Bernays Project: Text No. 21

Mathematics as both familiar and unknown (1954)

Die Mathematik als ein zugleich Vertrautes und Unbekanntes

(*Synthese* IX (1953–55), pp. 465–471;
repr. in *Abhandlungen*, pp. 107–112)

Translation by: *David Weberman*
Revised by: *Erich Reck*
Final revision by: *Charles Parsons*

When the mind feels weighed down or oppressed by the many mysteries of existence, by the impression of our extensive ignorance in so many areas, by the inadequacies of linguistic representation and communication, it often turns gladly to mathematics, where objects can be grasped clearly and precisely, and where gratifying insight can be attained through appropriate concepts. Here the human mind feels at home; here it experiences the triumph that the application and combination of quite elementary ideas—familiar to us from childhood play—yield significant, unexpected, and far-reaching results. Taking concrete matters as the starting-point, mathematical thinking is occupied in fixing its objects intuitively and imagining them; from there, by forming concepts and by mentally interweaving its findings, it goes on to results, which in turn can be applied to the concrete and show themselves impressively successful.

As a consequence, mathematical activity reveals its power and productivity in three ways. First of all, we have here a striking form of an original representation as a source for cognition, as well as for concept formation connected with it. Second, logical reasoning is here a powerful cognitive tool, indeed one that functions in a truly essential way only in this domain. But there is still a third respect: in mathematics we have not only the activity of intuition and logical reasoning, an activity which allows these powers residing in our inner nature to develop freely and productively; we also have the connection to familiar objects of everyday perception, and, beyond that, we have the remarkable confirmation which mathematics finds in the extended domain of experience where our ordinary perception no longer suffices for orientation.

These three kinds of success and satisfaction evinced by mathematics correspond roughly to the three aspects distinguished by Ferdinand Gonseth: to the intuitive, the theoretical, and the experimental aspect.

If we take a closer look at the development of mathematics and its applications, we do, of course, soon come to problematic features. If we begin with the application of mathematics to the explanation of nature, the historical development shows us a twofold disappointment in the following respect: mathematics was believed to yield a kind of familiarity with reality that it *de facto* does not provide.

This happened first in connection with the doctrine of the Pythagoreans, who discovered the reducibility of qualitative differences in perceptual objects to numerical relations as carried through in theoretical physics. The pursuit of this discovery gave rise to the hope that the concept of number might bring about an ultimate, penetrating understanding of, and thus, intellectual familiarity with, what is real. As is well known, this doctrine was fundamentally shaken by the discovery of irrational magnitudes. The Greeks soon learned to deal with irrational magnitudes in a correct deductive manner; but Eudoxus' procedure was quite abstract, and Euclidean geometry, which built on it, was in its axiomatic attitude much more restrained than the Pythagorean doctrine. It is here, too, that the purely mathematical was for the first time strictly separated from the natural sciences.

Hope for a mathematical understanding of reality arose for a second time at the beginning of the modern era. Under the influence of the powerful development of the theoretical natural sciences, and especially also of mathematics itself, that mechanistic view of nature emerged that captured many minds. Although this view of nature was paradoxical from the outset, Kantian philosophy then provided a mode, by opposing the real in itself to appearances, to carry through the mechanistic viewpoint for the domain of appearances and to view this domain as something governed by the manner of our intuitive representation. Thus nature, governed by our forms of intuition and structured mathematically, acquired the character of something familiar to us.

I need not speak about the fact, discussed so often and so much, that the contemporary development of theoretical physics has moved fundamentally away from this view. To be sure, in today's theoretical physics mathematical tools are used extensively and with great success. But we are no longer talking about a perspective of intuitive familiarity.

However, these difficulties concern the theoretical sciences, not mathematics itself. A brief survey of the development of mathematics presents us initially with the picture of an impressive triumphal march. It begins with the formal development of the infinitesimal calculus, which caused the so-called irrational in the theory of magnitudes to lose its character as an *apeiron*. The numerous beautiful and, in terms of laws, simple presentations of irrational magnitudes then moved them into the domain of the familiar. Yet initially the procedure of the infinitesimal calculus lacked sufficient methodological precision; this was achieved in the nineteenth century.

This time was also a period of massive expansion in mathematics, which deserves to be emphasized all the more since it has never come sufficiently to the consciousness of educated humanity. What developed was a freer mode of abstraction and a strengthened way of forming concepts. Consequently, new methods were developed, and a whole series of new mathematical disciplines emerged. In these disciplines the operation with mathematical concepts began to display great power, beauty, and an impressive richness of thought. A high level of rational understanding was reached here, and a new way of being intellectually familiar with enitities was gained.

Two important events in the history of ideas took place in connection with this development. The first was the discovery of non-Euclidean geometry. The second was the realization of the Leibnizian program in the domain of logic by establishing a logical calculus. This calculus might have appeared playful in its initial form, but it was later extended in such a way that it allowed for the formal representation of mathematical proofs.

