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Theses and remarks on philosophical questions and on the situation in logico-mathematical foundational research (1937)

Thesen und Bemerkungen zu den philosophischen Fragen und zur Situation der logisch-mathematischen Grundlagenforschung

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Abstract. – I. *Scientific philosophy* and *logical syntax*. Necessity of an interpretation. – II. *Logic* and *mathematics*. The Kantian "analytic"-"synthetic" distinction is replaced by a distinction between "formal" and "objective." Concerning mathematics and logic we focus especially on the objective side: in mathematics it consists in the existence of mathematical results independent of any formulation as a proposition and in the verifiability of the arithmetical laws; in logic it consists in the hidden relation between expressions and principles and certain traits of reality. – III. *Arithmetic* and *geometry* are distinguished with respect to considerations of what is discrete and what continuous. Formal precision of intuitive mathematical concepts. – IV. *On the problematic of the foundations*. Reflections and remarks concerning the current state of research.

1 Philosophy and syntax

1. Scientific philosophy consists of fundamental considerations of the organization resp. reorganization of the language of science and considerations concerning the possible fundamental interpretations and conceptions of scientific enterprises.

2. Syntax, as it is developed in Carnap's book *Logical Syntax of Language*[a] following Hilbert's meta-mathematics, the investigations by the Polish logicians, and those by Gödel on formalized languages, considers the mathematical properties of formalized languages of science.

3. If syntax is to contain assertions, it must take place in an interpreted language.

If a formal definition is to serve to make a philosophical concept formation precise, then either the formal definition has to be provided with an interpretation or this more precise rendering is achieved indirectly by demanding a syntactic property of the formal definition which itself has then to be determined in a way that can be interpreted.

4. That a formal language functions as a syntax-language using, for instance, Gödel's method of arithmetization, is based on the intuitive-concrete validity of arithmetic.

2 Logic and mathematics

1. Instead of the Kantian "analytic–synthetic" distinction, which in its general formulation suffers from fundamental problems, the introduction of a different kind of distinction recommends itself, a distinction between "*formally*" and "*objectively*" *motivated elements* of a theory, i. e., between elements (terms, axioms, modes of inference) that are introduced for the sake of the elegance, simplicity, and the rounding off of the system, and those that are introduced with regard to matters of fact of the domain in question.

Remark. This distinction admittedly does not yield a sharp classification, since formal and objective motives can overlap.

2. Systematic logic forms a domain of application for mathematical considerations. The connection between logic and mathematics in the systems

[a] *Vide* [**?**].

of logistic corresponds to that of physics and mathematics in the systems of theoretical physics.

3. What is mathematical can be found not only in connection with the sentential formalism of logic, rather we find mathematical relations also in intuitable objects; in particular, we find mathematical relationships in all domains of the physical and the biological.—The independence of the mathematical from language has been emphasized in particular by Brouwer.

4. We must acknowledge that numerical relations express objective facts. This becomes particularly clear in syntax: e. g., if a formula A is derivable in a formalism F, then this is a fact which as such can be exhibited and verified explicitly. On the other hand, this derivability is represented in the language of syntax by a numerical relation.

We also have a way of checking arithmetical statements of general form, e. g., the statement that every whole number can be represented as the sum of four or fewer squares can be confirmed in a sense analogous to physical laws, except that in the former case one is confronted with a computational situation and in the latter case with an experimental one; in both cases a particular result to be obtained is predicted by the law.

5. In both the logic of ordinary language and symbolic logic we have formally and objectively motivated elements side by side. An objective motivation is present in so far as the logical terms and principles refer in part to certain very general characteristics of reality. In particular, Paul Hertz has pointed out this objective side of logic. F. Gonseth also speaks of logic as a general "théorie de l'objet."

On the other hand, the fact remains that the scope and the problems of logic are oriented according to certain basic features of the structure of language.

3 On the question of mathematical intuition

1. In Kant's doctrine of pure intuition the assumption of a mathematical intuition is afflicted with various questionable additional aspects. We can leave aside all these additions, such as the claim that the intuition of space and time is required for physics and the distinction between "sensible" and "pure" intuition, and still acknowledge, however, that spatial relationships can be represented in an intuitive mathematical way, and at least to a certain extent, we can read off the properties of configurations, as it were, from

their intuitive representation. The kind of imagination involved need not be fundamentally different from that which a composer uses in the domain of music when he calls up combinations of tones in his mind.

2. It is advisable to distinguish between "arithmetical" and "geometrical" intuition, not according to spatial or temporal moments, but with regard to the distinction between discrete and continuous. Accordingly, the representation of a figure that is composed of discrete parts, in which the parts themselves are considered either only in their relation to the whole figure or according to certain coarser distinctive features that have been specially singled out, is arithmetical; furthermore, the representation of a formal process that is performed with such a figure and that is considered only with regard to the change that it causes is likewise arithmetical. By contrast, the representations of continuous change, of continuously variable magnitudes, moreover topological representations, like those of the shapes of lines and plains, are geometrical.

