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# **The philosophy of mathematics and Hilbert's proof theory (1930)**

## **Die Philosophie der Mathematik und die Hilbertsche Beweistheorie**

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## **Part I**

# **The nature of mathematical knowledge**

When we read and hear today about the foundational crisis in mathematics or the dispute between “formalism” and “intuitionism,” those unfamiliar with the activities of mathematical science may think that this science is shaken to its very foundations. In reality, mathematics has long been moving in such quiet waters that one notices instead the absence of stronger impulses, although there has been no shortage of significant systematic advances and brilliant achievements.

In fact, the current discussion of the foundations of mathematics does not spring at all from a predicament of mathematics itself. Mathematics is in a completely satisfactory state of methodological certainty. In particular, the concern raised by the set theoretic paradoxes has long been overcome, since it was recognized that, to avoid the known contradictions, it suffices to

impose restrictions that do not in the least impinge on the demands made on set theory by mathematical theories.

The problems, the difficulties, and the differences of opinion really begin only when one inquires not just about the mathematical facts, but about the epistemological foundation and the demarcation of mathematics. These philosophical questions have become particularly urgent since the transformation that the methodological approach of mathematics underwent towards the end of the nineteenth century.

The characteristic aspects of this transformation are: the emergence of the concept of set, by means of which the rigorous foundation of the infinitesimal calculus was achieved, and further the rise of existential axiomatics, that is, the method of developing a mathematical discipline as the theory of a system of things with certain relations whose properties constitute the content of the axioms. To these we must add, as a consequence of the two aforementioned aspects, a closer connection between mathematics and logic is established.

This development confronted the philosophy of mathematics with a completely new situation and entirely new insights and problems. Since then no agreement has been reached in the discussion of the foundations of mathematics. The present stage of this discussion is centered around the struggle with the difficulties that are caused by the role of the infinite in mathematics.

The problem of the infinite, however, is neither the first nor the most general question which one has to address in the philosophy of mathematics. Here, the first task is to gain clarity about what constitutes the specific nature of mathematical knowledge. We intend to address this question first, and also to recall the development of the different points of view, but only in a rough way and without their exact chronological order.

## **1 The development of conceptions of mathematics**

The older conception of mathematical knowledge proceeded from the division of mathematics into arithmetic and geometry; according to it mathematics was characterized as a theory of two particular kinds of domains, that of numbers and that of geometric figures. This division could no longer be maintained in the face of the rise to prominence of arithmetical methods in geometry. Also geometry was not restricted to the study of the prop-

erties of figures but was broadened to a general theory of manifolds. The completely changed situation of geometry found a particularly concise expression in Klein's Erlangen Program, which systematically summarized the various branches of geometry from a group-theoretical point of view.

In the light of this situation the possibility arose to incorporate geometry into arithmetic. And since the rigorous foundations of the infinitesimal calculus by Dedekind, Weierstrass, and Cantor reduced the more general concepts of number—as required by the mathematical theory of quantities (rational number, real number)—to the usual (“natural”) numbers  $1, 2, \dots$ , the conception emerged that the natural numbers constitute the true object of mathematics and that mathematics is precisely the *theory of numbers*.

This conception has many supporters. In its favor is the fact that all mathematical objects can be represented through numbers, or combinations of numbers, or through higher set formations obtained from the number sequence. From a foundational perspective the characterization of mathematics as a theory of numbers is already unsatisfactory, because it remains open what one considers here as essential to number. The question concerning the nature of mathematical knowledge is thereby shifted to the question concerning the nature of numbers.

To the proponents of the conception of mathematics as the science of numbers, however, his question appears to be completely idle. They proceed from the attitude common to mathematical thought, that numbers are a sort of things, which by their nature are completely familiar to us, so much so that an answer to the question concerning the nature of numbers could only consist in reducing something familiar to something less familiar. From this standpoint one sees the reason for the special status of numbers in the fact that numbers make up an essential component of the world order. This order is comprehensible to us in a rigorous scientific way just to the extent to which it is governed by the notion of number.

Opposing this view, according to which number is something completely absolute and final, there emerged soon, in the aforementioned epoch of the development of set theory and axiomatics, a completely different conception. This conception denies that mathematical knowledge is of a particular and characteristic kind, and holds that mathematics is to be obtained *from pure logic*. One was led naturally to this conception through axiomatics, on the one hand, and through set theory, on the other.

The new methodological turn in axiomatics consisted in giving prominence to the fact that for the development of an axiomatic theory the epis-

temic character of its axioms is irrelevant. Rigorous axiomatics demands that in the proofs no other knowledge from the given subject be used than what is expressly formulated in the axioms. This was intended already by Euclid in his axiomatics, even though at certain points the program is not completely carried through.

According to this demand, the development of an axiomatic theory shows the logical dependence of the theorems on the axioms. But for this logical dependence it does not matter whether the axioms placed at the beginning are true sentences or not. It represents a purely hypothetical connection: if things are as the axioms say, then the theorems hold. Such a separation of deduction from asserting the truth of the initial statements is in no way idle hair splitting. On the contrary, an axiomatic development of theories, without regard to the truth of the fundamental sentences taken as starting points, can be of great value for our scientific knowledge: in this way, on the one hand, it is possible to test, in relation to the facts, assumptions of doubtful correctness by systematic development of their logical consequences; furthermore, the *possibilities of a priori theory construction* can be investigated mathematically from the point of view of systematic simplicity and, as it were, to develop a supply in reserve. With the development of such theories, mathematics takes over the role of the discipline formerly called *mathematical natural philosophy*.

By completely ignoring the truth of the axioms of an axiom system, the content of the basic concepts also becomes irrelevant, and thus one is lead to completely *abstract from all intuitive content of the theory*. This abstraction is further supported by a second feature, which comes as an addition to the newer axiomatics, as it was developed above all in Hilbert's *Foundations of Geometry*, and which is, in general, essential for the formation of recent mathematics, namely, the *existential* conception of the theory.

Whereas Euclid always thinks of the figures under consideration as constructed ones, contemporary axiomatics proceeds from the idea of a *system of objects*, which is fixed in advance. In geometry, for example, one conceives of the points, lines, and planes in their totality as such a system of things. Within this system one considers the relations of incidence (a point lies on a line, or in a plane), of betweenness (a point lies between two others), and of congruence as being determined from the outset. Now, regardless of their intuitive meaning, these relations can be characterized purely abstractly as certain *basic predicates*. (We will use the term "predicate" also in the case of a relation between several objects, so that we also speak of predicates with

several subjects.)<sup>1</sup>

Thus, e.g., in Hilbert's system the Euclidean construction postulate, which demands the possibility of connecting two points with a line, is replaced by the existence axiom: for any two points there is always a straight line that belongs to each of the two points. "Belonging to" is here the abstract expression of incidence.

According to this conception of axiomatics, the axioms as well as the theorems of an axiomatic theory are statements about one or several predicates, which refer to the objects of an underlying system. And the knowledge provided to us by the proof of a theorem  $L$ , which is carried out by means of the axioms  $A_1 \dots A_k$  (for the sake of simplicity we will assume that we are dealing here with only *one* predicate) consists in the realization that, if the statements  $A_1 \dots A_k$  hold of a predicate, then so does the statement  $L$ .

What we have before us, however, is a very general proposition about predicates, that is, a proposition of pure logic. In this way, the results of an axiomatic theory, according to the purely hypothetical and existential understanding of axiomatics, present themselves as *theorems of logic*.

These theorems, though, are only significant if the conditions formulated in the axioms can be satisfied at all by a system of objects together with certain predicates concerning them. If such a satisfaction is inconceivable, that is, logically impossible, then the axiom system does not lead to a theory at all, and the only logically important statement about the system is then the observation that a contradiction results from the axioms. For this reason every axiomatic theory requires a proof of the *satisfiability*, that is, *consistency*, of its axioms.

Unless one can make do with direct finite model constructions, this proof is accomplished in general by means of the method of *reduction to arithmetic*, that is, by exhibiting objects and relations within the realm of arithmetic that satisfy the axioms to be investigated. As a result, one is again faced with the question of the epistemic character of arithmetic.

Even before this question became acute in connection with axiomatics, as just described, set theory and logistics had already taken a novel position on it. Cantor showed that the number concept, both in the sense of cardinal number and in the sense of ordinal number, can be extended to infinite

<sup>1</sup>This terminology follows a suggestion of Hilbert. It has certain advantages over the usual distinction between "predicates" and "relations" for the conception of what is logical in principle and also agrees with the usual meaning of the word "predicate."

sets. The theory of natural numbers and the theory of positive real numbers (analysis) were subsumed as parts under general set theory. Even if the natural numbers lost an essential aspect of their distinguished role, nonetheless, from Cantor's standpoint, the number sequence still constitutes something immediately given, the examination of which was the starting point of set theory.

But this was not the end of the matter; rather, the logicians soon adopted the stronger claim that sets are nothing but extensions of concepts, set theory is synonymous with the logic of extensions, and, in particular, the theory of numbers is to be derived from pure logic. With this thesis that mathematics is to be obtained from pure logic, an old and favorite idea of rational philosophy, which had been opposed by the Kantian theory of pure intuition, was revived.

The development of mathematics and theoretical physics had already shown that the Kantian theory of experience, in any case, was in need of a fundamental revision, and to the radical opponents of Kant's philosophy the moment seemed to have arrived for refuting this philosophy in its very starting point, namely the claim that mathematics is synthetic in character.

This, however, was not completely successful. A first symptom that the situation was more difficult and complicated than the leaders of the logistic movement had thought became apparent in the discovery of the famous *set-theoretic paradoxes*. Historically, this discovery was the signal for the beginning of the critique. If we want to discuss the situation philosophically today, it is more satisfactory to consider the matter directly without bringing in the dialectical argument involving the paradoxes.

