$

preserves ω-colimits, then P has an initial algebra.

Proof. Note that this generalizes a very similar result for posets already given above as proposition 5.34. And even the proof by "Newton's method" is essentially the same! Take the ω-sequence

$$0 \to P0 \to P^2 0 \to \cdots$$

and let I be the colimit

$$I = \varinjlim_n P^n 0.$$

Then, since P preserves the colimit, there is an isomorphism

$$P(I) = P(\varinjlim_n P^n 0) \cong \varinjlim_n P(P^n 0) = \varinjlim_n P^n 0 = I$$

which is seen to be an initial algebra for P by an easy diagram chase. □

Since (as the reader should verify) every polynomial functor P : **Sets** $\to$ **Sets** preserves ω-colimits, we have

Corollary 10.13. *Every polynomial functor P :* **Sets** $\to$ **Sets** *has an initial algebra.*

Finally, we ask, what is the relationship between algebras for endofunctors and algebras for monads? The following proposition, which is a sort of "folk theorem," gives the answer.

Proposition 10.14. *Let the category $\mathcal{S}$ has finite coproducts. Given an endofunctor $P : \mathcal{S} \to \mathcal{S}$, the following conditions are equivalent:*

1. *The P-algebras are the algebras for a monad. Precisely, there is a monad $(T : \mathcal{S} \to \mathcal{S}, \eta, \mu)$, and an equivalence*

   $$P\text{-Alg}(\mathcal{S}) \simeq \mathcal{S}^T$$

   between the category of P-algebras and the category $\mathcal{S}^T$ of algebras for the monad. Moreover, this equivalence preserves the respective forgetful functors to $\mathcal{S}$.
2. *The forgetful functor $U : P\text{-Alg}(\mathcal{S}) \to \mathcal{S}$ has a left adjoint*

   $$F \vdash U.$$

3. *For each object A of $\mathcal{S}$, the endofunctor*

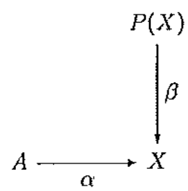
   $$P_A(X) = A + P(X) : \mathcal{S} \to \mathcal{S}$$

   has an initial algebra.

Proof. That (1) implies (2) is clear.

For (2) implies (3), suppose that U has a left adjoint $F : \mathcal{S} \to P\text{-Alg}$ and consider the endofunctor $P_A(X) = A + P(X)$. An algebra (X, γ) is a map $\gamma : A + P(X) \to X$. But there is clearly a unique correspondence between the following three types of things:

$$\gamma : A + P(X) \to X$$



$$\alpha : A \to U(X, \beta)$$

Thus, the P_A-algebras can be described equivalently as arrows of the form $\alpha : A \to U(X, \beta)$ for P-algebras (X, β). Moreover, a P_A-homomorphism $h : (\alpha, U(X, \beta)) \to (\alpha', U(X', \beta'))$ is just a P-homomorphism $h : (X, \beta) \to (X', \beta')$ making a commutative triangle with α and $\alpha' : A \to U(X', \beta')$. But an initial object in this category is given by the unit $\eta : A \to UFA$ of the adjunction $F \vdash U$, which shows (3).

Indeed, given just the forgetful functor $U : P\text{-Alg} \to \mathcal{S}$, the existence of initial objects in the respective categories of arrows $\alpha : A \to U(X, \beta)$, for each A, is exactly what is needed for the existence of a left adjoint F to U. So (3) also implies (2).

Before concluding the proof, it is illuminating to see how the free functor $F : \mathcal{S} \to P\text{-Alg}$ results from condition (3). For each object A in $\mathcal{S}$, consider the initial P_A-algebra $\alpha : A + P(I_A) \to I_A$. In the notation of recursive type theory,

$$I_A = \mu_X . A + P(X)$$

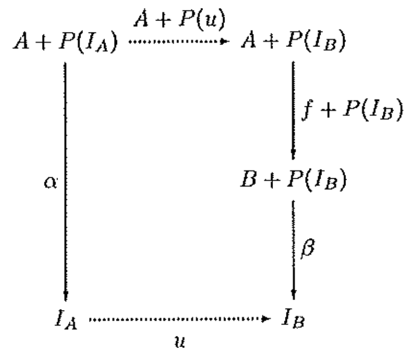
meaning it is the (least) solution to the "fixed point equation"

$$X = A + P(X).$$

Since α is a map on the coproduct $A + P(I_A)$, we have $\alpha = [\alpha_1, \alpha_2]$, and we can let

$$F(A) = (I_A,\ \alpha_2 : P(I_A) \to I_A)$$

To define the action of F on an arrow $f : A \to B$, let $\beta : B + P(I_B) \to I_B$ be the initial P_B-algebra and consider the diagram



The right-hand vertical composite $\beta \circ (f + P(I_B))$ now makes I_B into a P_A-algebra. There is thus a unique P_A-homomorphism u as indicated, and we can set

$$F(f) = u.$$

Finally, to conclude, the fact that (2) implies (1) is an easy application of Beck's Precise Tripleability Theorem, for which we refer the reader to section VI.7 of Mac Lane's *Categories Work* (1971). □

10.6 Exercises

1. Let $\mathbb{T}$ be the equational theory with one constant symbol and one unary function symbol (no axioms). In any category with a terminal object, a natural numbers object (NNO) is just an initial $\mathbb{T}$-model. Show that the natural numbers

   $$(\mathbb{N}, 0 \in \mathbb{N}, n + 1 : \mathbb{N} \to \mathbb{N})$$

   is an NNO in **Sets**, and that any NNO is uniquely isomorphic to it (as a $\mathbb{T}$-model).

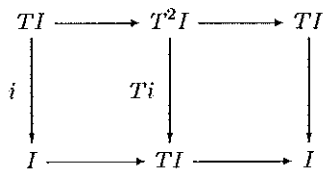
   Finally, show that $(\mathbb{N}, 0 \in \mathbb{N}, n + 1 : \mathbb{N} \to \mathbb{N})$ is uniquely characterized (up to isomorphism) as the initial algebra for the endofunctor $F(X) = X + 1$.

2. Let **C** be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor. A *T-algebra* consists of an object A and an arrow $a : TA \to A$ in **C**. A morphism $h : (a, A) \to (b, B)$ of T-algebras is a **C**-morphism $h : A \to B$ such that $h \circ a = b \circ T(h)$. Let **C** be a category with a terminal object 1 and binary coproducts. Let $T : \mathbf{C} \to \mathbf{C}$ be the evident functor with object-part $C \mapsto C + 1$ for all objects C of **C**. Show (easily) that the categories of T-algebras and $\mathbb{T}$-models ($\mathbb{T}$ as above) (in **C**) are equivalent:

   $$T\text{-Alg} \simeq \mathbb{T}\text{-Mod}.$$

Conclude that free T-algebras exist in **Sets**, and that an initial T-algebra is the same thing as an NNO.

3. ("Lambek's lemma") Show that for any endofunctor $T : \mathbf{C} \to \mathbf{C}$, if $i : TI \to I$ is an initial T-algebra, then i is an isomorphism. (Hint: consider a diagram of the following form, with suitable arrows.)



Conclude that for any NNO N in any category, there is an isomorphism $N + 1 \cong N$. Also, derive the usual recursion property of the natural numbers from initiality.

4. Given categories **C** and **D** and adjoint functors $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$ with $F \dashv U$, unit $\eta : 1_{\mathbf{C}} \to UF$, and counit $\epsilon : FU \to 1_{\mathbf{D}}$, show that

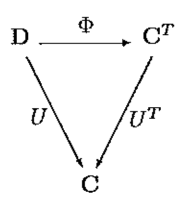
$$T = U \circ F : \mathbf{C} \to \mathbf{C}$$
$$\eta : 1_{\mathbf{C}} \to T$$
$$\mu = U\epsilon_F : T^2 \to T$$

do indeed determine a monad on **C**, as stated in the text.

5. Assume given categories **C** and **D** and adjoint functors

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U$$

with unit $\eta : 1_{\mathbf{C}} \to UF$ and counit $\epsilon : FU \to 1_{\mathbf{D}}$. Show that every D in **D** determines a $T = UF$ algebra $U\epsilon : UFUD \to UD$, and that there is a "comparison functor" $\Phi : \mathbf{D} \to \mathbf{C}^T$ which, moreover, commutes with the "forgetful" functors $U : \mathbf{D} \to \mathbf{C}$ and $U^T : \mathbf{C}^T \to \mathbf{C}$.



6. Show that $(P, s, \cup)$ is a monad on **Sets**, where

   - $P :$ **Sets** $\to$ **Sets** is the covariant powerset functor, which takes each function $f : X \to Y$ to the image mapping

$$P(f) = im(f) : P(X) \to P(Y)$$

- for each set X, the component $s_X : X \to P(X)$ is the singleton mapping, with

$$s_X(x) = \{x\} \subseteq X$$

for each $x \in X$;

- for each set X, the component $\cup_X : PP(X) \to P(X)$ is the union operation, with

$$\cup_X(\alpha) = \{x \in X \mid \exists_{U \in \alpha}.\ x \in U\} \subseteq X$$

for each $\alpha \subseteq P(X)$.

7. Determine the category of (Eilenberg–Moore) algebras for the $(P, s, \cup)$ monad on **Sets** defined in the foregoing problem. (Hint: consider complete lattices.)
8. Consider the free $\dashv$ forgetful adjunction

$$F : \mathbf{Sets} \rightleftarrows \mathbf{Mon} : \mathrm{U}$$

between sets and monoids, and let (T, η^T, μ^T) be the associated monad on **Sets**. Show that any T-algebra $\alpha : TA \to A$ for this monad comes from a monoid structure on A (exhibit the monoid multiplication and unit element).

