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Platonism in mathematics (1935)

Sur le platonisme dans les mathématiques

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With your permission, I shall now address you on the subject of the present situation in research in the foundations of mathematics. Since there remain open questions in this field, I am not in a position to paint a definitive picture of it for you. But it must be pointed out that the situation is not so critical as one could think from listening to those who speak of a foundational crisis. From certain points of view, this expression can be justified; but it could give rise to the opinion that mathematical science is shaken at its roots.

The truth is that the mathematical sciences are growing in complete security and harmony. The ideas of Dedekind, Poincaré, and Hilbert have been systematically developed with great success, without any conflict in the results.

It is only from the philosophical point of view that objections have been raised. They bear on certain ways of reasoning peculiar to analysis and set theory. These modes of reasoning were first systematically applied in giving a rigorous form to the methods of the calculus. According to them, the objects of a theory are viewed as elements of a totality such that one can reason as follows: for each property expressible using the notions of the theory, it is objectively determined whether there is or there is not an element of the totality which possesses this property. Similarly, it follows from this point of view that either all the elements of a set possess a given property, or there is at least one element which does not possess it.

An example of this way of setting up a theory can be found in Hilbert's axiomatization of geometry. If we compare Hilbert's axiom system to Euclid's, ignoring the fact that the Greek geometer fails to include certain necessary postulates, we notice that Euclid speaks of figures to be *constructed* whereas, for Hilbert, system of points, straight lines, and planes exist from the outset. Euclid postulates: one can join two points by a straight line; Hilbert states the axiom: given any two points, there exists a straight line on which both are situated. "Exists" refers here to existence in the system of straight lines.

This example already shows that the tendency of which we are speaking consists in viewing the objects as cut off from all links with the reflecting subject.

Since this tendency asserted itself especially in the philosophy of Plato, allow me to call it "platonism."

The value of platonistically inspired mathematical conceptions is that they furnish models of abstract imagination. These stand out by their simplicity and logical strength. They form representations which extrapolate from certain regions of experience and intuition.

Nonetheless, we know that we can arithmetize the theoretical systems of geometry and physics. For this reason, we shall direct our attention to platonism in arithmetic. But I am referring to arithmetic in a very broad sense, which includes analysis and set theory.

The weakest of the "platonistic" assumptions introduced by arithmetic is that of the totality of integers. The *tertium non datum* for integers follows from it; viz.: if P is a predicate of integers, either P is true of each number, or there is at least one exception.

By the assumption mentioned, this disjunction is an immediate consequence of the logical principle of the excluded middle; in analysis it is almost continually applied.

For example, it is by means of it that one concludes that for two real numbers a and b, given by convergent series, either $a = b$ or $a < b$ or $b < a$; and likewise: a sequence of positive rational numbers either comes as close as you please to zero or there is a positive rational number less than all the members of the sequence.

At first sight, such disjunctions seem trivial, and we must be attentive in order to notice that an assumption slips in. But analysis is not content with this modest variety of platonism; it reflects it to a stronger degree with respect to the following notions: set of numbers, sequence of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a "quasi-combinatorial" sense, by which I mean: in the sense of an analogy of the infinite to the finite.

Consider, for example, the different functions which assign to each member of the finite series $1, 2, \ldots, n$ a number of the same series. There are n^n functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions.

In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. We add to this the idea of the totality of these sets. Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. From this point of view, constructive definitions of specific functions, sequences, and sets are only ways to pick out an object which exists independently of, and prior to, the construction.

The axiom of choice is an immediate application of the quasi-combinatorial concepts in question. It is generally employed in the theory of real numbers in the following special form. Let

$$M_1, M_2 \ldots$$

be a sequence of non-empty sets of real numbers, then there is a sequence

$$a_1, a_2 \ldots$$

such that for every index n, a_n is an element of M_n.

The principle becomes subject to objections if the effective construction of the sequence of numbers is demanded.

A similar case is that of Poincaré's impredicative definitions. An impredicative definition of a real number appeals to the hypothesis that all real numbers have a certain property P, or the hypothesis that there exists a real number with the property T.

