Bernays Project: Text No. 17

Introduction
(1939)

Zur Einführung

(*Grundlagen der Mathematik*, Bd. 2, Berlin: Springer, pp. vi–ix)

Translation by: *Dirk Schlimm*
Revised by: *CMU*
Final revision by: *Bill Tait, CMU*

The aim of this book is to provide a thorough orientation to the current content of Hilbert's proof theory. Despite the fact that the achievements to date have been modest in comparison to the goals of the theory, there are still plenty of suggestive results, viewpoints, and proof ideas that seem worth reporting on.

The purpose of the book has resulted in two main themes for the contents of this second volume. On the one hand, to present in detail the principal proof-theoretic approaches of Hilbert that follow from the ϵ-symbol together with their implementations.

A substantial part of the investigations presented here have not yet been published at all, aside from some brief hints. Thus, not only is there interest in the subject matter, but there is also a scientific obligation on the part of the Hilbert-school to justify the various previous announcements of proofs by actually providing these proofs. This demand is all the more pressing in this case, since there was initially (until the year 1930) some error about the scope of the proofs by Ackermann and v. Neumann, which resulted from one of the approaches of Hilbert mentioned above.

These hitherto unpublished proofs are now presented in detail in §§ 1 and 2. In particular, the restriction which is here still imposed on the consistency proof for the number-theoretic formalism is made clearly apparent.

With the help of one of the methods presented here, a simple approach to a series of theorems also arises by which the proof-theoretic investigation of the predicate calculus is satisfactorily rounded off and which also allows for remarkable applications to axiomatics. A theorem of theoretical logic that was first formulated and proved by J. Herbrand, for which we obtain a natural and simple proof by the mentioned route, lies at the center of these considerations.

The discussion of the applications of this theorem also offers the opportunity for consideration of the decision problem. Following this, a proof-theoretic sharpening of Gödel's completeness theorem is proved in § 4.

The second main theme is represented by the considerations leading to the necessity of extending the limits of the contentual forms of inference allowed in proof-theory, beyond the previous demarcation of the "finitistic standpoint." Gödel's discovery that every sharply delineated and sufficiently expressive formalism is deductively incomplete stands at the center of these considerations. Both of Gödel's theorems which express this fact are discussed in detail with respect to their relation to the semantic paradoxes, to the conditions for their validity, and the implementation of their proofs—Gödel only hints at the proof for the second theorem—and to their applicability to the full number-theoretic formalism.

The discussion regarding the extension of the finitistic standpoint is followed by consideration of Gentzen's recent consistency proof for the number-theoretic formalism. Of course, only what is methodologically novel in this proof is presented in detail and discussed, namely the application of a particular version of Cantor's "transfinite induction."

The mainly external reason for not presenting the entire proof was that the newer, first really clear version of Gentzen's proof had not been published at the time of the printing of this volume. By the way, Gentzen's proof does not relate directly to the number-theoretic formalism discussed in the book. L. Kalmár recently succeeded in modifying this proof so that it becomes directly applicable to the number-theoretic formalism developed in our book (in § 8 of the first volume), whereby certain simplifications also arise.

W. Ackermann is currently extending his earlier consistency proof (presented in § 2 of the present volume) by applying the kind of transfinite induction that is used by Gentzen in order to make it valid for the full number-theoretic formalism.

If this succeeds—which seems quite likely—Hilbert's original approach would be rehabilitated with respect to its effectiveness. In any case, Gentzen's proof already justifies the view that the temporary crisis in proof theory was merely due to the fact that methodological requirements had been imposed on the theory which were too strong. To be sure, the final decision about the fate of proof theory will be based on the task of proving the consistency of analysis.

A few considerations that are distinct from to the train of thought developed in §§ 1–5 of the present volume are added as "supplements." Two of these complement the considerations in § 5: Supplement II is about making precise the notion of computable function (as has recently been done successfully by various methods) and presents the facts related to this circle of problems, and can be easily developed following the other considerations of the book. A. Church's theorem about the impossibility of a general solution to the decision problem for the predicate calculus is applied in this connection. In Supplement III some questions pertaining to deductive propositional logic are discussed, and some further remarks are made on the "positive logic" formulated in § 3 of the first volume.

Various deductive formalisms for analysis are set up in Supplement IV, and it is shown how the theory of the real numbers and also that of the numbers of the second number class are obtained from them.

Supplement I contains an overview of the rules of the predicate calculus and its application to formalized axiom systems, as well as remarks about possible modifications of the predicate calculus, and a compilation of various definitions and results from the first volume.

In view of the already enormous amount of material, various proof-theoretic themes could not be addressed in this book: in particular, the topic of multi-sorted predicate calculus, which was dealt with first in Herbrand's thesis[a] and recently in more detail by Arnold Schmidt (*vide* [**?**]).

Certain considerations that could be found in Hilbert's lectures and in discussions with Hilbert, but that only remained isolated remarks or that had not been sufficiently clarified, are not presented: in particular, the approaches regarding the definitions of numbers of the second number class by common (i. e., not transfinite) recursion, and those concerning the use of type symbols, in particular those that are introduced by explicit or recursive definitions.

The present volume follows the first volume closely; this connection is also strengthened by frequent references to page numbers. On the other hand, the compilation of terms and theorems from the first volume given in Supplement I and the recapitulation in part b), section 1 is intended to render the reading of the second volume largely independent of the first

[a] *Vide* [**?**].

one. The reader who is already somewhat familiar with logical formalization and with the questions addressed by proof theory will be able to follow the considerations of the second volume without knowledge of the first one.

In any case, it is recommended that the reader of the present volume start with § 1 of Supplement I. Furthermore, he should make use of the page references only when he feels the need to do so in the particular passages.

In addition to the remarks given in § 2 about possible omissions in the course of study, it may also be remarked here that the rather tedious section 2 of § 4 can be omitted.

Regarding references to paragraphs, the numbers from 1 to 5 refer to the present second volume if not otherwise indicated, while the numbers from 6 to 8 occur only in the first volume.