9. (a) Show that an adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$ and $\eta : UF \to 1_{\mathbf{C}}$ and $\epsilon : 1_{\mathbf{D}} \to FU$ also gives rise to a *comonad* (G, ϵ, δ) in $\mathbf{D}$, with

$$G = F \circ U : \mathbf{D} \to \mathbf{D}$$

$$\epsilon : G \to 1 \text{ the counit}$$

$$\delta = F\eta_U : G \to G^2$$

satisfying the duals of the equations for a monad.

(b) Define the notion of a *coalgebra* for a comonad, and show (by duality) that every comonad (G, ϵ, δ) on a category $\mathbf{D}$ "comes from" a (not necessarily unique) adjunction $F \dashv G$ such that $G = FU$ and ϵ is the counit.

(c) Let **End** be the category of sets equipped with an endomorphism, $e : S \to S$. Consider the functor $G : \mathbf{End} \to \mathbf{End}$ defined by

$$G(S, e) = \{x \in S \mid e^{(n+1)}(x) = e^{(n)}(x) \text{ for some } n\}$$

equipped with the restriction of e. Show that this is the functor part of a comonad on **End**.

10. Verify that the open and closed subsets of a topological space give rise to comonad and monad, respectively, on the powerset of the underlying pointset. Moreover, the categories of coalgebras and algebras are isomorphic.

11. (Kleisli category) Given a monad (T, η, μ) on a category $\mathbf{C}$, in addition to the Eilenberg–Moore category, we can construct another category $\mathbf{C}_T$ and an adjunction $F \dashv U$, $\eta : 1 \to UF$, $\epsilon : FU \to 1$ with $U : \mathbf{C}_T \to \mathbf{C}$ such that

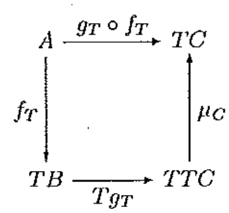
$$T = U \circ F$$
$$\eta = \eta \quad \text{(the unit)}$$
$$\mu = U\epsilon_F$$

This category $\mathbf{C}_T$ is called the *Kleisli category* of the adjunction, and is defined as follows:

- the objects are the same as those of $\mathbf{C}$, but written $A_T, B_T, \ldots$,
- an arrow $f_T : A_T \to B_T$ is an arrow $f : A \to TB$ in $\mathbf{C}$,
- the identity arrow $1_{A_T} : A_T \to A_T$ is the arrow $\eta_A : A \to TA$ in $\mathbf{C}$,
- for composition, given $f_T : A_T \to B_T$ and $g_T : B_T \to C_T$, the composite $g_T \circ f_T : A_T \to C_T$ is defined ∧ ins. to be

$$\mu_C \circ Tg_T \circ f_T$$

as indicated in the following diagram:



Verify that this indeed defines a category, and that there are adjoint functors $F : \mathbf{C} \to \mathbf{C}_T$ and $U : \mathbf{C}_T \to \mathbf{C}$ giving rise to the monad as $T = UF$, as claimed.

12. Let $P : \mathbf{Sets} \to \mathbf{Sets}$ be a polynomial functor,

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

with natural number coefficients C_k. Show that P preserves ω-colimits.

13. The notion of a *coalgebra* for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ on an arbitrary category $\mathcal{S}$ is exactly dual to that of a P-algebra. Determine the *final* coalgebra for the functor

$$P(X) = 1 + A \times X$$

for a set A. (Hint: Recall that the initial algebra consisted of *finite* lists $a_1, a_2, \ldots$ of elements of A.)

SOLUTIONS TO SELECTED EXERCISES

Chapter 1

1. (a) Identity arrows behave correctly, for if $f \subset A \times B$, then

$$\begin{aligned} f \circ 1_A &= \{\langle a,b\rangle \mid \exists a' \in A : \langle a,a'\rangle \in 1_A \wedge \langle a',b\rangle \in f\} \\ &= \{\langle a,b\rangle \mid \exists a' \in A : a = a' \wedge \langle a',b\rangle \in f\} \\ &= \{\langle a,b\rangle \mid \langle a,b\rangle \in f\} = f \end{aligned}$$

and symmetrically $1_B \circ f = f$. Composition is associative; if $f \subseteq A \times B$, $g \subseteq B \times C$, and $h \subseteq C \times D$, then

$$\begin{aligned} (h \circ g) \circ f &= \{\langle a,d\rangle \mid \exists b : \langle a,b\rangle \in f \wedge \langle b,d\rangle \in h \circ g\} \\ &= \{\langle a,d\rangle \mid \exists b : \langle a,b\rangle \in f \wedge \langle b,d\rangle \in \{\langle b,d\rangle \mid \exists c : \langle b,c\rangle \in g \wedge \langle c,d\rangle \in h\}\} \\ &= \{\langle a,d\rangle \mid \exists b : \langle a,b\rangle \in f \wedge \exists c : \langle b,c\rangle \in g \wedge \langle c,d\rangle \in h\} \\ &= \{\langle a,d\rangle \mid \exists b \exists c : \langle a,b\rangle \in f \wedge \langle b,c\rangle \in g \wedge \langle c,d\rangle \in h\} \\ &= \{\langle a,d\rangle \mid \exists c : (\exists b : \langle a,b\rangle \in f \wedge \langle b,c\rangle \in g) \wedge \langle c,d\rangle \in h\} \\ &= \{\langle a,d\rangle \mid \exists c : \langle a,b\rangle \in g \circ f \wedge \langle c,d\rangle \in h\} \\ &= h \circ (g \circ f). \end{aligned}$$

2. (a) $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$. The isomorphism functor (in both directions) takes an object A to itself, and takes a relation $f \subseteq A \times B$ to the *opposite relation* $f^{\mathrm{op}} \subseteq B \times A$ defined by $f^{\mathrm{op}} := \{\langle b,a\rangle \mid \langle a,b\rangle \in f\}$. It is straightforward to check that this is a functor $\mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ and $\mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$, and it is its own inverse.

   (b) $\mathbf{Sets} \not\cong \mathbf{Sets}^{\mathrm{op}}$. Consider maps into the empty set $\emptyset$; there is exactly one. If $\mathbf{Sets} \cong \mathbf{Sets}^{\mathrm{op}}$ held, there would have to be a corresponding set $\emptyset'$ with exactly one arrow out of it.

   (c) $P(X) \cong P(X)^{\mathrm{op}}$. The isomorphism takes each element U of the powerset to its complement $X - U$. Functoriality amounts to the fact that $U \subseteq V$ implies $X - V \subseteq X - U$.

3. (a) A bijection f from a set A to a set B, and its inverse f^{-1}, comprise an isomorphism; $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$, and so $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$, by definition of the inverse. If an arrow $f : A \to B$ in **Sets** is an isomorphism, then there is an arrow $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. The arrow f is an injection because $f(a) = f(a')$ implies $a = g(f(a)) = g(f(a')) = a'$, and f is surjective because every $b \in B$ has a preimage, namely $g(b)$, since $f(g(b)) = b$.

(b) Monoid homomorphisms that are isomorphisms are also isomorphisms in **Sets**, so by the previous solution they are bijective homomorphisms. It remains to show that bijective homomorphisms are isomorphisms. It is sufficient to show that the inverse mapping of a bijective homomorphism $f : M \to N$ is a homomorphism. But we have

$$\begin{aligned} f^{-1}(b \star_N b') &= f^{-1}(f(f^{-1}(b)) \star_N f(f^{-1}(b'))) \\ &= f^{-1}(f(f^{-1}(b) \star_M f^{-1}(b'))) \\ &= f^{-1}(b) \star_M f^{-1}(b') \end{aligned}$$

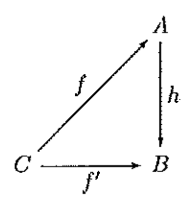
and $f^{-1}(e_N) = f^{-1}(f(e_M)) = e_M$.

(c) Consider the posets $A = (U, \leq_A)$ and $B = (U, \leq_B)$ given by $U = \{0, 1\}$, $\leq_A = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$, and $\leq_B = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\}$. The identity function $i : U \to U$ is an arrow $A \to B$ in **Posets**, and it is a bijection, but the only arrows $B \to A$ in **Posets** are the two constant functions $U \to U$, because arrows in **Posets** must be monotone. Neither is an inverse to i, which is therefore not an isomorphism.

Au: Please confirm the sequence of number is OK.

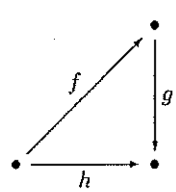
yes

6. The *coslice category* $C/\mathbf{C}$, the category whose objects are arrows $f : C \to A$ for $A \in \mathbf{C}$ and whose arrows $f \to f'$ are arrows h completing commutative triangles

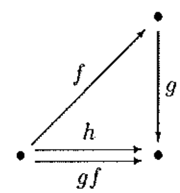


can equivalently be described as $(\mathbf{C}^{op}/C)^{op}$. For example, in the above diagram f, f' are arrows *into* C in the opposite category $\mathbf{C}^{op}$, so they are objects in the slice $\mathbf{C}^{op}/C$. The arrow h is $B \to A$ in $\mathbf{C}^{op}$ and $h \circ f = f'$ so it is an arrow $B \to A$ in $\mathbf{C}^{op}/C$, hence an arrow $A \to B$ in $(\mathbf{C}^{op}/C)^{op}$.