This kind of definition depends on the assumption of the existence of the totality of sequences of integers, because a real number is represented by a decimal fraction, that is to say, by a special kind of sequence of integers.

It is used in particular to prove the fundamental theorem that a bounded set of real numbers always has a least upper bound.

In Cantor's theories, platonistic conceptions extend far beyond those of the theory of real numbers. This is done by iterating the use of the quasi-combinatorial concept of a function and adding methods of collection. This is the well-known method of set theory.

The platonistic conceptions of analysis and set theory have also been applied in modern theories of algebra and topology, where they have proved very fertile.

This brief summary will suffice to characterize platonism and its application to mathematics. This application is so widespread that it is not an exaggeration to say that platonism reigns today in mathematics.

But on the other hand, we see that this tendency has been criticized in principle since its first appearance and has given rise to many discussions. This criticism was reinforced by the paradoxes discovered in set theory, even though these antinomies refute only extreme platonism.

We have set forth only a restricted platonism which does not claim to be more than, so to speak, an ideal projection of a domain of thought. But the matter has not rested there. Several mathematicians and philosophers interpret the methods of platonism in the sense of conceptual realism, postulating the existence of a world of ideal objects containing all the objects and relations of mathematics. It is this absolute platonism which has been shown untenable by the antinomies, particularly by those surrounding the Russell-Zermelo paradox.

If one hears them for the first time, these paradoxes in their purely logical form can seem to be plays on words without serious significance. Nonetheless one must consider that these abbreviated forms of the paradoxes are obtained by following out the consequences of the various requirements of absolute platonism.

The essential importance of these antinomies is to bring out the impossibility of combining the following two things: the idea of the totality of all mathematical objects and the general concepts of set and function; for the totality itself would form a domain of elements for sets, and arguments and values for functions.

We must therefore give up absolute platonism. But it must be observed that this is almost the only injunction which follows from the paradoxes. Some will think that this is regrettable, since the paradoxes are appealed to on every side. But avoiding the paradoxes does not constitute a univocal program. In particular, restricted platonism is not touched at all by the antinomies.

Still, the critique of the foundations of analysis receives new impetus from this source, and among the different possible ways of escaping from the paradoxes, eliminating platonism offered itself as the most radical.

Let us look and see how this elimination can be brought about. It is done in two steps, corresponding to the two essential assumptions introduced by platonism. The first step is to replace by constructive concepts the concepts of a set, a sequence, or a function, which I have called quasi-combinatorial. The idea of an infinity of independent determinations is rejected. One emphasizes that an infinite sequence or a decimal fraction can be given only by an arithmetical law, and one regards the continuum as a set of elements defined by such laws.

This procedure is adapted to the tendency toward a complete arithmetization of analysis. Indeed, it must be conceded that the arithmetization of analysis is not carried through to the end by the usual method. The conceptions which are applied there are not completely reducible, as we have seen, to the notion of integer and logical concepts.

Nonetheless, if we pursue the thought that each real number is defined by an arithmetical law, the idea of the totality of real numbers is no longer indispensable, and the axiom of choice is not at all evident. Also, unless we introduce auxiliary assumptions—as Russell and Whitehead do—we must do without various usual conclusions. Weyl has made these consequences very clear in his book *The Continuum* (*vide* [**?**]).

Let us proceed to the second step of the elimination. It consists in renouncing the idea of the totality of integers. This point of view was first defended by Kronecker and then developed systematically by Brouwer.

Although several of you heard an authentic exposition of this method by Professor Brouwer himself in March 1934, I shall allow myself a few words of explanation.

A misunderstanding about Kronecker must first be dispelled, which could arise from his often-cited aphorism that the integers were created by God, whereas everything else in mathematics is the work of man. If that were really Kronecker's opinion, he ought to admit the concept of the totality of integers.

In fact, Kronecker's method, as well as that of Brouwer, is characterized by the fact that it avoids the supposition that there exists a series of natural numbers forming a determinate ideal object.

