Bernays Project: Text No. 29

Preface (1974)

Vorwort

(*Abhandlungen*, pp. vii–x)

Translation by: *Wilfried Sieg*
Revised by: *CMU*
Final revision by: *Charles Parsons, Wilfried Sieg*

The *Wissenschaftliche Buchgesellschaft* has kindly offered to publish a collection of my essays on the philosophy of mathematics, which have appeared in various journals. I accept this offer gladly, in particular since several of these essays are not easily accessible.

The present volume can also serve as a temporary substitute for a comprehensive treatment of the philosophy of mathematics.[1] This is possible because, during the period in which these articles were published, my views on the relevant questions have changed almost exclusively in response to new insights gained from research in the foundations of mathematics.

The collection of these various essays should provide the reader with an adequate characterization of my views on mathematics.

Especially with regard to what has been called the "foundational crisis" of mathematics it will become clear, I hope, that according to my view we cannot justifiably speak of a crisis, at least not in the sense that classical mathematics has been shown to be questionable. Problematic aspects have, of course, presented themselves in various respects.

First of all, we have become conscious of the fact that the idea of the obviousness of mathematics is not justified unless we consider as obvious simply what has become familiar to us through use and practice. Even ideas that are not really trivial can become familiar to us in this sense, and in their

[1] A monograph on this topic, to be published by Duncker & Humblot, has long been planned (but is still not done).

use we can acquire practical certainty. The very idea of an absolute certainty is presumably illusory for human reason in any case.

Going beyond the trivial is involved especially in all those idealizations which are characteristic for mathematical concept formation. It has become clear that even the general concept of natural number and the related notion of the number series are based on an idealization.

Already here, we also meet with an opposition that calls for a restriction of methods of proof. For instance, the restricted methodology of Brouwer's "intuitionism" as well as that of the "finitist" standpoint—as Hilbert called it—avoid the inference according to which a numerical predicate either applies to all numbers or else there exists at least one number to which it does not apply. This kind of application of the *tertium non datur* is avoided here, all the more, for predicates of sets and of functions. However, even if only such a restricted methodology is accepted, the recursive generation of functions leads beyond what is concretely, computationally feasible.

Avoidance of the above-mentioned application of the *tertium non datur* has, incidentally, no essential impact on elementary number theory. For analysis, however, it amounts to a considerable restriction.

As far as classical analysis is concerned, it was initially believed that its foundation according to the methods of Dedekind and Cantor provided a complete arithmetization. However, viewed from the standpoint of the requirement of a strict arithmetization, classical analysis was soon criticized. And this critique grew under the influence of the demands of finitist and intuitionistic methodology. Various programs for a more strictly arithmetical treatment of analysis have since been developed within research in the foundations of mathematics.

It would be unjust not to recognize that these different versions of a more strongly arithmetized analysis are of definite mathematical interest. Yet, it should also be admitted that it is a prejudice to believe that it is absolutely necessary to arithmetize analysis completely. In analysis, geometrical ideas are made conceptually precise. The methods of Dedekind and Cantor, referred to above, succeed in connecting analysis to number theory, but not without the addition of *set-theoretical* concepts. If one understands clearly that these concepts are not completely arithmetical, then the procedure involves nothing objectionable.

To be sure, the methods of classical analysis contain strong idealizations. But these do not detract from practical certainty. Indeed, a kind of intuitiveness is gained here that confers great certainty on our considerations.—

Problems have also arisen in connection with the discovery of the formalizability of mathematical proofs by means of symbolic logic. This formalization can certainly be viewed as a sharpening of the axiomatic method. Indeed, formal systems have been successfully set up for number theory, analysis, and set theory. These formal systems consist of a symbolism and rules of deduction, and are set up in such a way that, within the framework of such a system, the known proofs of the respective theory can be formally represented.

Formal-deductive systems can also be set up independently of already existing theories, and then we have the reverse situation, namely that we can try to find contentual interpretations (models) for them. That is a topic for "semantics" which is generally concerned with the relations between theories and formal systems.

A different kind of research tied to the formalization of mathematical theories takes formalized proofs as the object of mathematical investigation. This is the aim of Hilbert's proof theory. It is above all a matter of investigating the internal consistency of formalized theories. For this purpose there arises the possibility of a methodological reduction in the case of the theories of the infinite, because formalized proofs are, after all, finite objects and because consistency can be formally characterized. Consistency proofs of this kind have actually been given successfully for formalized number theory and formalized analysis, but they do not use means as elementary as Hilbert had sought. He wanted to restrict the methods of such proofs to combinatorial ones in accord with the finitist standpoint. Stronger methods of giving constructive proofs had to be used.

This necessity of going beyond the elementary "finitist" methods in consistency proofs is related to another difficulty. Both came to light on the basis of results obtained by Kurt Gödel and Thoralf Skolem: It was shown that a formal system, if it is to satisfy the conditions of strict controllability, could not completely express its intended theory. This is particularly shown by the fact that the formal system, apart from its normal interpretation by the intended theory, also permits deviant interpretations, the so-called "non-standard" models.

Within foundational research this fact has been dealt with in different ways, either by studying non-standard models more closely or by considering possibilities of excluding non-standard models by an extended kind of formalization. For number theory two ways of extending the procedure of formalization have been considered: on the one hand "infinite induction" and on the other hand the admission of infinite conjunctions and disjunctions. In either case, the finitist character of proof figures is lost.

As regards fundamental reflections, it emerges that the role of formalization is not so simple as was originally intended and, at the same time, that we do not have to demand formalization so unconditionally. In any case, semantics, in keeping with its purpose, uses set-theoretic thinking that is not bound to a formal system.—

As we see, there is no dearth of problems for the philosophy of mathematics. Nevertheless, what I said in one of my essays[2] still holds: "If we adopt the conception . . . according to which mathematics is the science of idealized structures, then we have a position for foundational research in mathematics that saves us from exaggerated perplexities and forced constructions, and one that also will not be undermined if foundational research turns up various new surprises."

[2] "Schematic correspondence and idealized structures" (*vide* ch. **??**, p. **??**).