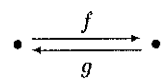
9. The free category on the graph



is

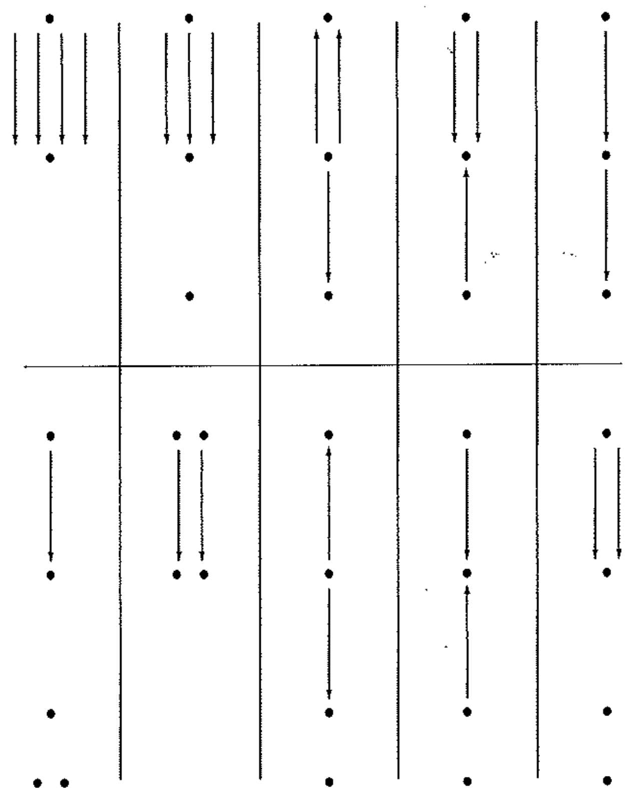


plus three identity arrows, one for each object. The free category on the graph



has infinitely many arrows, all possible finite sequence of alternating fs and gs—there are two empty sequences (i.e., identity arrows), one for each object.

10. The graphs whose free categories have exactly six arrows are the discrete graph with six nodes, and the following 10 graphs:



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11. (a) The functor $M : \mathbf{Sets} \to \mathbf{Mon}$ that takes a set X to the free monoid on X (i.e., strings over X and concatenation) and takes a function $f : X \to Y$ to the function $M(f)$ defined by $M(f)(a_1 \dots a_k) = f(a_1) \dots f(a_k)$ is a functor; $M(f)$ is a monoid homomorphism $MX \to MY$ since it preserves the monoid identity (the empty string) and the monoid operation (composition). It can be checked that M preserves identity functions and composition: $M(1_X)(a_1 \dots a_k) = 1_X(a_1) \dots 1_X(a_k) = a_1 \dots a_k$ and

$$M(g \circ f)(a_1 \dots a_k) = (g \circ f)(a_1) \dots (g \circ f)(a_k)$$
$$= g(f(a_1)) \dots g(f(a_k)) = M(g)(M(f)(a_1 \dots a_k))$$
$$= (M(g) \circ M(f))(a_1 \dots a_k).$$

12. Let $\mathbf{D}$ be a category and $h : G \to U(\mathbf{D})$ be a graph homomorphism. Suppose $\bar{h}$ is a functor $\mathbf{C}(G) \to \mathbf{D}$ such that

$$U(\bar{h}) \circ i = h \qquad (*).$$

From this equation, we see that $U(\bar{h})(i(x)) = h(x)$ for all vertices and edges $x \in G$. So the behavior of $\bar{h}$ on objects and paths of length one (i.e., arrows in the image of i) in $\mathbf{C}(G)$ is completely determined by the requirement $(*)$. But since $\bar{h}$ is assumed to be a functor, and so must preserve composition, its behavior on arrows in $\mathbf{C}(G)$ that correspond to longer paths in G is also determined, by a simple induction. Now it must be that $\bar{h}(f_1 \cdots f_k) = h(f_1) \circ \cdots h(f_k)$ if $\bar{h}$ is a functor, and similarly $\bar{h}(\varepsilon_A) = 1_A$, where ε_A is the empty path at A. So uniqueness of $\bar{h}$ is established, and it is easily checked that this definition is indeed a functor, so the ~~universal mapping property (UMP)~~ is satisfied.

UMP

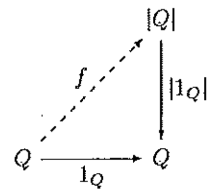
Chapter 2

1. Suppose $f : A \to B$ is epi and not surjective. Choose $b \in B$ not in the range of f. Define $g_1, g_2 : B \to \{0, 1\}$ as follows: $g_1(x) = 0$ for all $x \in B$, and $g_2(x) = 1$ if $x = b$, and 0 otherwise. Note that $g_1 \circ f = g_2 \circ f$ by choice of b, a contradiction. In the other direction, suppose f is surjective, and suppose $g_1, g_2 : B \to C$ are such that $g_1 \neq g_2$. Then there is $b \in B$ such that $g_1(b) \neq g_2(b)$. By assumption, b has a preimage a such that $f(a) = b$. So $g_1(f(a)) \neq g_2(f(a))$ and $g_1 \circ f \neq g_2 \circ f$.

4. (a) Iso: the inverse of h is $f^{-1} \circ g^{-1}$. Monic: If $h \circ k_1 = h \circ k_2$, then $g \circ f \circ k_1 = g \circ f \circ k_2$. Since g is monic, $f \circ k_1 = f \circ k_2$. Since f is monic, $k_1 = k_2$. Epi: dual argument.
   (b) If $f \circ k_1 = f \circ k_2$, then $g \circ f \circ k_1 = g \circ f \circ k_2$. Since h is monic, $k_1 = k_2$.
   (c) Dual argument to (b).
   (d) In $\mathbf{Sets}$, put $A = C = \{0\}$, $B = \{0, 1\}$, and all arrows constantly 0. h is monic but g is not.

5. Suppose $f : A \to B$ is an isomorphism. Then f is mono because $f \circ k_1 = f \circ k_2$ implies $k_1 = f^{-1} \circ f \circ k_1 = f^{-1} \circ f \circ k_2 = k_2$, and dually f is mono also. Trivially, f is split mono and split epi because $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. So we know $(a) \Rightarrow (b), (c), (d)$. If f is mono and split epi, then there is g such that $f \circ g = 1_B$. But since f is mono, $(f \circ g) \circ f = f \circ (g \circ f) = f = f \circ 1_A$ implies $g \circ f = 1_A$ and so g is in fact the inverse of f, and we have $(b) \Rightarrow (a)$. Dually, $(c) \Rightarrow (a)$. The fact that $(d) \Rightarrow (b), (c)$ needs only that split mono implies mono (or dually that split epi implies epi). If there is g such that $g \circ f = 1_A$, then $f \circ k_1 = f \circ k_2$ implies $k_1 = g \circ f \circ k_1 = g \circ f \circ k_2 = k_2$.
6. If $h : G \to H$ is injective on edges and vertices, and $h \circ f = h \circ g$ in **Graphs**, then the underlying set functions on edges and vertices are mono arrows in **Sets**, so the edge and vertex parts of f and g are equal, and so $f = g$. If $h : G \to H$ is mono in **Graphs**, and it is not injective on vertices, then there are two vertices v, w such that $h(v) = h(w)$. Let 1 be the graph with one vertex, and f, g be graph homomorphisms $1 \to G$ taking that vertex to v, w, respectively. Then, $h \circ f = h \circ g$. A similar argument holds for edges.
9. First, in the category **Pos**, an arrow is epi iff it is surjective: suppose that $f : A \to B$ is surjective and let $g, h : B \twoheadrightarrow C$ with $gf = hf$. In **Pos**, this means that g and h agree on the image of f, which by surjectivity is all of B. Hence $g = h$ and f is epi. On the other hand, suppose f is not epi and that $g, h : B \to C$ witness this. Since $g \neq h$, there is some $b \in B$ with $g(b) \neq h(b)$. But from this $b \notin f(A)$, and so A is not surjective.
   Next, the singleton set $\mathbf{1}$, regarded as a poset, is projective: suppose $f : \mathbf{1} \to Y$ and $e : X \twoheadrightarrow Y$ are arrows in **Pos**, with e epi. Then e is surjective, so there is some $x \in X$ with $e(x) = f(*)$. Any map $* \mapsto x$ witnesses the projectivity of $\mathbf{1}$.
10. Any set A is projective in **Pos**: suppose that $f : A \to Y$ and $e : X \twoheadrightarrow Y$ are arrows in **Pos**. Choose for each $y \in Y$ an element $x_y \in X$ with $f(x_y) = y$; this is possible since e is epi and hence surjective. Now define a map $\bar{f} : A \to X$ by $a \mapsto x_{f(a)}$. Since A is discrete this is necessarily monotonic, and we have $e\bar{f} = f$, so A is projective.
   For contrast, the two element poset $P = \{0 \leq 1\}$ is not projective. Indeed, we may take f to be the identity and X to be the discrete two-element set $\{a, b\}$. Then the surjective map $e : a \mapsto 0, b \mapsto 1$ is an epi, since it is surjective. However, any monotone map $g : P \to \{a, b\}$ must identify 0 and 1, since the only arrows in the second category are identities. But then $e \circ g \neq 1_P$. Thus, there is no function g lifting the identity map on P across e, so P is not projective.
   Moreover, every projective object in **Pos** is discrete: For suppose Q is projective. We can always consider the discretation $|Q|$ of Q, which has the same objects as Q and only identity arrows. We clearly get a map $|Q| \to Q$ which is surjective and hence epi. This means that we can complete the

diagram

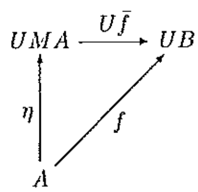


But the only object function that could possibly commute in this situation is the object identity. Then,

$$x \le x' \iff f(x) \le f(x') \iff f(x) = f(x') \iff x = x'.$$

But then the only arrows of Q are identity arrows, so Q is discrete, as claimed. Thus, the projective posets are exactly the discrete sets. Clearly, composition of maps and identity arrows of discrete posets are exactly those of **Set**, so **Set** is a subcategory of **Pos**. Moreover, every function between discrete sets is monotone, so this is a full subcategory.