According to Kronecker and Brouwer, one can speak of the series of numbers only in the sense of a process that is never finished, surpassing each limit which it reaches.

This point of departure carries with it the other divergences, in particular those concerning the application and interpretation of logical forms: neither a general judgment about integers nor a judgment of existence can be interpreted as expressing a property of the series of numbers. A general theorem about numbers is to be regarded as a sort of prediction that a property will present itself for each construction of a number; and the affirmation of the existence of a number with a certain property is interpreted as an incomplete communication of a more precise proposition indicating a particular number having the property in question or a method for obtaining such a number; Hilbert calls it a "partial judgment."

For the same reasons the negation of a general or existential proposition about integers does not have precise sense. One must strengthen the negation to arrive at a mathematical proposition. For example, it is to give a strengthened negation of a proposition affirming the existence of a number with a property P to say that a number with the property P cannot be given, or further, that the assumption of a number with this property leads to a contradiction. But for such strengthened negations the law of the excluded middle is no longer applicable.

The characteristic complications to be met with in Brouwer's "intuitionistic" method come from this.

For example, one may not generally make use of disjunctions like these: a series of positive terms is either convergent or divergent; two convergent sums represent either the same real number or different ones.

In the theory of integers and of algebraic numbers, we can avoid these difficulties and manage to preserve all the essential theorems and arguments.

In fact, Kronecker has already shown that the core of the theory of algebraic fields can be developed from his methodological point of view without appeal to the totality of integers.[1]

As for analysis, you know that Brouwer has developed it in accord with the requirements of intuitionism. But here one must abandon a number of the usual theorems, for example, the fundamental theorem that every continuous function has a maximum in a closed interval. Very few things in set theory remain valid in intuitionist mathematics.

[1] To this end, Kronecker set forth in his lectures a manner of introducing the notion of algebraic number which has been almost totally forgotten, although it is the most elementary way of defining this notion. This method consists in representing algebraic numbers by the changes of sign of irreducible polynomials in one variable with whole rationals as coefficients; starting from that definition, one introduces the elementary operations and relations of magnitude for algebraic numbers and proves that the ordinary laws of calculation hold; finally one shows that a polynomial with algebraic coefficients having values with different signs for two algebraic arguments a and b has a zero between a and b.

We would say, roughly, that intuitionism is adapted to the theory of numbers; the semiplatonistic method, which makes use of the idea of the totality of integers but avoids quasi-combinatorial concepts, is adapted to the arithmetic theory of functions, and the usual platonism is adequate for the geometric theory of the continuum.

There is nothing astonishing about this situation, for it is a familiar procedure of the contemporary mathematician to restrict his assumptions in each domain of the science to those which are essential. By this restriction, a theory gains methodological clarity, and it is in this direction that intuitionism proves fruitful.

But as you know, intuitionism is not at all content with such a role; it opposes the usual mathematics and claims to represent the only true mathematics.

On the other hand, mathematicians generally are not at all ready to exchange the well-tested and elegant methods of analysis for more complicated methods unless there is an overriding necessity for it.

We must discuss the question more deeply. Let us try to portray more distinctly the assumptions and philosophic character of the intuitionistic method.

What Brouwer appeals to is evidence. He claims that the basic ideas of intuitionism are given to us in an evident manner by pure intuition. In relying on this, he reveals his partial agreement with Kant. But whereas for Kant there exists a pure intuition with respect to space and time, Brouwer acknowledges only the intuition of time, from which, like Kant, he derives the intuition of number.

As for this philosophic position, it seems to me that one must concede to Brouwer two essential points: first, that the concept of integer is of intuitive origin. In this respect nothing is changed by the investigations of the logicists, to which I shall return later. Second, one ought not to make arithmetic and geometry correspond in the manner in which Kant did. The concept of number is more elementary than the concepts of geometry.

Still it seems a bit hasty to deny completely the existence of a geometrical intuition. But let us leave that question aside here; there are other, more urgent ones. Is it really certain that the evidence given by arithmetical intuition extends exactly as far as the boundaries of intuitionist arithmetic would require? And finally: is it possible to draw an exact boundary between what is evident and what is only plausible?