## 2 The mathematical element in logic – Frege's definitions of number

In order to see what is essential we in fact need only to consider the new discipline of theoretical logic itself, the intellectual achievement of the great logicians Frege, Schröder, Peano, and Russell, and see what it teaches us about the relation of the mathematical to the logical.

One sees immediately a peculiar two-sidedness in this relation which shows itself in a varying conception of the task of theoretical logic: Frege strives to subordinate mathematical concepts under the concept formations of logic, but Schröder, on the other hand, tries to bring to prominence the

mathematical character of logical relations and develops his theory as an “algebra of logic.”

But the difference here is only a matter of emphasis. In the various systems of logistic one never finds the specifically logical point of view dominating by itself; rather, in each case, it is imbued from the outset with the mathematical perspective. Just as in theoretical physics, here, too, the mathematical formalism and mathematical concept formation prove to be the appropriate means of representing interconnections and of gaining a systematic overview.

To be sure, it is not the usual formalism of algebra and analysis that is applied here, but a newly created calculus developed by theoretical logic on the basis of the formula language used to represent the logical connectives. No one familiar with this calculus and its theory will doubt its explicitly mathematical character.

Concerning this situation there arises first of all the requirement to delimit the concept of the mathematical, independently of the actual situation in the mathematical disciplines, by means of a principled characterization of the nature of mathematical knowledge. If we examine what is meant by the mathematical character of a deliberation, it becomes apparent that the distinctive feature lies in a certain kind of abstraction that is involved. This abstraction, which may be called *formal* or *mathematical abstraction*, consists in emphasizing and taking exclusively into account the structural aspects of an object, that is, the manner of its composition from parts; “object” is understood here in its widest sense. One can, accordingly, define mathematical knowledge as that which rests on the *structural* consideration of objects.

The study of theoretical logic teaches us, furthermore, that in the relationship between mathematics and logic, the mathematical point of view, in contrast to the contentual logical one, is under certain circumstances the more abstract one. The aforementioned analogy between theoretical logic and theoretical physics extends as follows: just as the mathematical laws of theoretical physics are contentually specialized by their physical interpretation, so the mathematical relationships of theoretical logic are also specialized through their contentual logical interpretation. *The laws of the logical relations appear here as a special model for a mathematical formalism.*

This distinctive relation between logic and mathematics—not only can mathematical judgments and inferences be subjected to logical abstraction, but also logical relationships can be subjected to mathematical abstraction—is based on the special role of the formal realm with respect to logic. Namely,

whereas in logic one can otherwise abstract from the specifics of a given subject, this is not possible in the formal realm, because *formal elements enter essentially into logic itself*.

This holds in particular for *logical inference*. Theoretical logic teaches that logical proofs can be “formalized.” The method of formalization consists first of all in representing the premises of the proof by specific formulas in the logical formula language, and furthermore in the replacement of the principles of logical inference by rules that specify determinate procedures, according to which one proceeds from given formulas to other formulas. The result of the proof is represented by an end formula, which, on the basis of the interpretation of the logical formula language, presents the proposition to be proved.

Here we use the fact that logical inference, considered as a process, is reducible to a limited number of logical elementary processes that can be exactly and completely enumerated. In this way it becomes possible to pursue questions of *provability* systematically. The result is a field of theoretical inquiry within which the theory of the different possible forms of categorical inference put forward in traditional logic deals with only a very specific special problem.

The typically mathematical character of the theory of provability reveals itself especially clearly through the role of the logical *symbolism*. The symbolism is here the *means for carrying out the formal abstraction*. The transition from the point of view of logical content to the formal one takes place when one ignores the original meaning of the logical symbols and makes the symbols themselves representatives of formal objects and connections.

For example, if the hypothetical relation

“if  $A$ , then  $B$ ”

is represented symbolically by

$$A \rightarrow B,$$

then the transition to the formal standpoint consists in abstracting from all meaning of the symbol  $\rightarrow$  and taking the connection by means of the “sign”  $\rightarrow$  itself as the object to be considered. To be sure one has here a specification in terms of figures instead of the original specification of the connection in terms of content; this, however, is harmless insofar as it is easily recognized as an accidental feature. Mathematical thought uses the symbolic figure to carry out the formal abstraction.



The method of formal consideration is not at all introduced here artificially; rather it is almost forced upon us when we inquire more closely into the effects of logical inference.

If we now consider why the investigation of logical inference is so much in need of the mathematical method, we discover the following fact. In proofs there are two essential features which work together: the elucidation of concepts, the feature of *reflection*, and the mathematical feature of *combination*.

Insofar as inference rests only on elucidation of meanings, it is analytic in the narrowest sense; progress to something new comes about only through mathematical combination.

This combinatorial element can easily appear to be so obvious that it is not viewed as a separate factor at all. With regard to deductively obtained knowledge, philosophers especially were in the habit of considering only the precondition of proof as epistemologically problematic and in need of discussion, namely, the fundamental assumptions and rules of inference. This standpoint, however, is insufficient for the philosophical understanding of mathematics; for the typical effect of a mathematical proof is achieved only after the fundamental assumptions and rules of inference have been fixed. The remarkable character of mathematical results is not diminished when we modify the provable statements contentually by introducing the ultimate assumptions of the theory as premises, and in addition explicitly state the rules of inference (in the sense of the formal standpoint).

To clarify the situation we can make use of Weyl's comparison of a proof conducted in a purely formal way with a game of chess; the fundamental assumptions correspond to the initial position in the game, the rules of inference to the rules of the game. Let us assume that for a certain initial position  $A$ , a clever chess master has discovered the possibility of checkmating his opponent in 10 moves. From the usual point of view we must then say that this possibility is *logically* determined by the initial position and the rules of the game. On the other hand, one can not maintain that the assertion of the possibility of a checkmate in 10 moves is contained in the meaning of the specification of the initial position  $A$  and the rules of the game. The appearance of a contradiction between these claims disappears, when we recognize that the "logical" effect of the rules of the game depends upon *combination* and therefore does not result from a simple analysis of meaning, but only through actual demonstration.

Every mathematical proof is in this sense a demonstration. We will show here by a simple special case how the combinatorial element comes into play

in a proof.

We have the rule of inference: “if  $A$  and if  $A$  implies  $B$ , then  $B$ .” In a formal translation of a proof this inference principle corresponds to the rule that the formula  $B$  can be obtained from the two formulas  $A$  and  $A \rightarrow B$ . Now let us apply this rule in a formal derivation, and we furthermore assume that  $A$  and  $A \rightarrow B$  do not belong to the initial assumptions. Then we have a sequence of inferences  $S$  leading to  $A$  and a sequence  $T$  leading to  $A \rightarrow B$  and according to the rule described the formulas  $A$  and  $A \rightarrow B$  yield the formula  $B$ .

If we want to analyze what is going on here, we must not prejudge the decisive point by the mode of presentation. The end formula of the sequence of inferences  $T$  is initially only given as such, and it is epistemologically a new step to recognize that this formula coincides with the one which arises by connecting with a “ $\rightarrow$ ” the formula  $A$  obtained in some other way and the formula  $B$  to be derived.

The determination of an identity is by no means always an identical or tautological determination. The coincidence to be noted in the present case can not be read off directly from the content of the formal rules of inference and the structure of the initial formulas; rather, it can be read off only from the structure that is obtained by application of the rules of inference, that is to say by the carrying out of the inferences. Thus, a combinatorial element is in fact present here.<sup>2</sup>

If in this way we become clear about the role of the mathematical in logic, then it will not seem astonishing that arithmetic can be subsumed within the system of theoretical logic. But also from the standpoint we have now reached this subsumption loses its epistemological significance. For we know in advance that the formal element is not eliminated by the inclusion of arithmetic in the logical system. With respect to the formal. however, we have found that the mathematical considerations represent a standpoint of higher abstraction than the conceptual logical ones. We therefore achieve no greater generality at all for mathematical knowledge as a result of its subsumption under logic; rather we achieve just the opposite; a *specialization by logical interpretation*, a kind of *logical clothing*.

<sup>2</sup>Paul Hertz defended the claim that logical inference contains “synthetic elements” in his essay “On thinking” (*vide* [?]). His grounds for this claim will be explained in an essay on the nature of logic, to appear shortly; they include the point developed here but also rest in addition on still other considerations.

A typical example of such logical clothing is the method by which Frege and, following him but with a certain modification, Russell defined the natural numbers.

Let us briefly recall the idea underlying Frege's theory. Frege introduces the numbers as cardinal numbers. His premises are as follows:

A cardinal number applies to a *predicate*. The concept of cardinal number arises from the concept of equinumerosity. Two predicates are called equinumerous if the things of which the one predicate holds can be correlated one-one with the things of which the other predicate holds.

If the predicates are divided into classes with respect to equinumerosity in such a way that all the predicates of a class are equinumerous with one another and predicates of different classes are not equinumerous, then every class represents the *cardinal number* which applies to the predicates belonging to it.

In the sense of this general definition of cardinal number, the particular finite numbers like 0, 1, 2, 3 are defined as follows:

0 is the class of predicates which hold of no thing. 1 is the class of "one-numbered" predicates; and a predicate  $P$  is called one-numbered if there is a thing  $x$  of which  $P$  holds and no other thing different from  $x$  of which  $P$  holds. Similarly, a predicate  $P$  is called two-numbered if there is a thing  $x$  and a thing  $y$  different from it such that  $P$  holds of  $x$  and  $y$  and if there is no thing different from  $x$  and  $y$  of which  $P$  holds. 2 is the class of two-numbered predicates. The numbers 3, 4, 5 etc. are to be explained as classes in an analogous way.

After he has introduced the concept of a number immediately following a number, Frege defines the general concept of finite number in the following way: a number  $n$  is called finite if every predicate holds of  $n$  which holds of 0 and which, if it holds of a number  $a$  holds of the immediately following number.