11. The UMP of a free monoid states that for any $f : A \to UB$, there is a unique $\bar{f} : MA \to B$ such that



(*)

commutes. For $\eta : A \to UM$ to be an initial object in A-**Mon**, it must be that for object $f : A \to UB$, there is a unique arrow $\bar{f}$ in A-**Mon** from η to f. But the definition of arrow in A-**Mon** is such that this arrow must complete exactly the commutative triangle $(*)$ above. Therefore, the two characterizations of the free monoid coincide.

13. Let P be the iterated product $A \times (B \times C)$ with the obvious maps $p_1 : P \to A$, $p_2 : P \to B \times C \to B$, and $p_3 : P \to B \times C \to C$. Define $Q = (A \times B) \times C$ and q_i similarly. By the UMP, we get a unique map $f_1 = p_1 \times p_2 : P \to A \times B$. Applying it again, we get a unique map $f = (p_1 \times p_2) \times p_3 : P \to Q$ with $q_i f = p_i$. We can run a similar argument to get a map g in the other direction. Composing, we get $gf : P \to P$ which respects the p_i. By the UMP, such a map is unique, but the identity is another such map. Thus they must be the same, so $gf = 1_P$. Similarly $fg = 1_Q$, so f and g are inverse and $P \cong Q$.

17. The pairing of any arrow with the identity is in fact split mono: $\pi_1 \circ \langle 1_A, f \rangle = 1_A$. There is a functor $G : \mathbf{Sets} \to \mathbf{Rel}$ which is constant on objects and takes $f : A \to B$ to $(\mathrm{im}\,\langle 1_A, f \rangle) \subseteq A \times B$. It preserves identities since

$G(1_A)\,\mathrm{im}\,\langle 1_A, 1_A\rangle = \{\langle a, a\rangle \mid a \in A\} = 1_A \in$ **Rel**. It preserves composition because for $g : B \to C$, we have

$$G(g \circ f) = \mathrm{im}\,\langle 1_A, g \circ f\rangle = \{\langle a, g(f(a))\rangle \mid a \in A\}$$

$$= \{\langle a, c\rangle \mid \exists b \in B. b = f(a) \wedge c = g(b)\}$$

$$= \{\langle b, g(b)\rangle \mid b \in B\} \circ \{\langle a, f(a)\rangle \mid a \in A\}$$

$$= G(g) \circ G(f).$$

Chapter 3

1. In any category **C**, the diagram

$$A \xleftarrow{\;c_1\;} C \xrightarrow{\;c_2\;} B$$

is a product diagram iff the mapping

$$\hom(Z, C) \longrightarrow \hom(Z, A) \times \hom(Z, B)$$

given by $f \mapsto \langle c_1 \circ f, c_2 \circ f\rangle$ is an isomorphism. Applying this fact to $\mathbf{C}^{\mathrm{op}}$, the claim follows.

2. Say i_{MA}, i_{MB} are the injections into the coproduct $MA + MB$, and η_A, η_B are the injections into the free monoids on A, B. Put $e = [U(i_{MA}) \circ \eta_A, U(i_{MB}) \circ \eta_B]$. Let an object Z and an arrow $f : A + B \to UZ$ be given. Suppose $h : MA + MB \to Z$ has the property that

$$Uh \circ e = f \qquad (*)$$

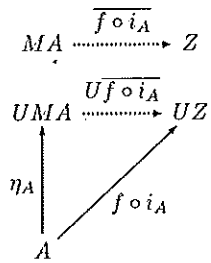
Because of the UMP of the coproduct, we have generally that $a \circ [b, c] = [a \circ b, a \circ c]$, and in particular

$$Uh \circ e = [Uh \circ U(i_{MA}) \circ \eta_A, Uh \circ U(i_{MB}) \circ \eta_B]$$

Because this is equal to f, which is an arrow out of $A + B$, and since functors preserve composition, we have

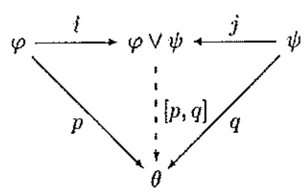
$$U(h \circ i_{MA}) \circ \eta_A = f \circ i_A$$
$$U(h \circ i_{MB}) \circ \eta_B = f \circ i_B$$

where i_A, i_B are the injections into $A + B$. But the UMP of the free monoid implies that $h \circ i_{MA}$ must coincide with the unique $\overline{f \circ i_A}$ that makes the triangle



commute. Similarly, $h \circ i_{MB} = \overline{f \circ i_B}$. Since its behavior is known on both injections, h is uniquely determined by the condition $(*)$; in fact, $h = [\overline{f \circ i_A}, \overline{f \circ i_B}]$. That is, the UMP of the free monoid on $A+B$ is satisfied by $MA+MB$. Objects characterized by UMPs are unique up to isomorphism, so $M(A+B) \cong MA + MB$.

5. In the category of proofs, we want to see that (modulo some identifications) the coproduct of formulas φ and ψ is given by $\varphi \vee \psi$. The intro and elim rules automatically give us maps (proofs) of the coproduct from either of its disjuncts, and from pairs of proofs that begin with each of the disjuncts into a single proof beginning with the disjunction. To see that this object really is a coproduct, we must verify that this is the unique commuting arrow.



But this is simple since composition is simply concatenation of proofs. Suppose we have another proof $r : \varphi \vee \psi \to \vartheta$ with $r \circ i = p$. Then by disjunction elimination, r necessarily has the form

$$\frac{\varphi \vee \psi \quad \begin{matrix}[\varphi]\\ \vdots \\ \vartheta\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \vartheta\end{matrix}}{\vartheta}\ .$$

Applying i on the right simply has the effect of bringing down part of the proof above, so that the quotienting equation now reads $r \circ i = \begin{matrix}[\varphi]\\ \vdots \\ \vartheta\end{matrix} = p$. Hence, up to the presence of more detours, we know that the proof appearing as part

$$\begin{array}{c}[\psi]\\ \vdots\\ \vartheta\end{array}$$

of r is exactly p. Similarly, we know that the second part of the proof must be q. Thus r is uniquely defined (up to detours) by p and q, $\varphi \vee \psi$ is indeed a coproduct.

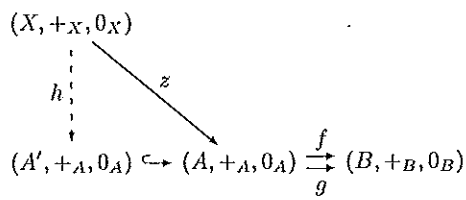
6. (Equalizers in **Ab**). Suppose we have a diagram

$$(A, +_A, 0_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, +_B, 0_B)$$

in **Ab**. Put $A' := \{a \in A \mid f(a) = g(a)\}$. It is easy to check that A' is in fact a subgroup of A, so it remains to be shown that

$$(A', +_A, 0_A) \hookrightarrow (A, +_A, 0_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, +_B, 0_B)$$

is an equalizer diagram.



If the triangle is to commute, $h(x) = z(x)$ for all $x \in X$, so h is uniquely determined. It is easily checked that h is a homomorphism, implying that **Ab** indeed has all equalizers.

14. (a) The equalizer of $f \circ \pi_1$ and $f \circ \pi_2$ is the relation $\ker(f) = \{\langle a, a' \rangle \in A \times A \mid f(a) = f(a')\}$. Symmetry, transitivity, and reflexivity of $\ker(f)$ follow immediately from the same properties of equality.

    (b) We need to show that a pair a, a' of elements are in the kernel of the projection $q : A \twoheadrightarrow A/R$ iff they are related by R. But this amounts to saying that $q(a) = q(a')$ iff aRa', where $q(x) = \{x \mid xRa\}$ is the equivalence class. But this is true since R is an equivalence relation.

    (c) Take any function $f : A \to B$ with $f(a) = f(a')$ for all aRa'. The kernel $\ker(f)$ of f is therefore an equivalence relation that contains R, so $\langle R \rangle \subseteq \ker(f)$. It follows that f factors through the projection $q : A \twoheadrightarrow A/\langle R \rangle$ (necessarily uniquely, since q is epic).

    (d) The coequalizer of the projections from R is the projection $q : A \twoheadrightarrow A/\langle R \rangle$, which has $\langle R \rangle$ as its kernel.

Chapter 4

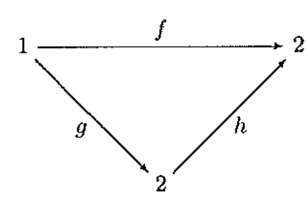
1. Given a categorical congruence $\sim$ on a group G, the corresponding normal subgroup is $N_\sim := \{g \mid g \sim e\}$. N is a subgroup; it contains the identity by reflexivity of $\sim$. It is closed under inverse by symmetry and the fact that $e \sim g$ implies $g^{-1} = g^{-1}e \sim g^{-1}g = e$. It is closed under product because if $g \sim e$ and $h \sim e$ then $gh \sim ge = g \sim e$, and by transitivity $gh \in N_\sim$. It is normal because
$$x\{g \mid g \sim e\} = \{xg \mid g \sim e\} = \{x(x^{-1}h) \mid x^{-1}h \sim e\} = \{h \mid h \sim x\}$$
and
$$\{g \mid g \sim e\}x = \{gx \mid g \sim e\} = \{(hx^{-1})x \mid hx^{-1} \sim e\} = \{h \mid h \sim x\}.$$
In the other direction, the categorical congruence $\sim_N$ corresponding to a normal subgroup N is $g \sim_N h :\iff gh^{-1} \in N$. The fact that $\sim_N$ is an equivalence follows easily from the fact that N is a subgroup. If $f \sim_N g$, then also $hfk \sim_N hgk$, since $fg^{-1} \in N$ implies $hfkk^{-1}g^{-1}h^{-1} = hfg^{-1}h^{-1} \in N$, because N was assumed normal, and so $N = hNh^{-1}$.
Since two elements g, h of a group are in the same coset of N precisely when $gh^{-1} = e$, the quotient G/N and the quotient $G/{\sim}$ coincide when N and $\sim$ are in the correspondence described above.