I believe that one must answer these two questions negatively. To begin with, you know that even scholars do not agree about evidence in general. Also, the same person sometimes rejects suppositions which he previously regarded as evident.

An example of a much-discussed question of evidence, about which there has been controversy up to the present, is that of the axiom of parallels. I think that the criticism which has been directed against that axiom is partly explained by the special place that it has in Euclid's system. Various other axioms had been omitted, so that the parallels axiom stood out from the others by its complexity.

In this matter I shall be content to point out the following: one can have doubts concerning the evidence of geometry, holding that it extends only to topological facts or to the facts expressed by the projective axioms. One can, on the other hand, claim that geometric intuition is not exact. These opinions are self-consistent, and all have arguments in their favor. But to claim that metric geometry has an evidence restricted to the laws common to Euclidean and Bolyai-Lobachevskian geometry, an exact metrical evidence which yet would not guarantee the existence of a perfect square, seems to me rather artificial. And yet it was the point of view of a number of mathematicians.

Our concern here has been to underline the difficulties to be encountered in trying to describe the limits of evidence.

Nevertheless, these difficulties do not make it impossible that there should be anything evident beyond question, and certainly intuitionism offers some such. But does it confine itself completely within the region of this elementary evidence? This is not completely indubitable, for the following reason: Intuitionism makes no allowance for the possibility that, for very large numbers, the operations required by the recursive method of constructing numbers can cease to have a concrete meaning. From two integers k, l one passes immediately to k^l; this process leads in a few steps to numbers which are far larger than any occurring in experience, e. g.,

$$67^{\left(257^{729}\right)}.$$

Intuitionism, like ordinary mathematics, claims that this number can be represented by an Arabic numeral. Could one not press further the criticism that intuitionism makes of existential assertions and raise the question: what does it mean to claim the existence of an Arabic numeral for the foregoing number, since in practice we are not in a position to obtain it?

Brouwer appeals to intuition, but one can doubt that the evidence for it really is intuitive. Isn't this rather an application of the general method of analogy, consisting in extending to inaccessible numbers the relations which we can concretely verify for accessible numbers? As a matter of fact, the reason for applying this analogy is strengthened by the fact that there is no precise boundary between the numbers which are accessible and those which are not. One could introduce the notion of a "practicable" procedure, and implicitly restrict the import of recursive definitions to practicable operations. To avoid contradictions, it would suffice to abstain from applying the principle of the excluded middle to the notion of practicability. But such abstention goes without saying for intuitionism.

I hope I shall not be misunderstood: I am far from recommending that arithmetic be done with this restriction. I am concerned only to show that intuitionism takes as its basis propositions which one can doubt and in principle do without, although the resulting theory would be rather meager.

It is therefore not absolutely indubitable that the domain of complete evidence extends to all of intuitionism. On the other hand, several mathematicians recognize the complete evidence of intuitionistic arithmetic and moreover maintain that the concept of the series of numbers is evident in the following sense: The affirmation of the existence of a number does not require that one must, directly or recursively, give a bound for this number. Besides, we have just seen how far beyond a really concrete presentation such a limitation would be.

In short, the point of view of intuitive evidence does not decide uniquely in favor of intuitionism.

In addition, one must observe that the evidence which intuitionism uses in its arguments is not always of an immediate character. Abstract reflections are also included. In fact, intuitionists often use statements, containing a general hypothesis, of the form "if every number n has the property $A(n)$, then B holds."

Such a statement is interpreted intuitionistically in the following manner: "If it is proved that every number n possesses the property $A(n)$, then B." Here we have a hypothesis of an abstract kind, because since the methods of demonstration are not fixed in intuitionism, the condition that something is proved is not intuitively determined.

It is true that one can also interpret the given statement by viewing it as a partial judgment, i. e., as the claim that there exists a proof of B from the given hypothesis, a proof which would be effectively given. (This is approximately the sense of Kolmogorov's interpretation of intuitionism.) In any case, the argument must start from the general hypothesis, which cannot be intuitively fixed. It is therefore an abstract reflection.