The concept of a number belonging to the series of numbers from 0 to  $n$  is explained in a similar way. The formulation of these concepts is followed by the derivation of the principles of number theory from the concept of finite number.

We now want to consider in particular Frege's definition of the individual finite numbers. Let us take the definition of the number 2, which is explained as the class of two-numbered predicates. It may at first be objected to this explanation that the belonging of a predicate to the class of two-numbered predicates depends upon extra-logical conditions and the class therefore con-

stitutes no logical object whatsoever.

This objection is, however, eliminated if we adopt the standpoint of Russell's theory with respect to the understanding of classes (sets or extensions of concepts). According to it classes (extensions of concepts) are not actual objects at all; rather they function only as dependent terms within a reformulated sentence. If, for example,  $K$  is the class of things with the property  $E$ , i. e. the extension of the concept  $E$ , then, according to Russell, the assertion that an thing  $a$  belongs to the class  $K$  is to be viewed only as a reformulation of the assertion that the thing  $a$  has the property  $E$ .

If we combine this conception with Frege's definition of cardinal number, we arrive at the idea that the number 2 is to be defined, not in terms of the class of two-numbered predicates but in terms of the concept the extension of which constitutes this class. The number 2 is then identified with the *property of two-numberedness* for predicates, i. e. with the property of a predicate of holding of an thing  $x$  and of an thing  $y$  different<sup>3</sup> from  $x$  but of no thing different from  $x$  and  $y$ .

For the evaluation of this definition it is essential to know how the process of defining is understood here and what claims are involved in it. What will be shown here is that this definition is not a correct reproduction of the true meaning of the cardinal number concept "two" by means of which this concept is revealed in its logical purity freed from all inessential features. Rather it will be shown that it is exactly the specifically logical element in the definition that is an inessential addition.

The two-numberedness of a predicate  $P$  means nothing else but that there are two things of which the predicate  $P$  holds. Here three distinct conceptual features are present: the concept "two things," the existential feature, and the fact that the predicate  $P$  holds. The content of the concept "two things" here does not depend on the meaning of either of the other two concepts. "Two things" means something already without the assertion of the existence of two things and also without reference to a predicate which holds of two things; it means simply "one thing and one more thing."

In this simple definition the concept of cardinal number shows itself to be an elementary *structural concept*. The appearance that this concept is reached from the elements of logic results, in the case of the logical definition of cardinal number under consideration, only from the fact that the concept is

<sup>3</sup>For the sake of simplicity we shall skip the considerations regarding the concept of difference, resp. its contradictory concept of identity.

conjoined with logical elements, namely the existential form and the subject-predicate relation, which are in themselves inessential for the concept of cardinal number. We thus indeed have before us here a formal concept in *logical clothing*.

The result of these considerations is that the claim of the logicians that mathematics is a purely logical field of knowledge shows itself to be imprecise and misleading when theoretical logic is examined more closely. That claim is sound only if the concept of the mathematical is taken in the sense of its historical demarcation and the concept of the logical is systematically broadened. But such a determination of concepts hides what is epistemologically essential and ignores the special nature of mathematics.

### 3 Formal abstraction

We have determined that formal abstraction, i. e. focusing on the structural side of objects, is the characteristic feature of mathematical reasoning and have thus demarcated the field of the mathematical in a fundamental way. If we want likewise to gain an epistemological understanding of the concept of the logical, then we are led to separate from the entire domain of the theory of concepts, judgments, and inferences, which is commonly called logic, a narrower subdomain, that of *reflective or philosophical logic*. This is the domain of knowledge which is *analytic in the genuine sense* and which stems from a pure *awareness of meaning*. This philosophical logic is the starting point of systematic logic, which takes its initial elements and its principles from the results of philosophical logic and, using mathematical methods, develops a theory from them.

In this way the extent of genuinely analytic knowledge is separated clearly from that of mathematical knowledge, and what is justified in *Kant's theory of pure intuition* on the one hand, and in the claim of the logicians on the other, comes into play. We can distinguish Kant's fundamental idea, that mathematical knowledge and indeed the *successful application of logical inference rest on intuitive evidence*, from the particular form that Kant gave to this idea in his theory of space and time. By doing so we also arrive at the possibility of doing justice to both the very elementary character of mathematical evidence and the high degree of abstraction of the mathematical point of view, emphasized in the claim about the logical character of mathematics.

Our conception also gives a simple account of the role of number in mathematics: we have explained mathematics as the knowledge which rests upon the formal (structural) consideration of objects. However, as cardinal numbers the numbers constitute *the simplest formal determinations* and as ordinal numbers the *simplest formal objects*.

Cardinality concepts present a special difficulty for philosophical explication because of their special categorial position, which also makes itself felt in language in the need for a unique species of number words. We do not have to bother here with more detailed explication, but we do have to observe that the determination of cardinal number involves the putting together of a given or imagined complex out of components, which is just what constitutes the structural side of an object. And indeed it is the most elementary structural characteristics that are conveyed by cardinal numbers. Thus cardinal numbers play a role in all domains to which formal considerations are applicable. In particular we encounter cardinal number within theoretical logic in a wide variety of ways: for example, as cardinal number of the subjects of a predicate (or as one says, as cardinal number of the arguments of a logical function); as cardinal number of the variable predicates involved in a logical sentence; as cardinal number of the applications of a logical operation involved in the construction of a concept or sentence; as cardinal number of the sentences involved in a mode of inference; as the type-number of a logical expression, i. e. the highest number of successive subject-predicate relations involved in the expression (in the sense of the ascent from the objects of a theory to the predicates, from the predicates to the predicates of the predicates, from these latter to their predicates, and so on).

Cardinal numbers, however, provide us only with formal *determinations* and not yet with formal objects. For example, in the conception of the cardinality three there is still no unification of three things into one object. The bringing together of several things into one object requires some kind of ordering. The simplest kind of order is that of mere succession, which leads to the concept of *ordinal number*. An ordinal number in itself is also not determined as an object; it is merely a place marker. We can, however, standardize it as an object, by choosing as *place markers the simplest structures deriving from the form of succession*. Corresponding to the two possibilities of beginning the sequence of numbers with 1 or with 0, two kinds of standardization can be considered. The first is based on a sort of things and a form of adjoining a thing; the objects are figures which begin and end with a thing of the sort under consideration, and each thing, which is not yet the

end of the figure, is followed by an adjoined thing of that sort. In the second kind of standardization we have an initial thing and a process; the objects are then the initial thing itself and in addition the figures that are obtained by beginning with the initial thing and applying the process one or more times.

If we want to have the ordinal numbers, according to either standardization, as unique objects free from all inessential features, then we must take as object in each case the bare schema of the respective figures obtained by repetition; this requires a very high degree of abstraction. However, we are free to represent these purely formal objects by concrete objects (“number signs” or “numerals”); these then possess inessential arbitrarily added characteristics, which, however, can be immediately recognized as such. This procedure is based on a certain agreement, which must be kept throughout one and the same deliberation.<sup>4</sup> Such an agreement, according to the first standardization, is the representation of the first ordinal numbers by the figures 1, 11, 111, 1111. According to an agreement corresponding to the second standardization, the first ordinal numbers are represented by the figures 0, 0', 0'', 0''', 0'''.

Having found a simple access to the numbers in this way by regarding them structurally, our conception of the character of mathematical knowledge receives a new confirmation. For, the dominant role of number in mathematics becomes clear on the basis of this conception; and our characterization of mathematics as a theory of structures seems to be an appropriate extension of the view mentioned at the beginning of this essay that numbers constitute the real object of mathematics.

The satisfactory features of the standpoint we have reached must not mislead us into thinking that we have already obtained all the fundamental insights required for the problem of the foundation of mathematics. In fact, until now we have only dealt with the preliminary question that we wanted to clarify first, namely, what is the specific character of mathematical knowledge? Now, however, we must turn to the problem that raises the main

<sup>4</sup>Philosophers are inclined to treat this relation of representation as a connection of meaning. One must notice, however, that there is an essential difference here from the usual relation of word and meaning; namely the representing thing contains in its constitution the essential properties of the object represented, so that the relationships to be investigated among the represented objects can also be found among the representatives and can be determined by consideration of the latter.

difficulties in the foundation of mathematics, the problem of the infinite.

## Part II

# The problem of the infinite and the formation of mathematical concepts

## 1 The postulates of the theory of the infinite – The impossibility of a basis in intuition – The finitist standpoint

The mathematical theory of the infinite is analysis (infinitesimal calculus) and its extension by general set theory. We can restrict ourselves here to consideration of the infinitesimal calculus because the step from it to general set theory requires only additional assumptions, but no fundamental change of philosophical conception.

The foundation given to the infinitesimal calculus by Cantor, Dedekind, and Weierstraß shows that a rigorous development of this theory succeeds if two things are added to the elementary inferences of mathematics:

1. The application of the method of existential inference to the integers, i. e., the assumption of the *system* of integers in the manner of a domain of objects of an axiomatic theory, as is explicitly done in *Peano's axioms for number theory*.
2. The conception of the totality of all sets of integers as a combinatorially surveyable multiplicity. A set of integers is determined by a distribution of the values 0 and 1 to the positions in the number series. The number  $n$  belongs to the set or not depending on whether the  $n$ th position in the distribution is 1 or 0. Just as the totality of possible distributions of the values 0, 1 over a finite number of positions, e. g. over five positions, is completely surveyable, by analogy the same is assumed also for the entire number series.



From this analogy follows in particular also the validity of *Zermelo's principle of choice* for collections of sets of numbers. However, for the time being we will put aside the discussion of this principle, it will fit in naturally at a later point.