6. (a)
$$1 \longrightarrow 2 \longrightarrow 3$$
No equations. (i.e., 3 is free)

(b)
$$1 \underset{g}{\overset{f}{\rightrightarrows}} 2 \longrightarrow 3$$
Equations: $f = g$

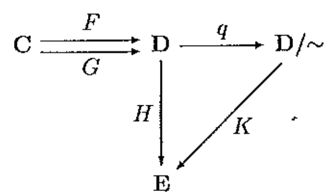
(c)
$$1 \underset{g}{\overset{f}{\rightrightarrows}} 2 \underset{k}{\overset{h}{\rightrightarrows}} 3$$
Equations: $f = g$, $h = k$

(d)



Equations: $f = h \circ g$

7. By definition of congruence, $f \sim f'$ implies $gf \sim gf'$ and $g \sim g'$ implies $gf' \sim g'f'$. By transitivity of $\sim$, we conclude $gf \sim g'f'$.
8. $\sim$ is an equivalence because equality is. For instance, if $f \sim g$, then for all $\mathbf{E}$ and $H : \mathbf{D} \to \mathbf{E}$ we have $HF = HG \Rightarrow H(f) = H(g)$. But under the same conditions, we have $H(g) = H(f)$, so $g \sim f$. Since H is assumed to be a functor, it preserves composition, and so $H(hfk) = H(h)H(f)H(k) = H(h)H(g)H(k) = H(hgk)$ for any H such that $HF = HG$ and any h, k, hence $\sim$ is a congruence.
   Let q be the functor assigning all the arrows in $\mathbf{D}$ to their $\sim$-equivalence classes in the quotient $\mathbf{D}/\sim$. We know q is indeed a well-defined functor by a previous exercise. Suppose we have H coequalizing F, G. By definition of $\sim$ any arrows that H identifies are $\sim$-equivalent, and therefore identified also by q. There can be at most one K making the triangle in



   commute, (for any $[f]_\sim \in \mathbf{D}/\sim$ it must be that $K([f]_\sim) = H(f)$) and the fact that q identifies at least as many arrows as H implies the existence of such a K. So q is indeed the coequalizer of F, G.

Chapter 5

1. Their UMPs coincide. A product in $\mathbf{C}/X$ of f and g is an object $h : A \times_X B \to X$ and projections $\pi_1 : h \to f$ and $\pi_2 : h \to g$ which is terminal among such structures. The pullback of f, g requires an object $A \times_X B$ and projections $\pi_1 : A \times_X B \to A$ and $\pi_2 : A \times_X B \to B$ such that $f \circ \pi_1 = g \circ \pi_2$, terminal among such structures. The commutativity requirements of the pullback are exactly those imposed by the definition of arrow in the slice category.
2. (a) If m is monic, then the diagram is a pullback; if $m \circ f = m \circ g$, then $f = g$, the unique mediating map being equivalently f or g. If the diagram is a pullback, suppose $m \circ f = m \circ g$. The definition of pullback implies the unique existence of h such that $1_M \circ h = f$ and $1_M \circ h = g$, but this implies $f = g$.

3.

$$\begin{array}{ccccc} Z & \overset{h}{\underset{k}{\rightrightarrows}} & M' & \xrightarrow{f'} & M \\ & & {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ & & A' & \xrightarrow[f]{} & A \end{array}$$

Let $h, k : Z \to M'$ be given. Suppose $m'h = m'k$. Then, $fm'h = fm'k$ and so $mf'h = mf'k$. Since m is assumed mono, $f'h = f'k$. The definition of pullback applied to the pair of arrows $m'k, f'k$ implies, there is exactly one arrow $q : Z \to M'$ such that $m' \circ q = m'k$ and $f' \circ q = f'k$. But both h, k can be substituted for q and satisfy this equation, so $h = k$.

4. Suppose $m : M \to A$ and $n : N \to A$ are subobjects of A. If $M \subseteq N$, then there is an arrow $s : M \to N$ such that $n \circ s = m$. If $z \in_A M$, then there is an arrow $f : Z \to M$ such that $m \circ f = z$. Then $s \circ f$ witnesses $z \in_A N$, since $n \circ s \circ f = m \circ f = z$. If for all $z : Z \to A$ we have $z \in_A M \Rightarrow z \in_A N$, then in particular this holds for $z = m$, and in fact $m \in_A M$ (via setting $f = 1_A$) so $m \in_A N$, in other words $M \subseteq N$.

7. We show that the representable functor $\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$ preserves all small products and equalizers; it follows that it preserves all small limits, since the latter can be constructed from the former. For products, we need to show that for any set I and family $(D_i)_{i \in I}$ of objects of $\mathbf{C}$, there is a (canonical) isomorphism,

$$\mathrm{Hom}(C, \prod_{i \in I} D_i) \cong \prod_{i \in I} \mathrm{Hom}(C, D_i).$$

But this follows immediately from the definition of the product $\prod_{i \in I} D_i$. For equalizers, consider an equalizer in $\mathbf{C}$,

$$E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B.$$

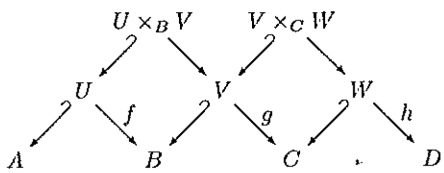
Applying $\mathrm{Hom}(C, -)$ results in the following diagram in **Sets**:

$$\mathrm{Hom}(C, E) \xrightarrow{e_*} \mathrm{Hom}(C, A) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \mathrm{Hom}(C, B),$$

which is clearly an equalizer: for, given $h : C \to A$ with $f_*(h) = g_*(h)$, we therefore have $fh = f_*(h) = g_*(h) = gh$, whence there is a unique $u : C \to E$ with $h = eu = e_*(u)$.

8. We have a putative category of partial maps. We need to verify identity and associativity. The first is easy. Any object is a subobject of itself, so we may

set 1_A in the category of partial maps to be the pair $(1_A, A)$. It is trivial to check that this acts as an identity.
For associativity, suppose U, V, and W are subobjects of A, B, and C, respectively, and that we have maps as in the diagram:



Now let P be the pullback of $U \times_B V$ and $V \times_C W$ over E and k the associated partial map. Since we can compose pullback squares, that means that P is also the pullback of U and $V \times_C W$ over B. Since the latter is the composition of g and h, this means $k = (h \circ g) \circ f$. Similarly, $k = h \circ (g \circ f)$. Hence the composition of partial maps is associative, and this setup does describe a category.

12. If we let the numeral n denote the initial segment of the natural number sequence $\{0 \leq 1 \leq \ldots \leq n\}$, we have a chain of inclusions in **Pos**:

$$0 \to 1 \to 2 \to \ldots \to n \to \ldots.$$

We would like to determine the limit and colimit of the diagram.
For the limit, suppose we have a cone $\zeta_n : Z \to n$. Since 0 is the initial object, ζ_0 is constant, and each map ζ_n has ζ_0 as a factor (this is the cone condition). But each such map simply takes 0 to itself, regarded as an element of n, so that ζ_n is also the constant zero map. So the limit of the diagram can be (anything isomorphic to) the object 0 together with the inclusions $0 \to n$.
Now suppose we have a co-cone $\psi_n : n \to Y$. The co-cone condition implies that ψ_n is simply the restriction of ψ_{m+n} to the subset $n \subseteq n + m$. If $m < n$, then

$$\psi(m) = \psi_m(m) = \psi_n(m) < \psi_n(n)$$

so this is a monotone function. For any other $\varphi : \mathbb{N} \to Y$, there is some n with $\varphi(n) \neq \psi(n) = \psi_n(n)$. Thus, ψ is the unique map factoring the co-cone on Y. Thus, $\omega = \{0 \leq 1 \leq 2 \leq \ldots\}$ together with the evident injections $n \to \omega$ is the colimit of the diagram.

Chapter 6

Notation: If $f : A \to B^C$, then $\mathrm{ev} \circ f \times C = \overline{f} : A \times C \to B$. If $f : A \times C \to B$, then $\lambda f : A \to B^C$.

2. These isomorphism are witnessed by the following pairs: $f : (A \times B)^C \to A^C \times B^C$ defined by $f = \langle \lambda(\pi_1 \circ \overline{1}_{(A\times B)^C}), \lambda(\pi_2 \circ \overline{1}_{(A\times B)^C}) \rangle$ and $f^{-1} : A^C \times B^C \to (A \times B)^C$ defined by $f^{-1} = \lambda\langle \overline{\pi_1}, \overline{\pi_2} \rangle$; and $g : (A^B)^C \to A^{B\times C}$

defined by $g = \lambda(\overline{\text{ev}} \circ \alpha_{(A^B)^C})$ and $g^{-1} : A^{B\times C} \to (A^B)^C$ defined by $g^{-1} = \lambda\lambda(\text{ev} \circ \alpha^{-1}_{A^{B\times C}})$, where α_Z is the evident isomorphism from associativity and commutativity of the product, up to isomorphism, $Z \times (B \times C) \to (Z \times C) \times B$.

3. The exponential transpose of ev is 1_{B^A}. The exponential transpose of $1_{A\times B}$ is the "partial pairing function" $A \to (A \times B)^B$ defined by $a \mapsto \lambda b : B.\langle a, b\rangle$. The exponential transpose of ev $\circ\ \tau$ is the "partial application function" $A \to B^{B^A}$ defined by $a \mapsto \lambda f : B^A.f(a)$.

