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Remarks on the philosophy of mathematics (1969)

Bemerkungen zur Philosophie der Mathematik

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When we compare mathematics with logic in regard to the role assigned to these two domains of knowledge within philosophical thinking, we find a disagreement among philosophers.

For some logic is singled out; for them, logic in the broader sense is the $\lambda\acute{o}\gamma o\varsigma$, that what is rational, and logic in the narrower sense is the inventory of elementary insights which should lie beneath all considerations, i. e., the inventory of those truths that hold independently of any particular factual content. Thus, logic in the narrower sense ("pure logic") has a primary epistemological status.

A different starting point takes the method of mathematics as exemplary for all scientific thinking. While for the first point of view the logical is what is the obvious and unproblematic, for the second point of view the mathematical is what is epistemologically unproblematic. Accordingly, understanding is ultimately mathematical understanding. The idea that all rational insight must be of a mathematical kind also plays a fundamental role particularly in the arguments of David Hume.

For this point of view Euclid's Elements counted for a long time as a paradigm of the mathematical method. But often it was not sufficiently clear that from the standpoint of axiomatics the Euclidean axiom system is special (the fact that early commentators had already come up with suggestions for replacing axioms by equivalent ones was an indication of this). Obviously many were of the opinion—although probably not the authors of the Greek work—that the possibility of a strict and successful proof in geometry is based on the evidence of the axioms.

Those who philosophized according to the axiomatic method, in particular in the school of Christian Wolff, at times understood evidence as conceptual evidence, so that they did not distinguish between the logical and the mathematical. The principle of contradiction (which mostly included the principle of excluded middle) was regarded as a magic wand so to speak, from which all scientific and metaphysical knowledge could be obtained with the help of suitable concept formations.

As you know, Kant has emphasized in opposition to this philosophy the moment of the intuitive in mathematics in his theory of pure intuition. But also for Kant the possibility of geometry as a successful deductive science is based on the evidence of the axioms, that is in his case, on the intuitive clarity and certainty of the postulates of existence. That the discovery of non-Euclidean, Boyai-Lobatschefskian geometry had such a revolutionary effect on philosophical doctrines is explained by the unclarity in epistemological judgment about Euclid's geometry.

But a fundamental change of aspect resulted also for the first of the two mentioned points of view from the development of mathematical logic. It became clear that logic as a discipline (which it was already with Aristotle) does not consist directly in establishing singular logical facts, but rather in investigating the possibilities of proofs in formally delimited domains of deduction, and should better be called metalogic. Furthermore, the method of such a metalogic is typically mathematical.

Thus it might seem appropriate to classify logic under mathematics. The fact that this has mostly not been done is explained by the lack of a satisfactory epistemological view of mathematics. The term "mathematical" was not, so to speak, a sufficiently familiar philosophical term. One tried to understand mathematics itself by classifying it under logic. This is particularly true of Gottlob Frege. You surely know Frege's definition of cardinal number in the framework of his theory of predicates. The method employed here is still important today for the reduction of number theory to set theory. Various objections can be raised, however, against the view that an *epistemological* reduction to pure logic has hereby been achieved (as might be discussed in a group of those interested) .

A different way of approaching the question of the relation between mathematics and logic is to regard both as being analytic—as is done in particular by R. Carnap. Thereby the Kantian concept of analyticity is fundamentally extended, which has been pointed out especially by E. W. Beth. For the most part the same character of obviousness that is ascribed to analytic sentences in the Kantian sense is attributed to analyticity in this extended sense.

As you know, W. V. Quine has fundamentally opposed the distinction between the analytic and the synthetic. Although his arguments contain much that is correct, they do not do justice to the circumstance that by the distinction between the analytic in the wide sense and the synthetic, a fundamental distinction is hit upon, namely, the distinction between mathematical facts and facts about natural reality. Just to mention something in this regard: mathematical statements are justified in a different sense from statements in physics. The mathematical magnitudes of analysis are relevant for physics only approximately. For example, the question whether the speed of light is measured in the centimeter-second-system by a rational or an irrational number has hardly any physical sense.

To be sure, the fundamental difference between what is mathematical and what is natural reality is not a sufficient reason to equate mathematics with logic. It appears natural to count as logic only what results from the general conditions and forms of discourse (concept and judgment). But mathematics is about possible structures, in particular about idealized structures.

Herewith, on the one hand the methodological importance of logic becomes apparent, but on the other hand also that its role is in some sense anthropomorphic. This does not hold in the same way for mathematics, where we are prompted to transcend the domain of what is surveyable in intuition in various directions. The importance of mathematics for science results already from the fact that we are concerned with structures in all areas of research (structures in society, structures in the economy, structure of the earth, structures of plants, of processes of life, etc.). The methodological importance of mathematics is also due to the fact that a kind of idealization of objects is applied in most sciences, in particular the theoretical ones. In this sense F. Gonseth speaks of the schematic character of the scientific description. What differentiates the theoretically exact from the concrete is emphasized especially also by Stephan Körner. As you know, science has succeeded in understanding the connections in nature largely structurally, and the applicability of mathematics to the characterization and explanation of the processes in nature reaches much further than humanity had once anticipated.

