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Considerations regarding the paradox of Thoralf Skolem (1957)

Betrachtungen zum Paradoxon von Thoralf Skolem

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Translation by: *Dirk Schlimm*
Revised by: *CMU*
Final revision by: *CMU*

About 35 years ago, on the occasion of a congress in Helsingfors, Thoralf Skolem pointed out a paradoxical consequence of a theorem by Leopold Löwenheim, for which he had presented a simplified proof two years earlier using the logical normal form named after him.

This well-known theorem by Löwenheim says that for every mathematical theory axiomatized in the framework of elementary predicate logic—i. e., without bound variables for predicates—there exists a model in which the individuals are natural numbers, provided that there is a model that satisfies it at all. The theorem can be extended to the case in which one or more axiom *schemata* occur in the axiom system besides the proper axioms. In the axiom schema an arbitrary predicate that can be constructed using the formation rules of the axiom system, resp. a set or function that is arbitrary in the same sense, occurs as parameter.

Now Skolem realized that this theorem can be applied to axiomatic set theory, provided it has been sharpened from the original formulation of Ernst Zermelo by a more precise concept of definite property. That it is possible to sharpen it in such a way, whereby the axiom system can be represented as a calculus by axioms and schemata, had been realized shortly before by Skolem and, in a different way, by Abraham Fraenkel. By the way, John von Neumann even succeeded in setting up a system of finitely many axioms (without schemata) for set theory.

The possibility arose thereby of such models for set theory, in which sets are represented by natural numbers. This possibility is quite paradoxical, because according to the theorems of set theory the cardinal numbers of the sets that occur rise to such dizzying heights that the infinity of the number sequence (the countably infinite) is by far exceeded.

That this does not constitute a proper contradiction follows, as is well-known, from the fact that the enumerations which work as such in the axiomatic framework do not yet exhaust all possible enumerations. The concept of set is restricted by the axiomatic specification in such a way that one can speak of "set" only relative to a particular framework, if one generally insists on the requirement of axiomatic precision. This relativization is extended to a series of other concepts that are closely connected to the concept of set, in particular the concept of a uniquely invertible mapping between two totalities and thereby also the concept of cardinality (generalized concept of number), and especially that of denumerability.

At first the impression arises that the detected paradox shows above all that differences of magnitudes are apparent and especially that the properly uncountable is an illusion. At the same time the thought is suggested that an operational construction of mathematics, and in particular of analysis, might be preferable to an axiomatic formulation, in the light of the ascertained relativity.

An operational understanding of mathematics is championed by many. It is characteristic that it does not regard the object of mathematics as something that is given in advance and that should be made accessible to our thought by formation of concepts and axiomatic descriptions, but that the mathematical operations themselves and the objects that are brought about by them are regarded as the topic of mathematics. Mathematics should, to some extent, create its own objects. Thereby the character of arithmetic is prescribed *eo ipso*, since the structures of the operational creation are not fundamentally more general than those of the number sequence.

Herein lies a strength of this standpoint on the one hand, and a weakness on the other. It possesses a strength insofar arithmetical (constructive, combinatorial) thinking has the methodical distinction of being elementary and intuitive. However, it is doubtful whether we can get by with it for mathematics and whether a, so to speak, monistic conception of mathematics in the sense of the operational view can do full justice to its content—even as it is now.

This idea is especially reinforced when we consider the enterprises of an operational construction of analysis, as they have been pursued in more recent times, following different programmatic points of view. All these kinds of constructions have in common that we are hindered by distinctions which are of no relevance for the geometrical idea of the continuum, and are not necessary for the consistent functioning of the concepts. The usual procedure of classical analysis proves to be vastly superior in this respect; and if the treatment of analysis had historically begun with an operational procedure, the detection of the possibility of the, so much simpler, classical methods would have been an eminent discovery, hardly less as it meant a *de facto* eminent progress in a different direction, namely compared to the vagueness of the former operations in analysis.

The sense of an appropriate formation of concepts for analysis apparently lies in a suitable compromise. We can make that plausible by the following. The conflicting aspects of the concept to be determined are, on the one hand, the intended homogeneity of the idea of the continuum and, on the other hand, the requirement of conceptual distinctness of the measures of magnitudes. From an arithmetical point of view, every element of the number sequence is an individual with its very specific properties; from a geometric point of view we have here only the succession of repeating similar things. The task of formulating a theory of the continuum is not simply descriptive, but a reconciliation of two diverging tendencies. In the operational treatment one of them is given too much weight, so that homogeneity comes up short.

The investigations about the effectiveness and the fine structure in the formation of number series and sets of numbers have an unquestionable importance for their specific aspects of the general question. But the insights that have been gained here do not constitute a definite indication that the usual procedure of analysis should be replaced by the more arithmetical methods.

