Introduction to Bernays Text No. 5, “Problems of Theoretical Logic”

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In the period 1914–1922, a significant amount of work on logic was carried out in Göttingen. Beginning in 1914, Hilbert and his students, in particular Heinrich Behmann, were studying Whitehead and Russell's *Principia Mathematica*, which resulted in Behmann's dissertation (1918, see Mancosu 1999). In the Fall of 1917, Paul Bernays joined Hilbert in Göttingen and from then on worked as his assistant, concentrating on logic and the foundations of mathematics. Hilbert taught several courses on these topics in Göttingen, and in each case, Bernays was actively involved in the preparation of the content itself as well as in the preparation of typewritten notes, which Bernays often amended and extended. The first course in this series, *Principles of Mathematics* (1917/18), was analyzed by Sieg (1999); it contained the first formal presentation of a first-order calculus and formed the basis of Hilbert and Ackermann's 1928 textbook *Principles of Theoretical Logic* (Hilbert and Ackermann, 1928). In 1920, this was followed by *Logical Calculus*, and in 1921/22 and 1922/23 by two courses on the foundations of mathematics (the latter co-taught with Bernays). At around the same time, Bernays (1918, 1926) studied the propositional calculus in detail and carried out proofs of completeness and decidability; these results were already contained in less explicit form in the 1917/18 lecture notes (see Zach 1999). Around 1920, Behmann began to work on the decision problem, and give a positive solution for first- and second-order monadic logic (1922). This result also follows from a paper by Löwenheim (1915), which, however, seems to have been unknown in the Hilbert school until the mid-1920s. Bernays himself contributed to this line of research, extending a result on a decidable class of first-order formulas which Moses Schönfinkel had obtained around 1923 (see Bernays and Schönfinkel 1928). During this time, he also continued to improve on the axiomatization of propositional and first-order logic which Hilbert and he began in 1917.

Bernays's “Problems of Theoretical Logic” is a survey of this early work on formal logic in Hilbert's school. The occasion of this paper was the *Versammlung Deutscher Philologen und Schulmänner* in Göttingen in September 1927, a biennial meeting of German high-school teachers. As would be appropriate for the occasion, Bernays's paper is not directed at specialists, but at a philosophically educated lay audience. It is intended to dispel the apparently then prevalent prejudice that mathematical logic is mere ‘idle play.’ To this end, Bernays shows how the machinery of propositional connectives and quantifiers can be used to capture all the logical relationships dealt with in Aristotelian syllogistic, the kind of logic his audience was familiar with. Bernays emphasizes not just the technical improvements arising from the new 'theoretical' logic, but also the conceptual advances over Aristotelian logic (replacing distinctions such as that between categorical and hypothetical judgments by formal distinction using quantification and propositional connectives, the limitations of one-place predicates, and the advantages of axiomatic development).

The various propositional connectives, the method of truth tables and the interdefinability relationships between propositional connectives (including the Sheffer stroke) are now included in any textbook treatment of logic, but were then relatively new and, in 1927, only known to those familiar with the recent literature in logic. Much of this literature was, moreover, only available in English (in particular, the work of Russell, Sheffer, and Post). Bernays begins with a discussion of propositional logic and truth functions (pp. 371–2). One major advance in the development of propositional logic, which Bernays discusses (p. 373), arose from his own work: the decidability of propositional logic. The question of whether a given propositional formula is valid (i.e., a tautology) can be decided by the truth table method alone. What Hilbert and Bernays showed in the 1917/18 lecture course and in Bernays's *Habilitationsschrift*, was that it can also be decided by transforming the formula into a disjunctive normal form. By carrying out this transformation in the propositional calculus, this method can be used to prove the completeness of the latter. This, however, requires an axiomatic formulation of the propositional calculus. Bernays shows (p. 374) in outline how such an axiomatization can be given, and how it relates to the principles of logical inference familiar from Aristotelian logic. His propositional calculus requires substitution and modus ponens as the only rules of inference.

Bernays also discusses two interesting details of the new, axiomatic way of treating propositional logic (on this, see also Zach 1999). One is the choice of axioms. Here Bernays indicates (p. 375) that it is advantageous to divide the axioms into groups governing individual connectives. This is in marked contrast to the approach of *Principia*, in which disjunction and negation are taken as the only primitives, and are given a joint axiomatization. Bernays's approach is instead to take disjunction, conjunction, negation, and implication as primitives and give separate groups of axioms for each. This suggests another question, which is the second detail which Bernays addresses: the characterization of the negation-free fragment of propositional logic. Both of these details were of importance to the application of the axiomatic development of logic in the service of Hilbert's program. Since negation is at the root of the difficulties arising from intuitionism's rejection of the law of excluded middle, it seemed necessary to develop an axiomatization of logic without it. This, of course, is not possible if one builds propositional formulas from negation and disjunction as the only primitives, as Russell's logic does. The focus on positive logic also suggests a study of the structural features of inference which does not rely on the excluded middle, and Bernays points here to the work of Paul Hertz, which in turn influenced Gentzen's later development of natural deduction and sequent calculi (see Schröder-Heister 2002).

Bernays concludes with a discussion of predicate logic. Again, he motivates this discussion by showing how Aristotelian propositions can be formalized using quantifiers in combination with propositional connectives. He then goes on to point out how the new logic goes beyond Aristotelian logic: many-place relation symbols, nested quantifiers, identity, and functions allow the formalization not just of syllogistic inference, but indeed of all the inferences in mathematics. Bernays points out the crucial question arising from this improvement in expressive power: the problem of deciding if a given formula of first-order (or indeed, second-order) logic is derivable from the axioms (p. 375). While acknowledging that "we are far from having a solution to this problem," he mentions the partial successes he and his colleagues had achieved by that time, in particular, Behmann's solution to the decision problem for monadic second-order logic.

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