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Category Theory

Second Edition

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Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Published in the United States
by Oxford University Press Inc., New York

First published 2010

British Library Cataloguing in Publication Data
Data available

Library of Congress Cataloging in Publication Data
Data available

Typeset by SPi Technologies, Chennai, India
Printed in Great Britain
on acid-free paper by
MPG Biddles Ltd., King's Lynn, Norfolk

ISBN 978–0–19–958736–0 (Hbk.)
ISBN 978–0–19–923718–0 (Pbk.)

1 3 5 7 9 10 8 6 4 2

in memoriam
Saunders Mac Lane

PREFACE TO THE SECOND EDITION

This second edition of *Category Theory* differs from the first in two respects: firstly, numerous corrections and revisions have been made to the text, including correcting typographical errors, revising details in exposition and proofs, providing additional diagrams, and finally adding an entirely new section on monoidal categories. Secondly, dozens of new exercises were added to make the book more useful as a course text and for self-study. To the same end, solutions to selected exercises have also been provided; for these, I am grateful to Spencer Breiner and Jason Reed.

Steve Awodey
Pittsburgh
September 2009

PREFACE

Why write a new textbook on Category Theory, when we already have Mac Lane's *Categories for the Working Mathematician*? Simply put, because Mac Lane's book is for the working (and aspiring) mathematician. What is needed now, after 30 years of spreading into various other disciplines and places in the curriculum, is a book for everyone else.

This book has grown from my courses on Category Theory at Carnegie Mellon University over the last 10 years. In that time, I have given numerous lecture courses and advanced seminars to undergraduate and graduate students in Computer Science, Mathematics, and Logic. The lecture course based on the material in this book consists of two, 90-minute lectures a week for 15 weeks. The germ of these lectures was my own graduate student notes from a course on Category Theory given by Mac Lane at the University of Chicago. In teaching my own course, I soon discovered that the mixed group of students at Carnegie Mellon had very different needs than the Mathematics graduate students at Chicago and my search for a suitable textbook to meet these needs revealed a serious gap in the literature. My lecture notes evolved over a time to fill this gap, supplementing and eventually replacing the various texts I tried using.

The students in my courses often have little background in Mathematics beyond a course in Discrete Math and some Calculus or Linear Algebra or a course or two in Logic. Nonetheless, eventually, as researchers in Computer Science or Logic, many will need to be familiar with the basic notions of Category Theory, without the benefit of much further mathematical training. The Mathematics undergraduates are in a similar boat: mathematically talented, motivated to learn the subject by its evident relevance to their further studies, yet unable to follow Mac Lane because they still lack the mathematical prerequisites. Most of my students do not know what a free group is (yet), and so they are not illuminated to learn that it is an example of an adjoint.

This, then, is intended as a text and reference book on Category Theory, not only for students of Mathematics, but also for researchers and students in Computer Science, Logic, Linguistics, Cognitive Science, Philosophy, and any of the other fields that now make use of it. The challenge for me was to make the basic definitions, theorems, and proof techniques understandable to this readership, and thus without presuming familiarity with the main (or at least original) applications in algebra and topology. It will not do, however, to develop the subject in a vacuum, simply skipping the examples and applications. Material at this level of abstraction is simply incomprehensible without the applications and examples that bring it to life.

Faced with this dilemma, I have adopted the strategy of developing a few basic examples from scratch and in detail—namely posets and monoids—and then carrying them along and using them throughout the book. This has several didactic advantages worth mentioning: both posets and monoids are themselves special kinds of categories, which in a certain sense represent the two "dimensions" (objects and arrows) that a general category has. Many phenomena occurring in categories can best be understood as generalizations from posets or monoids. On the other hand, the categories of posets (and monotone maps) and monoids (and homomorphisms) provide two further, quite different examples of categories in which to consider various concepts. The notion of a limit, for instance, can be considered both in a given poset and in the category of posets.

Of course, many other examples besides posets and monoids are treated as well. For example, the chapter on groups and categories develops the first steps of Group Theory up to kernels, quotient groups, and the homomorphism theorem, as an example of equalizers and coequalizers. Here, and occasionally elsewhere (e.g., in connection with Stone duality), I have included a bit more Mathematics than is strictly necessary to illustrate the concepts at hand. My thinking is that this may be the closest some students will ever get to a higher Mathematics course, so they should benefit from the labor of learning Category Theory by reaping some of the nearby fruits.

Although the mathematical prerequisites are substantially lighter than for Mac Lane, the standard of rigor has (I hope) not been compromised. Full proofs of all important propositions and theorems are given, and only occasional routine lemmas are left as exercises (and these are then usually listed as such at the end of the chapter). The selection of material was easy. There is a standard core that must be included: categories, functors, natural transformations, equivalence, limits and colimits, functor categories, representables, Yoneda's lemma, adjoints, and monads. That nearly fills a course. The only "optional" topic included here is cartesian closed categories and the calculus, which is a must for computer scientists, logicians, and linguists. Several other obvious further topics were purposely not included: two-categories, topoi (in any depth), and monoidal categories. These topics are treated in Mac Lane, which the student should be able to read after having completed the course.

Finally, I take this opportunity to thank Wilfried Sieg for his exceptional support of this project; Peter Johnstone and Dana Scott for helpful suggestions and support; André Carus for advice and encouragement; Bill Lawvere for many very useful comments on the text; and the many students in my courses who have suggested improvements to the text, clarified the content with their questions, tested all of the exercises, and caught countless errors and typos. For the latter, I also thank the many readers who took the trouble to collect and send helpful corrections, particularly Brighten Godfrey, Peter Gumm, Bob Lubarsky, and Dave Perkinson. Andrej Bauer and Kohei Kishida are to be thanked for providing Figures 9.1 and 8.6, respectively. Of course, Paul Taylor's macros for

commutative diagrams must also be acknowledged. And my dear Karin deserves thanks for too many things to mention. Finally, I wish to record here my debt of gratitude to my mentor Saunders Mac Lane, not only for teaching me Category Theory, and trying to teach me how to write, but also for helping me to find my place in Mathematics. I dedicate this book to his memory.

Steve Awodey
Pittsburgh
September 2005

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