If we now consider these requirements from the standpoint of our general characterization of mathematical knowledge, it seems at first that there is no fundamental difficulty in justifying them on that basis. For both in the case of the number series and in that of the sets derived from it, one deals with *structures*, which differ from those treated in elementary mathematics only in being structures of infinite multiplicity. The existential inference applied to numbers also seems to be justified by their objective character as formal objects the existence of which can not depend on accidental facts about people's conceptions of numbers.

Against this argumentation it is to be remarked, however, that it is premature to conclude from the character of formal objects, i. e. from their being free of accidental empirical features, that the formal entities must be related to a domain of existing formal things. As an argument against this conception we could put forward the set-theoretic paradoxes, but it is simpler to point out directly that primitive mathematical evidence does not assume such a domain of existing formal entities; instead, the connection with what is actually imagined is essential as a starting point for formal abstraction. In this sense the Kantian assertion that pure intuition is the form of empirical intuition is valid.

Correspondingly, existence assertions in disciplines that rest on elementary mathematical evidence do not have a proper meaning. In particular, in elementary number theory we only deal with existence assertions that refer to an explicit totality of numbers that can be exhibited, or to an explicit process that can be executed intuitively, or to both together, i. e. to a totality of numbers that can be obtained by such a process.

Examples of such existence claims are: "There is a prime number between 5 and 10," namely 7 is a prime number.

"For every number there is a greater one," namely if  $n$  is a number, then construct  $n + 1$ . This number is greater than  $n$ .

"For every prime number there is a greater one," namely if a prime number  $p$  is given, then construct the product of this number and all smaller prime numbers and add 1. If  $k$  is the number obtained in this way, then there must be a prime number among the numbers between  $p + 1$  and  $k$ .

In each of these cases the existence assertion is made more precise by a

further specification; the existence claim is restricted to explicit processes that can be carried out in intuition and makes no reference to a totality of all numbers. Following Hilbert, we will call this elementary point of view, restricted by the requirements imposed by intuitability in principle, the *finitist* standpoint; and in the same sense we will speak of finitist methods, finitist considerations, and finitist inferences.

It is now easy to see that existential reasoning goes beyond the finitist standpoint. This transcending of the finitist standpoint already takes place with any existence proposition that is put forward without a more exact determination of the existence claim, as for example with the theorem that there is at least one prime number in every infinite arithmetic sequence

$$a \cdot n + b \quad (n = 0, 1, 2, 3, \dots)$$

where  $a$  and  $b$  are relatively prime numbers.

An especially common and important case of transcending the finitist standpoint is the inference from the failure of an assertion to hold universally (for all numbers) to the existence of a counter-example, or, in other words, the principle according to which the following alternative holds for every number predicate  $P(n)$ : either the universal assertion that  $P(n)$  holds of all numbers is valid, or there is a number  $n$  of which  $P(n)$  does not hold. From the standpoint of existential reasoning this principle results as a direct application of the law of the excluded middle, i. e. from the meaning of negation. This logical consequence fails to hold for the finitist standpoint, because the assertion that  $P(n)$  holds for all numbers has here the purely hypothetical sense that the predicate holds for any given number, and thus the negation of this claim does not have the positive meaning of an existence assertion.

But this does not yet conclude the discussion of the possibilities of an insightful mathematical foundation for the assumptions of analysis. It must be admitted that the assumption of a totality of formal objects does not correspond to the standpoint of primitive mathematical evidence; however, the demands of the infinitesimal calculus can be motivated by the observation that the totalities of numbers and number sets one deals with are *structures of infinite sets*. In particular, the application of existential reasoning to numbers would thus not be inferred from the idea of the concept of formal numbers in the realm of formal objects, but rather from considering the structure of the number sequence in which the individual numbers occur as elements.

Indeed we have not yet considered the argument already mentioned that mathematical knowledge can also concern structures of infinite multiplicity.

We come herewith to the question of the *actual infinite*. For insofar as infinite multiplicities are concerned, the infinite is the true actual infinite in contrast to the “potential infinite;” by the latter is meant not an infinite object but merely the unboundedness of the progression from something finite to something that is again finite. For example, this unboundedness also holds from the finitist standpoint for numbers, since for every number a greater one can be constructed.

The question about the actual infinite which we have to ask first is whether it is given to us as an object of intuitive mathematical knowledge.

In harmony with what we have determined so far, one could be of the opinion that we really are capable of an intuitive knowledge of the actual infinite. For even if it is certain that we have a concrete conception only of finite objects, nevertheless an effect of formal abstraction could be exactly the following: that it frees itself from the restriction to the finite and passes to the limit, as it were, in the case of certain indefinitely continuable processes. In particular one may be tempted to invoke geometric intuition and to point to examples of intuitively given infinite multiplicities from the domain of geometric objects.

Now, in the first place, geometric examples are not conclusive. One is easily deceived here by interpreting the spatially intuitive in the sense of an existential conception. For example, a line segment is not intuitively given as an ordered multiplicity of points but as a uniform whole, although, to be sure, an extended whole within which *positions* are distinguishable. The idea of one position on the line segment is intuitive, but the totality of *all positions* on the line segment is merely a concept of thought. By means of intuition we here reach only the potential infinite since every position on the line segment corresponds to a division into two shorter segments each of which is in turn divisible into shorter segment yet.

Furthermore, one cannot point to infinitely extended things like infinite lines, infinite planes, or infinite space as objects of intuition. In particular, space as a whole is not given to us in intuition. We do indeed represent every spatial figure as situated in space. But this relationship of individual spatial figures to the whole of space is given as an object of intuition only to the extent that a spatial neighborhood is represented along with every spatial object. Beyond this representation, the position in the whole of space is conceivable *only in thought*. (In opposition to Kant, we must maintain this

view.)

The main argument that Kant gave in favor of the intuitive character of our representation of space as a whole, in fact proves only that one cannot attain the concept of a single inclusive space through mere generalizing abstraction. But that is not what is claimed by the assertion that our representation of the whole of space is only accessible in thought, i. e. that we are here dealing with a mere general concept.

Rather, we have in mind a more complicated situation: the representation of the whole of space involves two different kinds of thoughts both of which go beyond the standpoint of intuition and of reflective logic. One rests upon the thought that connecting things yields the world as a whole and therefore stems from our belief about what is real. The other is a *mathematical idea* which, to be sure, begins with intuition but does not remain in the domain of the intuitively representable; it is the representation of space as a manifold of points subject to the laws of geometry.<sup>5</sup>

In both of these ways of representing space as a whole this totality is not recognized as given, but rather is posited only tentatively. The representation of the whole of physical space is a fundamental problem; after all, it is exactly from the standpoint of contemporary physics that there is the possibility of giving this initially very vague thought a more restricted and precise formulation, whereby it becomes systematically significant and accessible to research. The geometric ideas of spatial manifolds are indeed precise from the very beginning, but require a proof of their consistency.

Thus we have no grounds for the assumption that we have an intuitive representation of space as a whole. We cannot point directly to such a representation, nor is it necessary to introduce that assumption as an explanation. But if we deny the intuitiveness of space as a whole, then we also cannot claim that infinitely extended spatial configurations are intuitively representable.

It should also be noted that the original intuitive conception of elementary Euclidean geometry does not in the least require a representation of infinite figures. After all, we are dealing here only with finitely extended

<sup>5</sup>Both of these representations of space are united in the view of nature found in Newtonian physics and are not clearly distinguished from one another. In Newtonian physics Euclidean geometry constitutes the law governing the spatial relation of things in the universe. Only the subsequent development of geometry and physics showed the necessity of distinguishing between space as a physical entity and space as an ideal manifold determined by geometric laws.

figures. Infinite multiplicities of points also never occur, since there are no underlying general existential assumptions; rather, every existential claim asserts a possible geometric construction. For example, from this standpoint the statement that every line segment has a midpoint only says that for every line segment a midpoint can be constructed.<sup>6</sup>

Thus the apparent possibility of displaying an actual infinity in the domain of objects of geometrical intuition is misleading. We can, however, also show in a more general way that there is no question of eliminating the condition of finitude via formal abstraction as would be required for an intuition of the actual infinite. Indeed, the requirement of finitude is no accidental empirical limitation but an essential characteristic of a formal object.

The empirical limitation still lies within the domain of the finite, where formal abstraction must help us to go beyond the boundaries of our actual power of representation. A clear example of this is the unlimited divisibility of a line segment. Our actual power of representation already fails when the division exceeds a certain degree of fineness. This boundary is physically accidental and it can be overcome with the help of optical equipment. But after a certain smallness all optical equipment becomes useless, and finally our spatial and metrical representations lose all physical meaning. Thus, in representing unlimited divisibility we already abstract from the requirements of actual representation as well as from the requirements of physical reality.

The situation is analogous in the case of the representation of unlimited addition in number theory. Here, too, there are limits to the execution of repetitions both with respect to actual representability and to physical realization. Let us consider as an example the number  $10^{(10^{1000})}$ . We can arrive at it in a finitist way as follows: we start from the number 10, which, according to the standardization given earlier, we represent by the figure,

1111111111.

Let  $z$  be an arbitrary number, represented by an analogous figure. If in the representation of 10 we replace each 1 with the figure  $z$ , there results, as we can see intuitively, another number-figure, which for purposes of communication is called “ $10 \times z$ .”<sup>7</sup> In this way we get the process of multiplying a

<sup>6</sup>In Euclid’s axiomatization this standpoint is of course not completely adhered to, since one finds here the notion of an *arbitrarily great extension* of a line segment. This notion can in fact be avoided; one need only formulate the axiom of parallels differently.

<sup>7</sup>Here we use a symbol “with meaning.”

number by 10. From this we obtain the process of transforming a number  $a$  into  $10^a$  by letting the first 1 in  $a$  correspond to the number 10 and every subsequent 1 to the process of multiplication by 10 until the end of the figure  $a$  is reached. The number obtained by the last process of multiplying by 10 is called  $10^a$ .