6. Here we consider the category **Sub**, whose objects are pairs $(A, P \subseteq A)$, and whose arrows $f : (A, P) \to (B, Q)$ are set functions $A \to B$ such that $a \in P$ iff $f(a) \in Q$. This means that an arrow in this category is essentially a pair of arrows $f_1 : P \to Q$ and $f_2 : A \setminus P \to B \setminus Q$; thus, this is (isomorphic to) the category **Sets**/2.
Now, **Sets**/2 is equivalent to the product category **Sets**×**Sets**, by a previous exercise. This latter category is cartesian closed, by the equational definition of CCCs, which clearly holds in the two factors. But equivalence of categories preserves cartesian closure, so **Sub** is also cartesian closed.

10. For products, check that the set of pairs of elements of ωCPOs A and B ordered pointwise, constitutes an ωCPO (with ω-limits computed pointwise) and satisfies the UMP of a product. Similarly, the exponential is the set of continuous monotone functions between A and B ordered pointwise, with limits computed pointwise. In *strict* ωCPOs, by contrast, there is exactly one map $\{\bot\} \to A$, for any object A. Since $\{\bot\} = 1$ is also a terminal object, however, given an exponential B^A there can be only one map $A \to B$, since $\text{Hom}(A, B) \cong \text{Hom}(1 \times A, B) \cong \text{Hom}(1, B^A)$.

11. (a) The identity

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

"holds" in any CC poset with joins, that is, this object is equal to the top element 1. Equivalently, from the definition of $\Rightarrow$, we have

$$((p \vee q) \Rightarrow r) \leq ((p \Rightarrow r) \wedge (q \Rightarrow r)),$$

as follows immediately from part (b), which shows the existence of such an arrow in any CCC.

(b) In any category where the constructions make sense, there is an arrow

$$C^{(A+B)} \to C^A \times C^B.$$

Indeed, by the definition of the coproduct, we have arrows $A \to A + B$ and $B \to A + B$, to which we apply the contravariant functor $C^{(-)}$ to obtain maps $C^{(A+B)} \to A^C$ and $C^{(A+B)} \to C^B$. By the UMP of the product, this gives a map $C^{(A+B)} \to C^A \times C^B$, as desired.

13. This can be done directly by comparing UMPs. For a different proof (anticipating the Yoneda Lemma), consider, for an arbitrary object X, the

bijective correspondence of arrows,

$$\frac{(A \times C) + (B \times C) \to X}{(A + B) \times C \to X}.$$

This is arrived at via the canonical isos:

$$\begin{aligned}
\mathrm{Hom}((A \times C) + (B \times C), X) &\cong \mathrm{Hom}(A \times C, X) \times \mathrm{Hom}(B \times C, X) \\
&\cong \mathrm{Hom}(A, X^C) \times \mathrm{Hom}(B, X^C) \\
&\cong \mathrm{Hom}(A + B, X^C) \\
&\cong \mathrm{Hom}((A + B) \times C, X).
\end{aligned}$$

Now let $X = (A \times C) + (B \times C)$, respectively $X = (A + B) \times C$, and trace the respective identity arrows through the displayed isomorphisms to arrive at the desired isomorphism

$$(A \times C) + (B \times C) \cong (A + B) \times C.$$

14. If $D = \emptyset$ then $D^D \cong 1$, so there can be no interpretation of $s : D^D \to D$. If $D \cong 1$ then also $D^D \cong 1$, so there are unique interpretations of $s : D^D \to D$ and $t : D \to D^D$. If $|D| \geq 2$ (in cardinality), then $|D^D| \geq |2^D| \geq |\mathcal{P}(D)|$, so there can be no such (split) mono $s : D^D \to D$, by Cantor's theorem on the cardinality of powersets. Thus, the only models can be $D \cong 1$, and in these, clearly all equations hold, since all terms are interpreted as maps into 1.

Chapter 7

1. Take any element $a \in A$ and compute

$$\begin{aligned}
(\mathcal{F}(h) \circ \phi_A)(a) &= \mathcal{F}(h)(\phi_A(a)) \\
&= \mathcal{F}(h)(\{\mathcal{U} \in \mathrm{Ult}(A) \mid a \in \mathcal{U}\}) \\
&= \mathcal{P}(\mathrm{Ult}(h))(\{\mathcal{U} \in \mathrm{Ult}(A) \mid a \in \mathcal{U}\}) \\
&= (\mathrm{Ult}(h))^{-1}(\{\mathcal{U} \in \mathrm{Ult}(A) \mid a \in \mathcal{U}\}) \\
&= \{\mathcal{V} \in \mathrm{Ult}(B) \mid a \in \mathrm{Ult}(h)(\mathcal{V})\} \\
&= \{\mathcal{V} \in \mathrm{Ult}(B) \mid h(a) \in \mathcal{V}\} \\
&= \phi_B(h(a)) \\
&= (\phi_B \circ h)(a).
\end{aligned}$$

4. Both functors are faithful. U is full because every monoid homomorphism between groups is a group homomorphism: if $h(ab) = h(a)h(b)$ then $e = h(a^{-1}a) = h(a^{-1})h(a)$ and symmetrically $e = h(a)h(a^{-1})$ and so $h(a^{-1})$ is the inverse of $h(a)$. V is not full; there are set functions between monoids that are not homomorphisms. Only V is surjective on objects (there

are, for example cyclic groups of every cardinality). Only U is injective on objects, since monoid structure uniquely determines inverses, if they exist.

5. It is easy to check that upward-closed sets are closed under unions and finite intersections. The arrow part of the functor A simply takes a monotone function $f : P \to Q$ to itself, construed as a function $f : A(P) \to A(Q)$. Preservation of identities and composition is therefore trivial, but we must check that f is in fact an arrow in **Top**. Let U be an open (that is, upward-closed) subset of $A(Q)$. We must show that $f^{-1}(U)$ is upward-closed. Let $x \in f^{-1}(U)$ and $y \in P$ be given, and suppose $x \leq y$. We know that $f(x) \in U$ and $f(x) \leq f(y)$ since f is monotone. Because U is upward-closed, we have $f(y) \in U$, so $y \in f^{-1}(U)$ and so f is continuous.
   A is trivially faithful. A is also full: Let f be a continuous function $P \to Q$. Put $D := \{q \in Q \mid f(x) \leq q\}$. Since f is continuous and D is upward-closed, $f^{-1}(D)$ is upward-closed. If $x \leq y$ then the fact that $x \in f^{-1}(D)$ implies $y \in f^{-1}(D)$ and so $f(y) \in D$. That is, $f(x) \leq f(y)$. Hence every continuous function $A(P) \to A(Q)$ is a monotone function $P \to Q$.
6. (a) Let the objects of $\mathbf{E}$ be those of $\mathbf{C}$, and identify arrows in $\mathbf{C}$ if they are identified by F, that is, let $\mathbf{E}$ be the quotient category of $\mathbf{C}$ by the congruence induced by F. The functor D is the canonical factorization of F through the quotient.
   (b) Let $\mathbf{E}$ be the subcategory of $\mathbf{D}$ whose objects are those in the image of F, and whose arrows are all the $\mathbf{D}$-arrows among those objects. Let D be the inclusion of $\mathbf{E}$ in $\mathbf{D}$ and E the evident factorization of F through $\mathbf{E}$.
   These factorizations agree iff F itself is injective on objects and full.
7. Suppose α is a natural isomoprhism $F \to G : \mathbf{C} \to \mathbf{D}$. Then it has an inverse α^{-1}. Since $\alpha^{-1} \circ \alpha = 1_F$ and $\alpha \circ \alpha^{-1} = 1_G$, it must be that $\alpha_C \circ \alpha_C^{-1} = 1_{GC}$ and $\alpha_C^{-1} \circ \alpha_C = 1_{FC}$. So the components of α are isomorphisms. If conversely all α's components are isomorphisms, then defining $\alpha_C^{-1} = (\alpha_C)^{-1}$ for all $C \in \mathbf{C}$ makes α^{-1} a natural transformation which is α's inverse. For $f : A \to B$, knowing $Gf \circ \alpha_A = \alpha_B \circ Ff$, we compose on the left with α_B^{-1} and on the right with α_A^{-1} to obtain $Ff \circ \alpha_A^{-1} = \alpha_B^{-1} \circ Gf$, the naturality of α^{-1}.
   The same does not hold for monomorphisms. Let $\mathbf{C}$ be the two-element poset $\{0 \leq 1\}$ and $\mathbf{D}$ the category

$$A \underset{y}{\overset{x}{\rightrightarrows}} B \xrightarrow{f} C$$

   such that $fx = fy$. Let F be the functor taking $0 \leq 1$ to $x : A \to B$ and G the functor taking it to $f : B \to C$. There is a natural transformation $\alpha : F \to G$ such that $\alpha_0 = x : A \to B$ and $\alpha_1 = f : B \to C$. The component f of α is not mono, but α itself is; there are no nontrivial natural transformations into F: any $\beta : H \to F$ would have to satisfy a naturality square

$$\begin{array}{ccc} H0 & \xrightarrow{H\leq} & H1 \\ {\scriptstyle \beta_0}\downarrow & & \downarrow{\scriptstyle \beta_1} \\ A & \xrightarrow[x]{} & B \end{array}$$

But $H0$ must be A and $\beta_0 = 1_A$. Then $H1$ must be either A or B, forcing β to either be the unique natural transformation to F from the functor taking $0 \leq 1$ to $1_A : A \to A$, or else the identity natural transformation on F.