3. The boundaries of what is intuitively representable are not sharp. This is what has led to the systematic sharpening of the arithmetical and geometrical concepts that are obtained by intuition, as has been done in part by the axiomatic method, in part by the introduction of formally motivated kinds of judgments and rules of inference. What is methodologically special in this case is that the formally motivated elements that were to be introduced had already been provided largely by logic, like the principle of *tertium non datur*, which is synonymous with the assumption that every statement can be negated in the sense of a strict contradictory opposite; and in addition the objectification of the concepts (predicates, relations) and extensions of concepts.

Remark. It is noteworthy historically, that in Aristotelian logic the *tertium non datur* is nowhere required in the well-known 19 modes of inference, because the general affirmative judgment must be understood as asserting the existence of objects that fall under the concept of subject. (Note the rule *ex mere negativis nihil sequitur*[b] from this point of view.)

4 On the problem of the foundations

[b]Translation: nothing follows from mere negative (judgments).

1. The method of sharpening mathematics by abstract means as it is applied in analysis and set theory has been opposed by some mathematicians, as is well known, from the very beginning. In its most distinctive form this opposition has the goal of replacing the usual method of introducing formally motivated elements by one that is performed completely within the framework of arithmetical evidence; geometric intuitiveness is to be eliminated and, moreover, all abstract concept formations and modes of inference that do not possess arithmetical intuitiveness are to be avoided.

2. The founding of a substantial part of existing mathematics that was begun by Kronecker and has been carried out by Brouwer according to the goal mentioned in 1. (of a mathematics aiming at arithmetical evidence) has not converted the mathematicians to accept the standpoint of arithmetical evidence. The reasons for this may be the following:

a) Those who are looking for intuitiveness in mathematics will feel the complete elimination of geometrical intuition to be unsatisfying and artificial. In fact, the reduction of the continuous to the discrete succeeds only in an approximate sense. On the other hand, those who are striving for sharp concepts will prefer those methods that are most beneficial from the systematic standpoint.

b) In Brouwer's approach, distinctions are introduced into the language of mathematics which play an essential role, but whose importance is only apparent from the standpoint of the syntax of this language. That the *tertium non datur* is invalid, as Brouwer claims, can only be stated as a *syntactic* matter of fact, but not as one of mathematical objectiveness itself.

Comment. Brouwer's idea of characterizing the continuum as a set of choice sequences is in itself independent of the rejection of the *tertium non datur*. Certainly no *tertium non datur* can hold with regard to indefinite predicates of choice sequences. But one could nevertheless choose a standpoint such that the *tertium non datur* is retained for number theoretic properties of regular sequences. In this manner one would obtain an extension of Weyl's theory of the continuum of 1918.

3. The standpoint that Hilbert adopts in his proof theory is characterized by the fact that it meets both the requirements of formal systematics and those of arithmetical evidence. As a way to unify these goals he employs the distinction between mathematics and meta-mathematics, which is modeled on the Kantian partitioning of philosophy into "critique" and "system." As is well known, the main task that Hilbert assigns to meta-mathematics as a critique of proof is to show the consistency of the usual practice of mathe-

matics. The problem is intended to be tackled in stages.

In the course of accomplishing this task, however, considerable difficulties arise, which are in part unexpected. An essential reason for difficulties which have not yet been overcome is that the difference between a formalism of intuitive arithmetic and that of usual mathematics is greater than Hilbert had presumed.

In the formalism of number theory the *tertium non datur* can be eliminated in a certain sense. The proofs of the consistency of the number theoretic formalism by Gödel and Gentzen are based on this fact. But as soon as one passes over to numerical textitfunctions such an elimination is no longer possible. This follows in particular from a theorem which was proved by S. C. Kleene after the concept of a "computable" function had made more precise; it says that there are numerical functions which are definable with the symbols of the number theoretic formalism (including a symbol for "the smallest number x that has the property $\mathfrak{P}(x)$"), but which are not computable.

Comment.—The concept of a computable function was made more precise in two independent ways: using the concept of a "generally recursive" function due to Herbrand and Gödel and by Church's concept of a "λ-definable" function; these concepts have been shown to be co-extensive by Church and Kleene.

4. While the task of a consistency proof for analysis is still an unsolved problem, in a different direction, namely in the domain of untyped formalisms of combinatory logic, proofs of consistency have succeeded. The theory of "combinators" which was formulated by H. B. Curry, following Schönfinkel, is such an untyped calculus, and so is the theory of "conversions" established by A. Church. Both these formal theories, whose close connection has been shown by J. B. Rosser, yield a far-reaching and logically satisfying formalism for definitions. The consistency of operating with combinators (in the sense of unambiguousness) was proved some time ago by Curry, that of the formalism of conversions more recently by Church and Rosser.

The untyped combinatory formalisms also yield a new clue as to how systems of logistic may be constructed. An integration of these domains may perhaps lead to a reform of the whole of logistic. To be sure, an adequate approach to such an integration is not available yet.