From an intuitive viewpoint this procedure offers no difficulty whatsoever. But, if we want to consider the process in detail our representation already fails in the case of rather small numbers. We can again get some further help from instruments or by making use of external objects, which involve the determination of very large numbers. But even with all of these we soon reach a limit: it is easy for us to represent the number 20;  $10^{20}$  far extends our actual power of representation, but is definitely within the domain of physical realizability; it is ultimately very questionable, however, whether the number  $10^{(10^{20})}$  occurs in any way in physical reality either as a relation between magnitudes or as a cardinal number.

But intuitive abstraction is not constrained by such limits on the possibility of realization. For limits are accidental from the formal standpoint. Formal abstraction finds no earlier place, so to speak, to make a principled distinction than at the difference between finite and infinite.

This difference is indeed a fundamental one. If we consider more precisely how an infinite multiplicity as such can be characterized at all, then we find that such a characterization is not possible by means of any intuitive presentation; rather it is possible only by means of the assertion (or assumption or determination) of a regular connection. Thus, infinite multiplicities are accessible to us only *in thought*. Such thinking is indeed also a kind of representation, by which a multiplicity is, however, not represented as an object; rather conditions are represented which a multiplicity satisfies (or has to satisfy).

The fact that formal abstraction is essentially tied to the element of finitude becomes especially apparent through the fact that the property of finitude is not a special limiting characteristic from the standpoint of intuitive evidence when considering totalities and figures. From this standpoint the limitation to the finite is observed immediately and, so to speak, *tacitly*. We do not need a special definition of finitude in this case, because the finitude of objects is taken for granted for formal abstraction. So, for example, the intuitive structural introduction of the numbers is suitable only for the *finite* numbers. From the intuitive formal standpoint, “repetition” is *eo ipso* finite repetition.

This representation of the finite, which is implicit in the formal point of view, contains the epistemological justification for the principle of complete induction and for the admissibility of recursive definition, both procedures here construed in their elementary form, as “finitist induction” and “finitist recursion.”

Drawing on this representation of the finite of course goes beyond the intuitive evidence that is necessarily involved in logical reasoning. It corresponds rather to the standpoint from which one *reflects* already on the general characteristics of intuitive objects. Furthermore, the use of the intuitive representation of the finite can be avoided in number theory if one does not insist on treating this theory in an elementary way. But the intuitive representation of the finite forces itself upon us as soon as a formalism itself is made the object of examination, thus in particular in the systematic theory of logical inferences. This brings to the fore the fact that finiteness is an essential feature of the figures of any formalism whatsoever. The limits of any formalism, however, are none other than those of representability of intuitive complexes in general.

Thus our answer to the question whether the actual infinite is intuitively knowable turns out to be negative. A further consequence is that the method of finitist examination is the appropriate one for the standpoint of intuitive mathematical knowledge.

In this way, however, we can not verify the already mentioned assumption for the infinitesimal calculus.

## 2 Intuitionism – Arithmetic as a theoretical framework

How should we now proceed, in the light of these facts? Concerning this question the opinions are divided. We find here a conflict of views similar to that over the question of characterizing mathematical knowledge. The proponents of the standpoint of primitive intuitiveness conclude immediately from the fact that the postulates of analysis and set theory transcend the finitist standpoint that these mathematical theories must be abandoned in their present form and revised from the ground up. The proponents of the standpoint of theoretical logic, on the other hand, either try to logically justify the postulates of the theory of the infinite, or they deny that these

postulates are problematic at all by disputing the fundamental significance of the difference between finite and infinite.

The former view was already held by Kronecker when the method of existential inference first emerged; he was probably the first person to pay close attention to the methodological standpoint that we call finitist and to emphasize most strongly its importance. His attempts to satisfy this methodological requirement in analysis remained fragmentary, however; a more precise philosophical presentation of this standpoint was also lacking. Thus in particular Kronecker's oft quoted dictum, that God has created the whole numbers, but everything else is the work of man, is not at all suited for motivating Kronecker's requirement:<sup>8</sup> if the whole numbers are created by God, one would think that it is permissible to apply existential inference to them, whereas it is just the existential point of view that Kronecker excludes even in the case of the whole numbers.

Brouwer has extended Kronecker's standpoint in two directions: on the one hand with respect to philosophical motivation, by putting forward his theory of "intuitionism,"<sup>9</sup> and on the other hand by showing how one can apply the finitist standpoint in analysis and set theory, and finitistically justify at least a considerable portion of these theories by fundamentally revising the formation of concepts and the methods of inference.

The result of this investigation does have its negative side, however; for it turns out that, in the process of treating analysis and set theory finitistically, one must not only accept great complications, but also serious losses with respect to systematization.

The complications appear already in connection with the first concepts of the infinitesimal calculus, such as boundedness, convergence of a number sequence, the difference between rational and irrational. Let us take for example the concept of boundedness of a sequence of integers. According to the usual view one of the following alternatives holds: either the sequence exceeds every bound, and then the sequence is unbounded, or all numbers in the sequence are below some given bound, and then the sequence is bounded. In order to determine here a finitist concept we must sharpen the definition

<sup>8</sup>The methodological standpoint appropriate to this dictum is the one adopted by Weyl in his book *The Continuum* (*vide* [?]).

<sup>9</sup>In the interest of clarifying the discussion it seems to me advisable to use the term "intuitionism" to refer to a philosophical view in contrast to the term "finitist," which refers to a particular method of inference and concept formation.



of boundedness and unboundedness as follows: a sequence is called bounded if we can indicate a bound for the numbers in the sequence, either directly or by giving a procedure for producing one; the sequence is called unbounded if there is a law according to which every bound is necessarily exceeded by the sequence, i. e., the assumption that the sequence has a bound leads to an absurdity.

With this formulation of the concepts the definitions do indeed have a finitist character, but we no longer have a complete disjunction between the cases of boundedness and unboundedness. We therefore cannot infer that a sequence is bounded from a refutation of the assumption that the sequence is unbounded. Likewise we cannot consider a claim as established when it is proved, on the one hand, under the assumption that a certain sequence of numbers is bounded and, on the other hand, under the assumption that it is unbounded.

In addition to such complications, which permeate the entire theory, there is a yet more essential disadvantage, namely that many of the general theorems, through which mathematics obtains its systematic clarity, fail. So, for example, in Brouwer's analysis even the theorem that every continuous function has a maximum value on a finite closed interval is not valid.

It seems to be an unjustified and unreasonable demand that philosophy is making of mathematics, to give up its simpler and more fruitful method in favor of a cumbersome one, which is also inferior from a systematic point of view, without being forced to do so by an inner necessity. This requirement makes us suspicious of the standpoint of intuitionism.

Let us see what are the main points of this philosophical view, which was developed by Brouwer. It includes, first of all, a characterization of mathematical evidence. Our earlier discussion of formal abstraction agrees in essential points with this characterization, in particular with regard to their connection with Kant's theory of pure intuition.

Admittedly there is a divergence insofar as according to Brouwer's view the temporal aspect is an essential feature of the objects of mathematics. But it is not necessary to go into a discussion of this point here, since a decision concerning it is of no consequence for the question of mathematical methodology: what for Brouwer arises as a consequence of the connection between time and the objects of mathematics is nothing other than what is obtained by us from the connection of formal abstraction with its concrete, intuitive starting point, namely the methodological restriction to finitist procedures.

The decisive consequences of intuitionism result initially from the further

assertion that all mathematical thought with a claim to scientific validity must be carried out on the basis of mathematical evidence, so that the limits of mathematical evidence are at the same time limits for mathematical thought in general.

This demand that mathematical thought be limited to the intuitively evident at first appears to be completely justified. Indeed it corresponds to our familiar conception of mathematical certainty. We must, however, keep in mind that this familiar conception of mathematics was originally accompanied by a philosophical view, according to which the intuitive evidence of the foundations of the infinitesimal calculus was not in question. However, we have departed from such a view since we found that the postulates of analysis cannot be verified by intuition, that in particular the representation of infinite totalities, which is fundamental in analysis, cannot be grasped in intuition but only through the *formation of ideas*.

Now we can not expect this new view of the limits of intuitive evidence to fit directly with the received conception of the epistemological character of mathematics. Rather, on the basis of what we have determined it seems likely that the generally accepted conception of mathematics represents the situation too simply and that we can not do justice to what goes on in mathematics from the standpoint of evidence alone; we must *acknowledge that thought has its own distinctive role*.

Thus we arrive at a distinction between the standpoint of elementary mathematics and a systematic standpoint that goes beyond it. This distinction is by no means artificial or merely *ad hoc*; rather it corresponds to the two different starting points from which one is led to arithmetic: on the one hand, the combinatorial consideration of relations between discrete entities, and on the other, the theoretical demand placed on mathematics by geometry and physics.<sup>10</sup> The system of arithmetic by no means arises only from an activity of construction and intuitive consideration, but also, in large part,

<sup>10</sup>It is remarkable that Jakob Friedrich Fries, who still ascribed mathematical evidence to a domain going far beyond the finite (in particular, according to his view “the continuous sequence of larger and smaller” is given in pure intuition), nevertheless made a methodological distinction between, on the one hand, “arithmetic as a theory,” which conceptualizes and scientifically develops the intuitive representation of magnitude, and, on the other, “combinatory theory or syntactic,” which rests only on the postulate of arbitrary ordering of given elements and its arbitrary repeated applications, and which needs no axioms since its operations are “immediately comprehensible in themselves.” (Cf. J. F. Fries, *Mathematical Philosophy of Nature* (*vide* [?] p. ■).)

from the task of precisely conceptualizing and theoretically mastering the geometric and physical representations of quantity, area, impact, velocity, and so on. The method of arithmetization is a means to this end. But in order to serve this purpose, arithmetic must *extend* its methodological standpoint from the original elementary standpoint of number theory *to a systematic perspective* in the sense of the postulates discussed.