8. Put $(F \times G)(C) = FC \times GC$, and $(F \times G)(f) = Ff \times Gf$. Define $(\pi_1)_C = \pi_1^{FC\times GC} : FC \times GC \to FC$ and $(\pi_2)_C = \pi_2^{FC\times GC} : FC \times GC \to GC$. It is easy to check that π_1, and π_2 are natural. Let a functor $Z : \mathbf{C} \to \mathbf{D}$ and natural transformations $\alpha : Z \to F$ and $\beta \to F$ be given. By the UMP of the product, there are unique arrows $h_C : ZC \to FC \times GC$ such that $(\pi_1)_C \circ h_C = \alpha_C$ and $(\pi_2)_C \circ h_C = \beta_C$. We need to verify that

$$\begin{array}{ccc} ZC & \xrightarrow{h_C} & FC \times GC \\ {\scriptstyle Zf}\downarrow & & \downarrow{\scriptstyle Ff \times Gf} \\ ZD & \xrightarrow[h_D]{} & FD \times GD \end{array}$$

But
$$\begin{aligned} \pi_1^{FD\times GD} \circ Ff \times Gf \circ h_C &= Ff \circ \pi_1^{FC\times GC} \circ h_C \\ &= Ff \circ \alpha_C = \alpha_D \circ Zf = \pi_1^{FD\times GD} \circ h_D \circ Zf. \end{aligned}$$

And similarly with the second projection, using the naturality of β.

10. To satisfy the bifunctor lemma, we need to show that for any $f : C \to C' \in \mathbf{C}^{\mathrm{op}}$ and $g : D \to D' \in \mathbf{C}$ the following commutes:

$$\begin{array}{ccc} \hom(C, D) & \xrightarrow{\hom(f, D)} & \hom(C', D) \\ {\scriptstyle \hom(C,g)}\downarrow & & \downarrow{\scriptstyle \hom(C',g)} \\ \hom(C, D') & \xrightarrow[\hom(f, D')]{} & \hom(C', D') \end{array}$$

But either path around the square takes an arrow $h : C \to D$ and turns it into $g \circ h \circ f : C' \to D'$; thus the associativity of composition implies that the square commutes.

12. If $\mathbf{C} \simeq \mathbf{D}$, then there are functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ and natural isomorphisms $\alpha : 1_{\mathbf{D}} \to FG$ and $\beta : GF \to 1_{\mathbf{C}}$. Suppose $\mathbf{C}$ has products, and let $D, D' \in \mathbf{D}$ be given. We claim that $F(GD \times GD')$ is a product object of D and D', with projections $\alpha_D^{-1} \circ F\pi_1^{GD \times GD'}$ and $\alpha_{D'}^{-1} \circ F\pi_2^{GD \times GD'}$. For suppose we have an object Z and arrows $a : Z \to D$ and $a' : Z \to D'$ in $\mathbf{D}$. There is a unique $h : GZ \to GD \times GD' \in \mathbf{C}$ such that $\pi_1^{GD \times GD'} \circ h = Ga$ and $\pi_2^{GD \times GD'} \circ h = Ga'$. Then the mediating map in $\mathbf{D}$ is $Fh \circ \alpha_Z$. We can calculate

$$\begin{aligned}\alpha_D^{-1} \circ F\pi_1^{GD \times GD'} \circ Fh \circ \alpha_Z &= \alpha_D^{-1} \circ F(\pi_1^{GD \times GD'} h) \circ \alpha_Z \\ &= \alpha_D^{-1} \circ FGa \circ \alpha_Z \\ &= \alpha_D^{-1} \circ \alpha_D \circ a \\ &= a\end{aligned}$$

and similarly for the second projection.
Uniqueness of the map $Fh \circ \alpha_Z$ follows from that of h.

16. Let $\mathbf{C}$ be given. Choose one object $D_{[C]_{\cong}}$ from each isomorphism class $[C]_{\cong}$ of objects in $\mathbf{C}$ and call the resulting full subcategory $\mathbf{D}$. For every object C of $\mathbf{C}$ choose an isomorphism $i_C : C \to D_{[C]_{\cong}}$. Then, $\mathbf{C}$ is equivalent to $\mathbf{D}$ via the inclusion functor $I : \mathbf{D} \to \mathbf{C}$ and the functor F defined by $FC = D_{[C]_{\cong}}$ and $F(f : A \to B) = i_B \circ f \circ i_A^{-1}$ (F is a functor because the i_Cs are isomorphisms) and i construed as a natural isomorphism $1_{\mathbf{D}} \to FI$ and $1_{\mathbf{C}} \to IF$. Naturality is easy to check:

$$\begin{array}{ccc} A & \xrightarrow{i_A} & IFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_B f i_A^{-1}} \\ B & \xrightarrow[i_B]{} & IFB \end{array}$$

So $\mathbf{C}$ is equivalent to the skeletal category $\mathbf{D}$.

Chapter 8

1. Let $f : C \rightleftarrows C' : g$ be an iso. Then, clearly, $Ff : FC \rightleftarrows FC' : Fg$ is also one. Conversely, if $p : FC \rightleftarrows FC' : q$ is an iso, then since F is full there are $f : C \rightleftarrows C' : g$ with $Ff = p$ and $Fg = q$. Then $g \circ f = 1_C$ since $F(g \circ f) = Fg \circ Ff = 1_{FC} = F(1_C)$, and F is faithful. Similarly, $f \circ g = 1_{C'}$.

2. Given two natural transformations $\varphi, \psi : P \to Q$, where $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, assume that for each $C \in \mathbf{C}$ and $\theta : yC \to P$, we have $\varphi \circ \theta = \psi \circ \theta$. In other words,

$$\varphi_{C}* = \psi_{C}* : \hom(yC, P) \to \hom(yC, Q).$$

The Yoneda Lemma gives us a bijection $\hom(yC, P) \cong PC$ for each C, and these bijections are natural in P, so the following diagram commutes:

$$\begin{array}{ccc} \hom(yC,P) & \xrightarrow{\cong} & PC \\ \Big\downarrow \varphi_C* = \psi_C* & & \varphi_C \Big\downarrow \Big\downarrow \psi_C \\ \hom(yC,Q) & \xrightarrow[\cong]{} & QC \end{array}$$

But then both φ_C and ψ_C must be given by the single composition through the left side of the square, so that $\varphi = \psi$.

3. The following isos are natural in Z:

$$\begin{aligned} \hom_{\mathbf{C}}(Z, A^B \times A^C) &\cong \hom_{\mathbf{C}}(Z, A^B) \times \hom_{\mathbf{C}}(Z, A^C) \\ &\cong \hom_{\mathbf{C}}(Z \times B, A) \times \hom_{\mathbf{C}}(Z \times C, A) \\ &\cong \hom_{\mathbf{C}}((Z \times B) + (Z \times C), A) \\ &\cong \hom_{\mathbf{C}}(Z \times (B + C), A) \\ &\cong \hom_{\mathbf{C}}(Z, A^{B+C}). \end{aligned}$$

Hence $A^B \times A^C \cong A^{B+C}$, since the Yoneda embedding is full and faithful. The case of $(A \times B)^C \cong A^C \times B^C$ is similar.

6. Limits in functor categories $\mathbf{D}^{\mathbf{C}}$ can be computed "pointwise": given $F : \mathbf{J} \to \mathbf{D}^{\mathbf{C}}$ set

$$(\varprojlim_{j \in \mathbf{J}} Fj)(C) = \varprojlim_{j \in \mathbf{J}} (Fj(C)).$$

Thus, it suffices to have limits in $\mathbf{D}$ in order to have limits in $\mathbf{D}^{\mathbf{C}}$. Colimits in $\mathbf{D}^{\mathbf{C}}$ are limits in $(\mathbf{D}^{\mathbf{C}})^{\mathrm{op}} = (\mathbf{D}^{\mathrm{op}})^{\mathbf{C}^{\mathrm{op}}}$.

7. The following are natural in C:

$$\begin{aligned} y(A \times B)(C) &\cong \hom(C, A \times B) \\ &\cong \hom(C, A) \times \hom(C, B) \\ &\cong y(A)(C) \times y(B)(C) \\ &\cong (y(A) \times y(B))(C), \end{aligned}$$

so $y(A \times B) \cong y(A) \times y(B)$. For exponentials, take any A, B, C and compute:

$$\begin{aligned} y(B)^{y(A)}(C) &\cong \hom(yC, yB^{yA}) \\ &\cong \hom(yC \times yA, yB) \\ &\cong \hom(y(C \times A), yB) \\ &\cong \hom(C \times A, B) \\ &\cong \hom(C, B^A) \\ &\cong y(B^A)(C). \end{aligned}$$

11. (a) For any poset $\mathbf{P}$, the subobject classifier Ω in $\mathbf{Sets}^{\mathbf{P}}$ is the functor:

$$\Omega(p) = \{F \subseteq \mathbf{P} \mid (x \in F \Rightarrow p \leq x) \wedge (x \in F \wedge x \leq y \Rightarrow y \in F)\},$$

that is, $\Omega(p)$ is the set of all upper sets above p. The action of Ω on $p \leq q$ is by "restriction": $F \mapsto F|_q = \{x \in F \mid q \leq x\}$. The point $t : 1 \to \Omega$ is given by selecting the maximal upper set above p,

$$t_p(*) = \{x \mid p \leq x\}.$$

In $\mathbf{Sets}^{\mathbf{2}}$, the subobject classifier is therefore the functor $\Omega : \mathbf{2} \to \mathbf{Sets}$ defined by

$$\begin{aligned} \Omega(0) &= \{\{0, 1\}, \{1\}\} \\ \Omega(1) &= \{\{1\}\}, \end{aligned}$$

together with the natural transformation $t : 1 \to \Omega$ with

$$\begin{aligned} t_0(*) &= \{0, 1\} \\ t_1(*) &= \{1\}. \end{aligned}$$

In $\mathbf{Sets}^{\omega}$, the subobject classifier is the functor $\Omega : \omega \to \mathbf{Sets}$ defined by

$$\begin{aligned} \Omega(0) &= \{\{0, 1, 2, \dots\}, \{1, 2, 3, \dots\}, \{2, 3, 4, \dots\}, \dots\} \\ \Omega(1) &= \{\{1, 2, 3, \dots\}, \{2, 3, 4, \dots\}, \{3, 4, 5, \dots\}, \dots\} \\ \vdots &= \vdots \\ \Omega(n) &= \{\{n, n+1, n+2, \dots\}, \{n+1, n+2, n+3, \dots\}, \dots\}, \end{aligned}$$

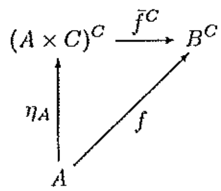
with the transition maps $\Omega(n) \to \Omega(n+1)$ defined by taking $\{n, n+1, n+2, \dots\}$ to $\{n+1, n+2, n+3, \dots\}$ and like sets to themselves, together with the natural transformation $t : 1 \to \Omega$ with

$$\begin{aligned} t_0(*) &= \{0, 1, 2, \dots\} \\ t_1(*) &= \{1, 2, 3, \dots\} \\ t_n(*) &= \{n, n+1, n+2, \dots\}. \end{aligned}$$

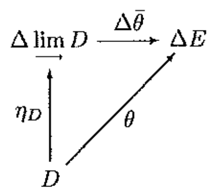
(b) One can check directly that all of the topos operations—pullbacks, exponentials, subobject classifier—construct only finite set-valued functors when applied to finite set-valued functors.