In the example just considered, the abstract part is rather limited. The abstract character becomes more pronounced if one superposes hypotheses; i. e., when one formulates propositions like the following: "If from the hypothesis that $A(n)$ is valid for every n, one can infer B, then C holds," or "If from the hypothesis that A leads to a contradiction, a contradiction follows, then B," or briefly "If the absurdity of A is absurd, then B." This abstractness of statements can be still further increased.

It is by the systematic application of these forms of abstract reasoning that Brouwer has gone beyond Kronecker's methods and succeeded in establishing a general intuitionistic logic, which has been systematized by Heyting.

If we consider this intuitionistic logic, in which the notions of consequence are applied without reservation, and we compare the method used here with the usual one, we notice that the characteristic general feature of intuitionism is not that of being founded on pure intuition, but rather that of being founded on the relation of the reflecting and acting subject to the whole development of science.

This is an extreme methodological position. It is contrary to the customary manner of doing mathematics, which consists in establishing theories detached as much as possible from the thinking subject.

This realization leads us to doubt that intuitionism is the sole legitimate method of mathematical reasoning. For even if we admit that the tendency away from the thinking subject has been pressed too far under the reign of platonism, this does not lead us to believe that the truth lies in the opposite extreme. Keeping both possibilities in mind, we shall rather aim to bring about in each branch of science, an adaptation of method to the character of the object investigated.

For example, for number theory the use of the intuitive concept of a number is the most natural. In fact, one can thus establish the theory of numbers without introducing an axiom, such as that of complete induction, or axioms of infinity like those of Dedekind and Russell.

Moreover, in order to avoid the intuitive concept of number, one is led to introduce a more general concept, like that of a proposition, a function, or an arbitrary correspondence, concepts which are in general not objectively defined. It is true that such a concept can be made more definite by the axiomatic method, as in axiomatic set theory, but then the system of axioms is quite complicated.

You know that Frege tried to deduce arithmetic from pure logic by viewing the latter as the general theory of the universe of mathematical objects. Although the foundation of this absolutely platonistic enterprise was undermined by the Russell-Zermelo paradox, the school of logicists has not given up the idea of incorporating arithmetic in a system of logic. In place of absolute platonism, they have introduced some initial assumptions. But because of these, the system loses the character of pure logic.

In the system of *Principia Mathematica*, it is not only the axioms of infinity and reducibility which go beyond pure logic, but also the initial conception of a universal domain of individuals and of a domain of predicates. It is really an *ad hoc* assumption to suppose that we have before us the universe of things divided into subjects and predicates, ready-made for theoretical treatment.

But even with such auxiliary assumptions, one cannot successfully incorporate the whole of arithmetic into the system of logic. For, since this system is developed according to fixed rules, one would have to be able to obtain by means of a fixed series of rules all the theorems of arithmetic. But this is not the case; as Gödel has shown, arithmetic goes beyond each given formalism. (In fact, the same is true of axiomatic set theory.)

Besides, the desire to deduce arithmetic from logic derives from the traditional opinion that the relation of logic to arithmetic is that of general to particular. The truth, it seems to me, is that mathematical abstraction does not have a lesser degree than logical abstraction, but rather a different direction.

These considerations do not detract at all from the intrinsic value of that research of logicists, which aims at developing logic systematically and formalizing mathematical proofs. We were concerned here only with defending the thesis that for the theory of numbers, the intuitive method is the most suitable.

On the other hand, for the theory of the continuum, given by analysis, the intuitionist method seems rather artificial. The idea of the continuum is a geometrical idea which analysis expresses in terms of arithmetic.

Is the intuitionist method of representing the continuum better adapted to the idea of the continuum than the usual one?