Arithmetic, which comprises the encompassing framework within which the geometric and physical disciplines find their place, consists not only of the elementary, intuitive treatment of numbers, but it has itself the character of a *theory* in that it builds on the representation of the totality of numbers as a system of things as well as the totality of sets of numbers. This systematic arithmetic achieves its aim in the best possible way, and there are no grounds in its procedures for objections, so long as it is clear that we are not here taking the standpoint of elementary intuitiveness, but that of a thought construction, i. e., the standpoint that Hilbert calls the *axiomatic* one.

The charge of arbitrariness against this axiomatic approach is also unjustified, for in the foundations of systematic arithmetic we are not dealing with an arbitrary axiom system, put together according to need, but with a *natural systematic extrapolation from elementary number theory*. However, the analysis and set theory which develop on this foundation constitute a theory which is already *distinguished in pure thought*, and which is suited to be taken as the theory  $\kappa\alpha\tau'\blacksquare'\epsilon\xi\omicron\chi\acute{\eta}\nu$ , into which we incorporate the doctrines and theoretical approaches of geometry and physics.

Thus we cannot acknowledge the veto that intuitionism directs against the method of analysis. The observation, on which we agree with intuitionism, that the infinite is not given to us intuitively does indeed require us to modify our philosophical conception of mathematics, but not to transform mathematics itself.

Of course, the problem of the infinite returns again. For in taking a thought construction as the starting point for arithmetic, we have introduced something problematic. A thought construction, may it be ever so plausible and natural from a systematic point of view, contains in and of itself no guarantee that it can be carried out consistently. In apprehending the idea of the infinite totality of numbers and of sets of numbers, the possibility is not excluded that this idea could lead to a contradiction in its consequences. Thus it remains to investigate the question of the freedom from contradiction,

or “consistency,”<sup>11</sup> of the system of arithmetic.

Intuitionism wants to spare us these tasks by restricting mathematics to the domain of finitist considerations; but the price for this elimination of the difficulties is too high: the problem goes away, but the systematic simplicity and clarity of analysis is also lost.

### **3 The problems with logicism – The value of the logicistic reduction of arithmetic**

The proponents of the standpoint of logicism believe that they can deal with this problem in a completely different way. In discussing this standpoint we connect with our earlier consideration of logicism. There it was important to recognize that intuitive evidence even plays a role in deductive logic, and that the logical definition of cardinal number does not establish the specifically logical nature of the concept of cardinal number (as a concept of pure reflection) but rather is only a logical normalization of elementary structural concepts.

These reflections concern the demarcation of what is logical in the narrow sense from what is formal. The recognition of the formal element in logic, however, by no means resolves the methodological question of logicism. Logicism is concerned not only with the theoretical development of the science of inference, but, as already explained, it takes as its further task the reduction of all arithmetic to the formalism of logic. This reduction proceeds first via the introduction of cardinal numbers as properties of predicates, as already described, and then (as will not be described more precisely here) by expressing the construction of sets of numbers in terms of the logical formalism, replacing each set with a defining predicate. Thus the totality of predicates of numbers replaces the totality of sets of numbers.

In this way one in fact succeeds in assigning to every arithmetical sentence a sentence from the domain of theoretical logic in which, aside from variables, only “logical constants” occur, i. e. basic logical operations like conjunction, negation, the form of generality, etc.

Now it is clear that the problem of the infinite can not be solved solely by this translation of arithmetic into the formalism of logic. If theoretical logic

<sup>11</sup>We suggest here using this expression, which Cantor used specifically with respect to the construction of sets, more generally with respect to any theoretical approach.

deductively obtains the system of arithmetic, then its procedures must include either explicit or hidden assumptions through which the actual infinite is introduced.

The justification that is given for these assumptions, and the position adopted with respect to them, has been the weak point of logicism from the start. Indeed, Frege and Dedekind, whose proofs and discussions displayed extreme precision and rigor everywhere else, were relatively unconcerned about the supposed self-evident assumptions they took as the basis for the standpoint of general logic, namely the idea of a closed totality of all conceivable logical objects whatsoever.

If this idea were tenable, it would of course be more satisfactory from a systematic point of view than the more specialized postulates of arithmetic. But, as is well known, it had to be dropped, because of the contradictions to which it lead. Since then logicism has forgone proving the existence of an infinite totality, and has instead explicitly postulated an *axiom of infinity*.

This axiom of infinity, however, is not a sufficient assumption for obtaining arithmetic as logically construed. We could only obtain with it what follows from our first postulate, the admissibility of existential inference with respect to the integers. To conform with our second postulate we still require something further, namely, the application of existential inference *with respect to predicates*. The justification of this way of proceeding might at first seem to be logically self-evident, and in fact it is not questioned under the conception of Frege and Dedekind. But once the idea of the totality of all logical objects is given up, the idea of the totality of all predicates becomes problematic as well, and here closer inspection reveals a particular, fundamental difficulty.

Namely, in accordance with the genuine logicist standpoint, we construe the totality of predicates as a totality which essentially first comes into existence in the framework of the system of logic by applying logical constructions to certain initial, prelogical predicates, e. g., predicates taken from intuition. Further predicates are now obtained by reference to the totality of predicates. An example is the already mentioned Fregean definition of finite number: “a number  $n$  is called finite if every predicate holds of  $n$  that holds of the number 0 and that, if it holds of a number  $a$  also holds of the succeeding number.” The predicate of finiteness is defined here by reference to the totality of all predicates.

Definitions of this kind, called “impredicative,”<sup>12</sup> occur everywhere in the foundation of arithmetic, and indeed, right in the crucial places.

Now there is really no objection to determining a thing from a totality by means of a property that refers to this totality. So, for example, in the totality of numbers a particular number is defined by the property of being the greatest prime number, such that its product with 1000 is greater than the product of the preceding prime number with 1001.<sup>13</sup>

But it is required here that the totality in question is determined *independently* of the definitions referring to it; otherwise we enter a vicious circle.

However, precisely in the case of the totality of predicates and the impredicative definitions referring to it, this precondition cannot be taken as directly satisfied. For the totality of predicates is determined according to the conception discussed here by the laws for logical constructions, and these include also impredicative definitions.

In order to avoid the vicious circle it would of course suffice to show that every predicate introduced by an impredicative definition can also be defined in a “predicative” way. Indeed, one could even get by with a weaker claim. Since in the logical foundation of arithmetic a predicate is always considered just with respect to its extension, i. e., with respect to the set of things of which it holds, we would only need to know that every predicate introduced by an impredicative definition *is extensionally equal* to a predicatively defined predicate.

This postulate, called the “axiom of reducibility,” was imposed along with the axiom of infinity by Russell, who recognized with total clarity the difficulty involved in impredicative definitions.

But how is this axiom of reducibility to be understood? From its for-

<sup>12</sup>The term is due to Poincaré who—in contrast to the other critics of set theory, almost all of whom concerned themselves just with the axiom of choice—brought the issue of impredicative definition into the discussion and put the emphasis on it. His criticism was disputable, however, because he made the use of impredicative definitions appear to be a novelty introduced by set theory. Zermelo could reply to him that impredicative definitions already occur essentially in the usual modes of inference in analysis that Poincaré in fact accepted.

Since then Russell and Weyl in particular have thoroughly discussed and completely clarified the role of impredicative definition in analysis.

<sup>13</sup>The example is chosen in such a way that the reference to the totality of numbers can not be eliminated directly as is the case in most of the simpler examples.

mulation it is not clear whether it expresses a logical law or an extra-logical assumption.

If, in the first case, in which the axiom of reducibility would be the expression of a logical law, its validity would have to be independent of the basic domain of prelogical initial predicates—at least assuming that this domain satisfies the axiom of infinity. But this would mean that the domain of predicates of an axiomatic theory in which the forms of the universal and the existential judgment (the existential reasoning) are applied only to objects and not to predicates cannot be enlarged by the introduction of impredicative definitions, provided only that the axiom system requires for its satisfaction an infinite system of objects.

But the correctness of such a statement is out of the question. One can easily construct examples which refute this claim.

Dedekind's introduction of the concept of number furnishes such an example. Dedekind starts with a system in which a thing 0 is distinguished and which permits a one-one mapping onto a subset not containing the thing 0. Suppose we represent this mapping by a predicate with two subjects and formulate the required properties of this predicate as axioms; we then get an elementary axiom system that contains in its axioms no reference to the totality of predicates and that, moreover, can be satisfied only by an infinite system of objects. Let us now consider Dedekind's concept of number; if we translate his definition from the language of set theory into that of the theory of predicates, it can be formulated in full analogy to Frege's definition of finite number: "a thing  $n$  of our system is a number if every predicate holds of  $n$  which holds of 0 and which, if it holds of a thing  $a$  in our system, holds also of the thing to which  $a$  is correlated by the one-one mapping." This definition is impredicative; and one can see that it is not possible to obtain a predicate that is extensionally equal to the hereby defined concept of "being a number," by a predicative definition from the basic elements of the theory.<sup>14</sup>

We find, therefore, that for the axiom of reducibility, only the second interpretation comes into consideration, according to which it expresses a *condition on the initial domain of prelogical predicates*.

By introducing such an assumption one abandons the conception that the domain of predicates is generated by logical processes. The aim of a genuinely logical theory of predicates is then given up.

If one decides to do this, then it seems more natural and appropriate to

<sup>14</sup>Another example was given by Waismann in a note on "The nature of the axiom of reducibility" (*vide* [?]). This, however, requires some modification.

return to the conception of a *logical function* that corresponds to Schröder's standpoint: one construes a logical function as an assignment of the values "true" and "false" to the objects of the domain of individuals. Each predicate defines such an assignment; but the totality of assignments of values is construed, in analogy with the finite, as a *combinatorial multiplicity which exists independently of conceptual definitions*.