Chapter 9

3. η_A takes an element $a \in A$ and returns the function $(c \mapsto \langle a, c\rangle) \in (A \times C)^C$.



4. For any small index category $\mathbf{J}$, the left adjoint of $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ is the functor taking a diagram in $\mathbf{C}^{\mathbf{J}}$ to its colimit (if it exists), and the right adjoint to its limit. Indeed, suppose $D : \mathbf{J} \to \mathbf{C}$ is a functor.



Define the natural transformation η_D to take an object $J \in \mathbf{J}$ to the injection $i_J : DJ \to \varinjlim D$. The commutativity condition on the colimit guarantees that η_D is natural. Suppose E and $\theta : D \to \Delta E$ are given. That is, suppose θ is a co-cone from the diagram D to the object E. Then there exists a unique arrow out of $\bar{\theta} : \varinjlim D \to E$ making the above diagram commute. Therefore, $\varinjlim \dashv \Delta$. Dually, $\Delta \dashv \varprojlim$.
It follows that for $\mathbf{J} = \mathbf{2}$, the left adjoint is binary coproduct and the right adjoint is binary product.

5. Right adjoints preserve limits, and left adjoints preserve colimits.

8. The first adjunction is equivalent to the statement:

$$\mathrm{im}(f)(X) \subseteq Y \iff X \subseteq f^{-1}(Y),$$

for all $X \subseteq A, Y \subseteq B$. Here,

$$\mathrm{im}(f)(X) = \{b \mid b = f(x) \text{ for some } x \in X\}$$

$$f^{-1}(Y) = \{a \mid f(a) \in Y\}$$

If $\mathrm{im}(f)(X) \subseteq Y$ then for any $x \in X$, we have $f(x) \in Y$, and so $X \subseteq f^{-1}(Y)$. Conversely, take $b \in \mathrm{im}(f)(X)$, so there is some $x \in X$ with $f(x) = b$. If $X \subseteq f^{-1}(Y)$ then $b = f(x) \in Y$.

For the right adjoint, set

$$f_*(X) = \{b \mid f^{-1}(\{b\}) \subseteq X\}.$$

We need to show

$$f^{-1}(Y) \subseteq X \iff Y \subseteq f_*(X).$$

Suppose $f^{-1}(Y) \subseteq X$ and take any $y \in Y$, then $f^{-1}(\{y\}) \subseteq f^{-1}(Y) \subseteq X$. Conversely, given $Y \subseteq f_*(X)$, we have $f^{-1}(Y) \subseteq f^{-1}(f_*(X)) \subseteq X$, since $b \in f_*(X)$ implies $f^{-1}(\{b\}) \subseteq X$.

9. We show that $\mathcal{P} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ has itself, regarded as a functor $\mathcal{P}^{\mathrm{op}} : \mathbf{Sets} \to \mathbf{Sets}^{\mathrm{op}}$, as a (left) adjoint:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sets}}(A, \mathcal{P}(B)) &\cong \mathrm{Hom}_{\mathbf{Sets}}(A, 2^B) \cong \mathrm{Hom}_{\mathbf{Sets}}(B, 2^A)\\ &\cong \mathrm{Hom}_{\mathbf{Sets}}(B, \mathcal{P}(A)) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathrm{op}}}(\mathcal{P}^{\mathrm{op}}(A), B).\end{aligned}$$

10. A right adjoint to $U : \mathbf{C}/C \to \mathbf{C}$ is given by products with C,

$$A \mapsto (\pi_2 : A \times C \to C),$$

so U has a right adjoint iff every object A has such a product.
To have a left adjoint, U would have to preserve limits, and in particular the terminal object $1_C : C \to C$. But $U(1_C) = C$, so C would need to be terminal, in which case $\mathbf{C}/C \cong \mathbf{C}$.

11. (a) In a Heyting algebra, we have an operation $b \Rightarrow c$ such that

$$a \leq b \Rightarrow c \iff a \wedge b \leq c.$$

We define a coHeyting algebra by duality, as a bounded lattice with an operation a/b satisfying

$$a/b \leq c \iff a \leq b \vee c$$

In a Boolean algebra, we know that $b \Rightarrow c = \neg b \vee c$. By duality, we can set $a/b = a \vee \neg b$.

(b) In intuitionistic logic, we have two inference rules regarding negation:

$$\varphi \wedge \neg\varphi \vdash \bot$$

$$\varphi \vdash \neg\neg\varphi$$

We get inference rules for the conegation $\sim p = 1/p$ by duality

$$\top \vdash \varphi \vee \sim \varphi$$

$$\sim\sim \varphi \vdash \varphi$$

For the boundary $\partial p = p \wedge \sim p$, we have the inference rules derived from the rules for $\wedge$:

$$q \vdash \partial p \quad \text{iff} \quad q \vdash p \text{ and } q \vdash \sim p$$

(c) We seek a biHeyting algebra P which is not Boolean. The underlying lattice of P will be the three-element set $\{0, p, 1\}$, ordered $0 \leq p \leq 1$. Now let

$$x \Rightarrow y = \begin{cases} 1 & x \leq y \\ y & \text{o.w.} \end{cases}$$

This is easily checked to satisfy the required condition for $x \Rightarrow y$, thus P is a Heyting algebra. But since P is self-dual, it is also a coHeyting algebra, and co-implication must be given by

$$x/y = \begin{cases} 0 & x \geq y \\ y & \text{o.w.} \end{cases}$$

To see that P is not Boolean, observe that $\neg x = x \Rightarrow 0 = 0$, so $\neg\neg x = 1 \neq x$.
Note that P is the lattice of lower sets in the poset **2**. In general, such a lattice is always a Heyting algebra, since it is completely distributive, as is easily seen. It follows that such a lattice is also coHeyting, since its opposite is isomorphic to the lower sets in the opposite of the poset.

19. The right adjoint $\mathbf{Rel} \to \mathbf{Sets}$ is the powerset functor, $A \mapsto \mathcal{P}(A)$, with action on a relation $R \subseteq A \times B$ given by

$$\begin{aligned} \mathcal{P}(R) : \mathcal{P}(A) &\to \mathcal{P}(B) \\ X &\mapsto \{b \mid xRb \text{ for some } x \in X\}. \end{aligned}$$

The unit $\eta_A : A \to \mathcal{P}(A)$ is the singleton mapping $a \mapsto \{a\}$, and the counit is the (converse) membership relation $\ni_A \subseteq \mathcal{P}(A) \times A$.

Chapter 10

2. Let $\mathbf{C}$ be a category with terminal object 1 and binary coproducts, and define $T : \mathbf{C} \to \mathbf{C}$ by $TC = 1 + C$. Let $\mathbb{T}$ be the equational theory of a set equipped with a unary operation and a distinguished constant (no equations). We want to show that the following categories are equivalent:

T-algebras	Objects :	$(A \in \mathbf{C}, a : 1 + A \to A)$
	Arrows :	$h : (A, a) \to (B, b)$ s.t. $h \circ a = b \circ T(h)$
$\mathbb{T}$-algebras	Objects :	$(X \in \mathbf{Sets}, c_X \in X, s_X : X \to X)$
	Arrows :	$f : X \to Y$ s.t $fc_X = c_Y$ and $f \circ s_X = s_Y \circ f$

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ital.

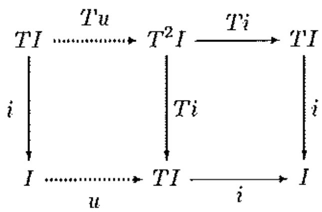
We have the functor $F : T\text{-Alg} \to \mathbb{T}\text{-Alg}$ sending

$$(A, a) \mapsto (A, a_1 : 1 \to A, a_2 : A \to A)$$

where $a = [a_1, a_2]$ as a map from the coproduct $1 + A$.
Conversely, given $(X, c \in X, s : X \to X)$, we can set $f = [c, s] : 1 + X \to X$ to get a T-algebra. The effect on morphisms is easily seen, as is the fact that these are pseudo-inverse functors.

Since free $\mathbb{T}$-algebras exist in **Sets** and such existence is preserved by equivalence functors, it follows that **Sets** has free T-algebras. In particular, an initial T-algebra in **Sets** is the initial $\mathbb{T}$-algebra $\mathbb{N}$, which is an NNO.

3. Let $i : TI \to I$ be an initial T-algebra. By initiality, we can (uniquely) fill in the dotted arrows of the following diagram:



Composing the squares, we have a map of T-algebras $I \to I$, which by uniqueness must be the identity. But then $i \circ u = 1_I$, and $u \circ i = Ti \circ Tu = T(i \circ u) = 1_{TI}$, so i is an isomorphism. A natural numbers object N is initial for the endofunctor $TC = 1 + C$, so it follows that $N \cong 1 + N$ for any NNO.

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