The method on which the procedures in classical analysis are based consists, in its logical means, in the application of a contentful "second-order" logic, in which the general concepts like "proposition," "set," "series," "function" etc. are used in an unbounded way that is not further specified. This second-order logic shows its strength not only in its application to the theory of the continuum, but that it generally allows for the characterization of mathematical structures, which may even be uncountable, by explicit definitions. Namely, to what is usually called an "implicit definition" of mathematical objects there corresponds an explicit definition of a whole structure wherein those objects occur as dependent components. The model theoretic concepts of satisfiability and categoricity also find here their unproblematic application.

To be sure, second-order logic is reproached for having a certain imprecision in the concepts, and it is the aim of the new sharper form of the axiomatic approach to repair this defect. Logic and axiomatic set theory have developed the methods for this. The phenomena of the relativity of the higher general concepts discussed above is evidence that this has not succeeded in a completely adequate way to make the concepts precise.

Let us again consider this with an example. The property that an ordering has no gaps is expressed in second-order logic by the condition that every proper initial segment of the ordering which has no last element possesses an immediately succeeding one. The general concept of set appears here by means of the proper initial segment. If this is now made precise by giving certain conditions on how to obtain sets, the manifold of the initial segments under consideration is narrowed, and thereby the condition is weakened. This means that some orderings are admitted as being gapless that can no longer count as being such if the concept of set is sufficiently expanded (i. e., if further processes are admitted for the formation of sets).

The difficulty considered here, which is related to the task of making a theory formally precise, not only occurs in the characterization of uncountable structures, but especially also in the characterization of the structure of the sequence of numbers. We can explain, in Dedekind's sense, that a set M has the structure of the sequence of numbers with regard to a mapping φ (from M to itself) if φ is uniquely invertible and there is an element a of M that is not in the image of φ, and which has the property that no proper subset of M exists that contains a as well as $\varphi(c)$ for each of its elements c. Here again to stipulate a narrower concept of subset can have the consequence that the above condition is satisfied by models to which we would not attribute the structure of the number sequence based on the unrestricted condition. This state of affairs results likewise if we use an axiom system to characterize the number sequence instead of the explicit definition of the structure. In the usual form of such an axiom system one has the axiom of complete induction in which the general concept of proposition (or predicate) occurs. If the axiomatics are formally sharpened, this axiom is replaced by a formal inference principle in which the range of the allowed predicates is formally delimited by a substitution rule. This restriction also allows for the possibility of models for number theory that satisfy all statements provable within the formal framework, but that deviate from the structure of the number sequence when they are considered on their own. Again it was Skolem who pointed out this state of affairs of the "non-characterizability" of the number sequence by a formalized axiom system, by using drastic examples.

On the whole, given what has been said so far, the success of attempting to make a theory sharper and more precise using axioms might appear highly questionable. But the circumstance is not taken into consideration thereby that there are frameworks that classical mathematics has no reasons to transgress—as has been shown by the axiomatic and logical analysis of mathematical theories. The domain of sets and functions, e. g., as it is provided by the axioms of set theory, is closed in such a way that the formal axiomatic restriction is hardly palpable when forming concepts and conducting proofs.

Furthermore, the set theoretic theorems are not affected by the relativity that holds for the general concepts. This relativity of course does not mean that the continuum is shown to be uncountable in *one* framework for set theory and countable in *another*. The discrepancy consists rather only in the fact that the totality of things that are represented in a set theoretic system, e. g., the set of subsets of the number sequence, can be countable in a more comprehensive system; but then it does not there act as a representation of that set of subsets, and thus it is impossible to map the numbers uniquely to the sets of numbers. In such a way, the cardinality theorems of Cantor's set theory are invariant with respect to the axiomatic framework, despite the relativity of the set concepts.

Of course, it must be conceded that this relativity brings the circumstance more forcefully into our attention that the higher cardinalities in set theory are only intended, so to speak, but not properly constructed. In this sense the levels of cardinalities are in a certain way unreal.

The awareness of this state of affairs is often explained by saying that everything in mathematics is countable in "actuality." But this formulation is misleading in so far as it does not take into account the fundamental fact that is expressed in both operational mathematics and in the consideration of formal axiom systems, namely that mathematical thinking in principle transcends every countable system. The framework for the mathematical formation of concepts is the open, contentual second number class, both when proceeding constructively and within a theory of types, if these are not restricted in an arbitrary fashion, or also in the sequence of the ascending systems of axiomatic set theory. It represents something that is in the proper sense uncountable, and to be sure, it cannot be addressed as a particular mathematical structure.

We are reminded here of the fact that the number sequence is also presented to us originally as an open domain compared to which the number sequence that we address as a structure is somehow unreal. The difference with respect to the second number class is that the openness of the number sequence is only due to the incompletedness of the iterations of a single process, whereas the openness of the second number class is due to the incompletedness of the formations of concepts.

That the unreal character of particular uncountable structures is much more noticeable than the unreal character that lies in the conception of the number sequence as a structure is due to the fact that our concept of a formal theory tends toward exactly the same kind of infinity as that of the number sequence.