This conception removes the circularity of the impredicative definitions of theoretical logic; we have only to replace any statement about the totality of predicates by the corresponding statement about the totality of logical functions. The axiom of reducibility is thus dispensable.

This step was actually taken by the logicist school at the suggestion of Wittgenstein and Ramsey. These two maintained in particular that in order to avoid the contradictions connected with the concept of the set of all mathematical objects it is not necessary to distinguish predicates by their definitions, as Whitehead and Russell had done in *Principia Mathematica*. Rather, they maintained, it suffices to delimit clearly the domains of definition of predicates, so that one distinguishes between the predicates of individuals, the predicates of predicates, the predicates of predicates of predicates, and so on.

In this way one has returned from the type theory of *Principia Mathematica* to the simpler conceptions of Cantor and Schröder.

However, one should not be deceived over the fact that with this change one has moved far away from the standpoint of logical self-evidence. The assumptions on which theoretical logic is then based are in principle of exactly the same kind as the basic postulates of analysis, and are also completely analogous to them in content. The axiom of infinity in the logical theory corresponds to the conception of the number sequence as an infinite totality; and in the logical theory one postulates the concept of all logical functions instead of the concept of all sets of numbers, whereby the functions refer to the "domain of individuals" or to a determinate domain of predicates.

Thus, when arithmetic is incorporated into the system of theoretical logic, nothing is saved in terms of assumptions. Contrary to what one might at first think, this incorporation by no means has the significance of a reduction of the postulates of arithmetic to lesser assumptions; its value is rather in the fact that the mathematical theory is placed on a broader basis by joining it with the logical formalism.

In this way the theory attains, first of all, a higher degree of methodological distinction, as follows. Not only do its assumptions result from a



natural extrapolation of intuitive numbers, but they are also obtained by *extrapolating the logic of extensions* to infinite totalities.

Moreover, by joining arithmetic with theoretical logic we gain an insight into the connection of the processes of set formation with the fundamental operations of logic; and the logical structure of concept formation and of inferences becomes clearer.

Thus, in particular, the meaning of the Principle of Choice becomes fully comprehensible only by means of the formalism of logic. We can express this principle in the following form: if  $B(x, y)$  is a two-place predicate (defined in a certain domain) and if for every thing  $x$  in the domain of definition there is at least one thing  $y$  in this domain for which  $B(x, y)$  holds, then there is (at least) one function  $y = f(x)$ , such that for every thing  $x$  in the domain of definition of  $B(x, y)$  the value  $f(x)$  is again in this domain, and is such that  $B(x, f(x))$  holds.

Let us consider what this assertion claims in the special case of a two element domain, the things of which we can represent by the numbers 0, 1. In this case there are only four different courses of values of functions  $y = f(x)$  to consider. Then the assertion is a simple application of one of the *distributive laws* governing the relation between conjunction and disjunction, i. e. the following theorem of elementary logic: “If  $A$  holds and if, in addition,  $B$  or  $C$  holds, then either  $A$  and  $B$  holds or  $A$  and  $C$  holds.”<sup>15</sup>

Also in the case of a subject domain consisting of any determinate finite number of things, the assertion of the Principle of Choice follows from this distributive law. The general assertion of the Principle of Choice is therefore nothing but the extension of a law of elementary logic for conjunction and disjunction to infinite totalities. And thus, the Principle of Choice supplements the logical rules governing universal and existential judgments, i. e. the rules of existential inference, for their application to infinite totalities signifies in the same way that certain elementary laws for conjunction and disjunction are being carried over to the infinite.

The Principle of Choice has a distinctive position with respect to these rules of existential inference only insofar as its formulation requires the *concept of function*. This concept, in turn, receives its sufficient implicit characterization only by means of the Principle of Choice.

This concept of function corresponds to the concept of logical function;

<sup>15</sup>“Or” is in both cases meant not in the sense of the exclusive “or” but in the sense of the Latin “vel.” But of course the theorem also holds for the exclusive “or.”

the only difference is that the values of the former are not taken to be “true” and “false” but the things of the subject domain. The totality of the functions that are being considered here is therefore the totality of all possible “self-assignments” of the subject domain.

According to this concept of function, the existence of a function with the property  $E$  in no way means that one can form a concept that uniquely determines a definite function with the property  $E$ . Consideration of this circumstance invalidates the usual objections to the Principle of Choice, which rests mostly on the fact that one is misled by the name “Principle of Choice” to the view that this principle asserts the possibility of a choice.

Moreover, we see that the assumption expressed by the Principle of Choice essentially respects the understanding upon which we must already base the method of theoretical logic, in order to interpret it in a non-circular way without introducing an axiom of reducibility.

Of course, we can also give the opposite emphasis to this observation: the controversial character of the Principle of Choice, the formulation of which is in line with the systematic elaboration of the standpoint of theoretical logic, brings most strongly to the fore what is problematic about this standpoint.

When we considered the logicist foundation of arithmetic we were also led to this result: the incorporation of arithmetic into theoretical logic provides indeed a broader foundation for the arithmetic theory and contributes to the contentual motivation of its assumptions; but it does not lead beyond the methodological standpoint of the conceptual approach, i.e. beyond the standpoint of axiomatics.

In this way the problem of the infinite is formulated, but it is not solved. For there remains the open question whether the analogies between the finite and the infinite, postulated as assumptions for the development of analysis and set theory, constitute an admissible approach, i.e. one which can be carried out consistently.

Intuitionism tries to avoid this question by excluding the problematic assumptions, while most logicians dispute its legitimacy by denying a fundamental difference between the finite and the infinite; *Hilbert’s proof theory* begins to address this question in a positive way.

## 4 Hilbert's proof theory

In order to better grasp the leading ideas of proof theory let us first call to mind once again the character of the problem to be solved here. At issue is proving the consistency of the mathematical idealization on which the theoretical edifice of arithmetic rests.

On the philosophical side, the question has frequently been raised whether a proof of consistency alone provides a justification for this idealization. This way of putting the question is however misleading; it does not take into account the fact that the scientific motivation for the theoretical approach of arithmetic has in essence already been provided by science, and that the proof of consistency is the only *desideratum* that remains to be fulfilled.

The edifice of arithmetic is built upon the foundation of ideas that are of greatest relevance for scientific systematization in general: namely the principle of *conservation* (“*permanence*”) of laws, which occurs here as the postulate of the unlimited applicability of the usual logical forms of judgment and inference, and the demand for a purely *objective* formulation of the theory, by which it is freed from all reference to our *cognition*.

The fundamental methodological significance of these requirements yields the *inner* motivation and distinctive character of the approach of the arithmetic theory.

In addition to this inner motivation we have the splendid corroboration of the conceptual system of arithmetic in the form of its deductive fruitfulness, its systematic success, and the coherence of its consequences. This conceptual system is clearly suited in a truly remarkable way to treating the relations of numbers and of magnitudes. The systematicity of this magnificent theory, obtained by combining function theory with number theory and algebra, has no equal. And as a comprehensive conceptual apparatus for the construction of scientific theories, arithmetic proves to be suited not only to the formulation and development of laws, but it has also been used with great success, and to an extent which had not been anticipated, in the search for laws.

Regarding the coherence of the consequences, it has been most strongly corroborated by the intensive theoretical development of analysis and its many numerical applications.

What is still lacking here is only this: that the merely empirical trust, gained by many trials, in the consistency of the arithmetic theory, i. e. in the thorough coherence of its results, be replaced by a real insight into this

consistency; to effect this is the purpose of a proof of consistency.

Thus it is not the case that the conceptual system of arithmetic is first to be established by means of a proof of its consistency. Rather, the sole purpose of this proof is to provide, with regard to this conceptual system (which is already motivated on internal systematic grounds and has proved itself as an intellectual tool in its applications), the evident certainty that it cannot be undermined by an inconsistency of its consequences.

If this succeeds, we will know that the idea of the actual infinite can be developed systematically. And we can rely on the results of applying the basic arithmetic postulates just as if we were in the position to verify them intuitively. For when we recognize the consistency of the application of these postulates, it follows immediately that their consequences, if they are intuitively, i. e. finitistically, meaningful, can never contradict an intuitively recognizable fact. In the case of finitist sentences, the ascertainment of their non-refutability is equivalent to the ascertainment of their truth.

From this consideration of the need for and the purpose of a consistency proof, it follows in particular that for such a proof only one thing matters; namely to recognize, in the literal sense of the word, the freedom from contradictions of the arithmetic theory, i. e., the *impossibility of its immanent refutation*.

The novel feature of Hilbert's approach was that he limited himself to this problem; previously, one had always carried out consistency proofs for axiomatic theories by positively exhibiting the simultaneous satisfaction of the axioms by certain objects. There was no basis for this method of exhibition in the case of arithmetic; in particular, Frege's idea of taking the objects to be exhibited from the domain of logic does not succeed, because, as we have recognized, the application of ordinary logic to the infinite is just as problematic as arithmetic, the consistency of which was to be shown. Indeed, the basic postulates of the arithmetic theory concern exactly the extended application of the usual forms of judgment and inference.

By focusing on this aspect, we are led directly to the *first guiding principle of Hilbert's proof theory*: it says that, in proving the consistency of arithmetic, we must consider the laws of logic as applied in arithmetic to be in the domain of what is to be shown consistent; thus, the proof of consistency covers *logic and arithmetic together*.

The first essential step in carrying out this idea is already taken by incorporating arithmetic into the system of theoretical logic. In virtue of this incorporation the task of proving the consistency of arithmetic reduces to

establishing the consistency of theoretical logic, or, in other words, determining the consistency of the axiom of infinity, of impredicative definitions, and of the Principle of Choice.

In this connection it is advisable to replace Russell's axiom of infinity with Dedekind's characterization of the infinite.