But the success and scope of mathematics is something entirely different from its so-called obviousness. The concept of obviousness is philosophically questionable in general. We can speak of something being relatively obvious in the sense in which, for example, the mathematical facts are obvious for the physicist, the physical laws for the geologist, and the general psychological properties of man for the historian. It may be clearer to speak here of the procedurally prior (according to Gonseth's expression "préalable") instead of the obvious.

At all events mathematics is not obvious in the sense that it has no problems, or at least no fundamental problems. But consider for instance, that there was no clear methodology for analysis for a long time despite its great formal success, but the researchers had to rely more or less upon their instinct. Only in the 19th century were precise and clear methods achieved here. Considered from a philosophical point of view the theory of the continuum of Dedekind and Cantor, which brought the justifications of these methods to an end, is not at all simple. It is not a matter of the bringing to consciousness of an *a priori* cognition. One might rather say that here a compromise between the intuitive and the demands of precise concepts has been achieved which succeeded very well. You also know that not all mathematicians agree with this theory of the continuum and that the Brouwerian Intuitionism advocates a different description of the continuum—of which one can surely find that it overemphasizes the viewpoint of strict arithmetization at the expense of the geometrically satisfying.

The problems connected with the antinomies of set theory are especially well known and often discussed. As you know, different suggestions have been brought forward to repair the antinomies. In particular, axiomatic set theory should be mentioned, which shows that such a small restriction of the set theoretic procedure suffices to avoid the antinomies that all of Cantor's proofs can be maintained. Zermelo's original axiom system for set theory has been, as you surely know, on the one hand extended, on the other hand made formally sharper. The procedure of solving the antinomies using axiomatic set theory can be interpreted philosophically as meaning that the antinomies are taken as an indication that mathematics as a whole is not a mathematical object and therefore mathematics can only be understood as an open manifold.

The application to set theory of the methods of formally precision resulted in a split of the set theoretic considerations into the formulation and deductive development of formal systems, and a model theory. As a result of this split the semantic paradoxes, which could be disregarded at first for the resolution of the purely set theoretic paradoxes, received new formulation and importance. So today we face a new set of fundamental problems, which, to be sure, does not bother mathematics in its proper research, just as the set-theoretic antinomies did not earlier. Rather, mathematics unfolds in the different disciplines with great success.

The above remarks suggest the following viewpoints for philosophy of mathematics, which are also relevant for epistemology in general:

1. It appears appropriate to ascribe to mathematics factual content, which is different from that of natural reality. That other kinds of objectivity are possible than the objectivity of natural reality is already obvious from the objectivity in the phenomenological regions. As has been said before, mathematics is not phenomenological insofar as it is about idealized structures on the one hand, and on the other hand it is governed by the method of deduction. With idealization, intuitiveness comes into contact with conceptualization. (Therefore, it is not appropriate to oppose intuitiveness and conceptualization so heavily as it is done in Kantian philosophy).

The significance of mathematics for theoretical physics consists in the fact that the processes of nature are approximated therein by mathematical entities.

2. It does not follow from the difference between mathematics and empirical research that we have knowledge in mathematics that is secured at the outset (*a priori*). It seems necessary to concede that we also have to learn in the fields of mathematics and that we here, too, have an experience sui generis (we might call it "intellectual experience"). This does not diminish the rationality of mathematics. Rather, the assumption that rationality is necessarily connected with certainty appears to be a prejudice. We almost nowhere have certain knowledge in the simple, full sense. This is the old Socratic insight which is emphasized today especially also in the philosophies of F. Gonseth and K. Popper.

We certainly have to admit that in mathematical considerations, in particular in those of elementary mathematics, we possess a particular kind of security, because on the one hand the objects are intuitively clear and, on the other hand, almost everything that could lead to subjectivity is stripped away by the idealization of the objects. But when we talk about the certainty of $2 \cdot 2 = 4$ in the popular sense, we think of the concrete applications of this statement. But the application of arithmetical statements to the concrete is based on empirical conditions, and for their compliance we only have an empirical, even if practically sufficient, certainty.

By dropping the coupling of rationality and certainty we gain, among other things, the possibility to appreciate the *heuristic rationality*, which plays an essential role for scientific inquiry.

The acknowledgment of heuristic rationality provides in particular the solution to the epistemological difficulty that has been made a problem by David Hume: we can acknowledge the rational character of the assumption of necessary connections in nature, without having to claim that the basic approach of assuming such connections guarantees success; with regard to this success we in fact depend on experience.