While mathematics was reaching up to new forms and spheres of understanding, its character of familiarity was lost in some respects, especially since what was once the starting point and center lost this position. Not only did Euclidean geometry lose its privileged position, and thus its role as the evident theory of space, but the arithmetical theory of magnitudes, too, now seemed to be just the theory of one structure among others. The dominant point of view had become that of the general formal theory of structures. But this led to difficulties in two different ways: first, in terms of antinomies, which resulted from the fact that some totalities of possible structures, while presenting themselves as mathematical entities in a manner analogous to the number series, cannot be understood in that way on pain of contradiction; second, in terms of the strange aspect of Cantor's set theory that an immense progression of infinite cardinal numbers appeared that dwarfed both the infinite number series and the manifold of the mathematical continuum, indeed to a fundamentally greater extent than that in which the size of our earth is dwarfed by astronomical expansions. This led many to begin to doubt the justification and meaningfulness of the methods applied, and the call could be heard "Back to the concrete!" Concepts and modes of inference which had previously been recognized and used were no longer accepted. Various developments of new frameworks in mathematics were undertaken. A particular example of such a new framework is, of course, that of Brouwer's intuitionism. Hilbert, on the other hand, had the idea of connecting mathematics more strongly to concrete representation by utilizing the formalization of mathematical reasoning.

In his recent talk at the Brussels Congress, Mr. Heyting discussed the current state of research concerning the foundations of mathematics. In addressing the question of the object of mathematics, he found that it cannot be answered in a satisfying way for classical mathematics (i. e., the mathematics developed in the nineteenth century). The reason is, according to Heyting, that in classical mathematics intuitive and formal elements are combined without clear distinction. Then again, he believes that a more precise working out of these two elements, as given in Brouwer's intuitionism for the intuitive element and in Hilbert's proof theory for the formal element, is also not satisfactory for an exhaustive treatment of the epistemological problems at hand. This, he believes, indicates that the question of the object of mathematics is ill-phrased and has to be replaced by a more adequate formulation.

We can certainly agree with that conclusion. Indeed, it is easy to tie the question of the object of mathematics to a non-trivial presupposition, namely, that in scientific inquiry the object must be given to us prior to it. A study of the sciences shows, however, that an exact determination of the objects of theoretical disciplines generally grows only out of their conceptualization. We also do not need to view the combination of intuitive and formal elements in classical mathematics, noted by Mr. Heyting, as a defect. Indeed, often the role of important conceptual and methodical approaches lies exactly in the fact that they offer a kind of balance between intuitive and theoretical-formal intentions.

Such a balance is already present in the basic perspective of number theory, even in an elementary ("finitist") treatment of it. We should be clear here that even in finitist number theory we are no longer in the sphere of the genuinely concrete; large numbers cannot be exhibited in imagination or perception. From the standpoint of an approach which aims to remain in the genuinely concrete, it is then in particular not clear what a universal statement ranging over arbitrary numbers could mean. The attempt to interpret such a universal statement by appealing to the existence of a proof does not lead to its objective. Indeed, if a contentual proof, for instance in the intuitionistic sense, is intended, it consists of a certain procedure that must be exhibited. But then it must be clear that this procedure realizes the *desideratum* in each individual case; and such a claim is again a universal number-theoretic statement. If, on the other hand, a formal derivation within a deductive system is intended, then one must convince oneself that the deductive formalism functions appropriately; and this leads again to a claim that takes the form of a universal number-theoretic statement.

We can think about the intellectual steps which lead to the specifically number-theoretic point of view roughly in the following way: First we are conscious of the freedom we have to advance from one position arrived at in the process of counting to the next one. But then we take the step of a connection, through which a function that associates a successor with each and every number is posited. Hence a *progressus in infinitum* replaces the *progressus in indefinitum*. But it is not immediately obvious that this idea of the infinite number series can be realized; the intellectual experience of its successful realization is then essential for developing a feeling of familiarity, even of obviousness, as an acquired evidence.

The philosophy of mathematics generally tends not to appeal to such acquired evidence, but replaces it with a evidence *ab ovo*. Thus one is misled to make one of two mistakes: either to exaggerate the reach of this evidence by trying to include all possibly attainable levels, which leads to the antinomies; or to posit a particular level of evidence as absolute, which results in requiring a restriction of mathematics in such a way that we unnecessarily lose the freedom of making intellectual decisions.

We can avoid these unacceptable consequences if we give up the view that mathematics is something obvious. The element of familiarity that we find in mathematics, especially in elementary mathematics, is an acquired familiarity. To be sure, mathematics is above all a grasping, not something to be grasped. But the possibility of successfully extrapolating, by way of strict mathematical laws, intuitive relations of numbers and figures is basically as non-obvious as the possibility of discovering physical laws of nature. In this respect we must return to the wisdom of Socrates, that is, we must recognize our own ignorance. Kant's belief that the structure of our own cognition must be determinable *a priori* for us is clearly based on an illusion. The structure of our mental organization is as transcendent for our consciousness as is the character of external nature.

It is also hardly true that a mathematical element enters into our investigation of nature only through the manner of our intuitive representation. Yet we can clearly accept that the world of mathematics confronts us as a phenomenal realm. Consequently one can speak in relation to mathematics of a phenomenology of the mind, in a sense different from Hegel's. However, the phenomenal in this sense goes certainly beyond what we may assume to be innate in the individual, if for no other reason than that it is structurally open. And if one speaks of "mind," "reason," or "form of intuition" in the sense of something that goes beyond concrete psychic constitution, there is no longer a clear difference between what belongs to the subject and any element of the world order.

Philosophical speculation about mathematics leads, indeed, to such lofty regions. When we consider mathematics not from the standpoint of its immediate application, where it provides us with the experience of the familiar and evident, but want to pursue its roots philosophically instead, we must avoid too simplistic a conception of mathematics.