Russell's axiom of infinity requires the existence of an  $n$ -numbered predicate for every finite number  $n$  (in the sense of Frege's definition of finite cardinal) and thus implicitly requires also that the domain of individuals (the basic domain of things) be infinite. Now it is an unnecessary, and from a principled standpoint objectionable, complication that here three infinities in different layers run concurrently: that of the infinitely many things in the domain of individuals, that of the infinitely many predicates, and then that of the resulting infinitely many cardinals, which are after all defined as predicates of predicates.

We can avoid this multiplicity by determining the infinity of the domain of individuals not by an infinite series of unary predicates, but rather by a single binary predicate, namely a predicate that provides a one-one mapping of the domain of individuals onto a proper subdomain, i. e. a subdomain which excludes at least one thing. This characterization of the infinite, due to Dedekind, can be introduced in the most simple and elementary way if we do not postulate the one-one mapping by means of an existence axiom, but introduce it explicitly from the start by taking as basic elements of the theory an initial object and a basic process.

In this way, the numbers already occur as things in the domain of individuals, rather than as predicates of predicates of things.

However, this consideration already refers to the particular form of the systematic development, and there are several ways of pursuing it. But we must first orient ourselves as to how in general a proof of consistency in the intended sense can be carried out at all. This possibility is not immediately obvious. For how can one survey all possible consequences that follow from the assumptions of arithmetic or of theoretical logic?

Here the investigation of mathematical proofs by means of the logical calculus comes into play in a decisive way. This has shown that the methods of forming concepts and making inferences which are used in analysis and set theory are reducible to a limited number of processes and rules; thus, one succeeds in completely formalizing these theories in the framework of a precisely specified symbolism.

From the possibility of this formalization, which was originally done only for the sake of a more precise logical analysis of proof, Hilbert inferred the *second guiding idea of his proof theory*, namely that the task of proving the consistency of arithmetic is a *finitist problem*.

An inconsistency in the contentual theory must indeed show itself by means of the formalization in the following way: two formulas are derivable

according to the rules of the formalism, one of which results from the other through the process that is the formal image of negation. The claim of consistency is therefore equivalent to the claim that two formulas standing in the above relation can not be derived by the rules of the formalism. But this claim has fundamentally the same character as any general statement of finitist number theory, e. g., the statement that it is impossible to produce three integers  $a, b, c$  (different from 0) such that  $a^3 + b^3 = c^3$ .

The proof of consistency for arithmetic thus becomes in fact a finitist problem of the theory of inferences. The finitist investigation, having formalized theories of mathematics as its object, is called *metamathematics* by Hilbert. The task falling to metamathematics *vis-à-vis* the system of mathematics is analogous to the one which Kant ascribed to the critique of reason *vis-à-vis* the system of philosophy.

In accord with this methodological program, proof theory has already been developed to a substantial degree;<sup>16</sup> but there are still considerable mathematical difficulties to be overcome. The proofs of Ackermann and von Neumann secure the consistency of the first postulate of arithmetic, i. e., the applicability of existential reasoning to the integers. Ackermann developed in some detail an approach to the further problem of the consistency of the general concept of a set (resp. numerical function) of numbers together with a corresponding Principle of Choice.

If this problem were solved, then almost the entire domain of existing mathematical theories would be proved to be consistent.<sup>17</sup> This proof would in particular be sufficient to recognize the consistency of the geometric and physical theories.

One can also extend the problem still further and investigate the consistency of more inclusive systems, e. g., axiomatic set theory. Axiomatic set theory, as first formulated by Zermelo and supplemented and extended by

<sup>16</sup>Hilbert gave a first sketch of a theory of proofs already in his 1904 Heidelberg lecture “On the foundations of logic and arithmetic” (*vide* [?]). The first guiding idea of a joint treatment of logic and arithmetic is expressly formulated here; the methodological principle of the finitist standpoint is also intended, but not yet explicitly stated. The investigation of Julius Koenig, *New Foundations for Logic, Arithmetic, and Set Theory* (*vide* [?]), falls between this lecture and Hilbert’s more recent publications on proof theory; it comes very close to Hilbert’s standpoint and already gives a proof of consistency which is in full accord with proof theory. This proof covers only a very narrow domain of formal operations and is therefore only of methodological significance.

<sup>17</sup>Cantor’s theory of numbers of the second number class is also included here.

Fraenkel and von Neumann, with its construction processes, already goes far beyond what is actually used in mathematics; and the proof of its consistency would also establish the consistency of the system of theoretical logic.

This does not achieve an absolute completion of this formation of concepts, because formalized set theory motivates metamathematical considerations which have the formal constructions of set theory as their object and in this way go beyond these constructions.<sup>18</sup>

In spite of this possibility of extending the concept formation a formalized theory can nevertheless be closed in the following sense: no new results are obtained in the domain of the laws that can be formulated in terms of the concepts of the theory by extending the concept formation.

This condition is satisfied whenever the theory is *deductively closed*, i. e., when it is impossible to add a new axiom, which is expressible in terms of the concepts of the theory but not already derivable, without producing a contradiction, or, what amounts to the same thing: if every statement that can be formulated within the framework of the theory is either provable or refutable.<sup>19</sup>

We believe that number theory as delimited through Peano's axioms with the addition of definition by recursion is deductively closed in this sense, but the problem of giving an actual proof of this is still entirely unsolved. The question becomes even more difficult if we go beyond the domain of number theory and ascend to analysis and the further set theoretic concept formations.

In the realm of these and related questions there lies a considerable field of open problems. But these problems are not of such a kind that they represent an objection to the standpoint we have adopted. We must only keep in mind that the formalism of theorems and proofs that we use to represent our ideas does not coincide with the formalism of the structure

<sup>18</sup>The more detailed discussion of this point is connected to the *Richard paradox*, of which Skolem has recently given a more precise formulation. These considerations are not conclusive since they are made in the framework of a non-finitist metamathematics. A final answer to the question discussed here would be obtained only if one succeeded in producing in a finitist way a set of numbers which could be shown not to occur in axiomatic set theory.

<sup>19</sup>Notice that this requirement of being deductively closed does not go as far as the requirement that every question of the theory be *decidable*. The latter says that there should be a procedure for deciding for any arbitrarily given pair of contradictory claims belonging to the theory which of the two is provable ("correct").

that we intend in thought. The formalism suffices to formulate our ideas of infinite multiplicities and to draw the logical consequences from them; but in general it is not able to produce the multiplicity combinatorially out of itself, so to speak.

The position we have reached concerning the theory of the infinite can be viewed as a form of the philosophy of “as if.” It differs fundamentally from the Vaihinger’s philosophy thus designated, however, by placing weight on the consistency and the permanence of ideas; in contrast, Vaihinger considers the demand for consistency to be a prejudice and indeed claims that the contradictions in the infinitesimal calculus are “not only not to be disavowed, but . . . are precisely the means by which progress was attained.”<sup>20</sup>

Vaihinger’s considerations are focused exclusively on scientific *heuristics*. He considers only the “fictions” that occur as mere temporary aids for thinking. In introducing these fictions, thought does itself violence and their contradictory character (if we are dealing with “genuine fictions”) can be rendered harmless only by a skillful adjustment for the contradictions.

Idealizations in our sense are a permanent possession of the mind. They are distinguished forms of systematic extrapolation and of idealizing approximation to what is real. They are also by no means arbitrary nor yet forced upon thought; on the contrary, they constitute a world in which our thought is at home, and from which the human mind, absorbed in this world, gains satisfaction and joy.

### *Postscript*

In light of various insights that have been gained since the publication of the above essay, some of the considerations presented here have to be corrected.

First of all, as far as intuitionism is concerned, it was initially believed that the methodology of intuitionistic proofs agreed with that of Hilbert’s “finitist standpoint.” It has become clear, however, that the methods of intuitionism go beyond the finitist proof procedures intended by Hilbert. In particular, Brouwer uses the general concept of a contentual proof, to which the concept of “absurdity” is also connected, but which is not employed in finitist reasoning.

<sup>20</sup>Vaihinger, *The Philosophy of “As if”*, 2nd edition, ch. XII (*vide* [?], p. ■).



Then, as far as Hilbert's proof theory is concerned, the view that the consistency proof for arithmetic amounts to a finitist problem is justified only in the sense that the consistency statement can be formulated finitistically. This does not imply at all that the problem can be solved with finitist methods. By a theorem of Gödel the possibility of a finitist solution was made most implausible, though not directly excluded, already for number theory; moreover, it turned out that the above mentioned consistency proofs that were available at the time did not extend to the full formalism of number theory. The methodological standpoint of proof theory was consequently broadened, and various consistency proofs were carried out, first for formalized number theory and then also for formal systems of analysis; their methods, although not restricted to finitist, i. e., elementary combinatorial considerations, require neither the usual methods of existential reasoning, nor the general concept of contentual proof.

In connection with the theorem of Gödel mentioned above, the assumption that number theory, when axiomatically delimited and formalized, is deductively complete turned out to be incorrect. Even more generally, Gödel showed that formalized theories satisfying certain very general conditions of expressiveness and formal rigor cannot be deductively complete as long as they are consistent.

On the whole the situation is as follows: Hilbert's proof theory, together with the discovery of the possibility of formalizing mathematical theories, has opened a rich area of research, but the epistemological perspective which motivated its formulation has become problematic.

This suggests revising the epistemological remarks in the above essay. Of course, the positive remarks are hardly in need of revision, in particular those exhibiting the mathematical element in logic and emphasizing the evidence of elementary arithmetic. However, the sharp distinction between the intuitive and the non-intuitive, which was employed in the treatment of the problem of the infinite, apparently cannot be drawn so strictly, and the reflections on the formation of mathematical ideas still need to be worked out in more detail in this respect. Various considerations for this are contained in the following essays.<sup>a</sup>

<sup>a</sup>This refers to the remaining essays in the collection [?].