Weyl would have us believe this. He reproaches ordinary analysis for decomposing the continuum into single points. But isn't this reproach better addressed to semiplatonism, which views the continuum as a set of arithmetical laws? The fact is that for the usual method there is a completely satisfactory analogy between the manner in which a particular point stands out from the continuum and the manner in which a real number defined by an arithmetical law stands out from the set of all real numbers, whose elements are in general only implicitly involved, by virtue of the quasi-combinatorial concept of a sequence.

This analogy seems to me to agree better with the nature of the continuum than that which intuitionism establishes between the fuzzy character of the continuum and the uncertainties arising from unsolved arithmetical problems.

It is true that in the usual analysis the notion of a continuous function, and also that of a differentiable function, have a generality going far beyond our intuitive representation of a curve. Nevertheless in this analysis, we can establish the theorem of the maximum of a continuous function and Rolle's theorem, thus rejoining the intuitive conception.

The intuitionist method, even though it begins with a much more restricted notion of a function, does not arrive at such simple theorems; they must instead be replaced by more complex ones. This stems from the fact that on the intuitionistic conception, the continuum does not have the character of a totality, which undeniably belongs to the geometric idea of the continuum. And it is this characteristic of the continuum which would resist perfect arithmetization.

These considerations lead us to observe that the duality of arithmetic and geometry is not unrelated to the opposition between intuitionism and platonism. The concept of number appears in arithmetic. It is of intuitive origin, but then the idea of the totality of numbers is superimposed. On the other hand, in geometry the platonistic idea of space is primordial, and it is against this background that the intuitionist procedures of constructing figures take place.

This suffices to show that the two tendencies, intuitionist and platonist, are both necessary; they complement each other, and it would be doing oneself violence to renounce one or the other.

But the duality of these two tendencies, like that of arithmetic and geometry, is not a perfect symmetry. As we have noted, it is not proper to make arithmetic and geometry correspond completely: the idea of number is more immediate to the mind than the idea of space. Likewise, we must recognize that the assumptions of platonism have a transcendent character which is not found in intuitionism.

It is also this transcendent character which requires us to take certain precautions in regard to each platonistic assumption. For even when such a supposition is not at all arbitrary and presents itself naturally to the mind, it can still be that the principle from which it proceeds permits only a restricted application, outside of which one would fall into contradiction.

We must be all the more careful in the face of this possibility, since the drive for simplicity leads us to make our principles as broad as possible. And the need for a restriction is often not noticed.

This was the case, as we have seen, for the principle of totality, which was pressed too far by absolute platonism. Here it was only the discovery of the Russell-Zermelo paradox which showed that a restriction was necessary.

Thus it is desirable to find a method to make sure that the platonistic assumptions on which mathematics is based do not go beyond permissible limits. The assumptions in question reduce to various forms of the principle of totality and of the principle of analogy or of the permanence of laws. And the condition restricting the application of these principles is none other than that of the consistency of the consequences which are deduced from the fundamental assumptions.

As you know, Hilbert is trying to find ways of giving us such assurances of consistency, and his proof theory has this as its goal.

This theory relies in part on the results of the logicists. They have shown that the arguments applied in arithmetic, analysis, and set theory can be formalized. That is, they can be expressed in symbols and as symbolic processes which unfold according to fixed rules. To primitive propositions correspond initial formulae, and to each logical deduction corresponds a sequence of formulae derivable from one another according to given rules. In this formalism, a platonistic assumption is represented by an initial formula or by a rule establishing a way of passing from formulae already obtained to others. In this way, the investigation of the possibilities of proof reduces to problems like those which are found in elementary number theory. In particular, the consistency of the theory will be proved if one succeeds in proving that it is impossible to deduce two mutually contradictory formulae A and $\overline{A}$ (with the bar representing negation). This statement which is to be proved is of the same structure as that, for example, asserting the impossibility of satisfying the equation $a^2 = 2b^2$ by two integers a and b.

Thus by symbolic reduction, the question of the consistency of a theory reduces to a problem of an elementary arithmetical character.

Starting from this fundamental idea, Hilbert has sketched a detailed program of a theory of proof, indicating the leading ideas of the arguments for the main consistency proofs. His intention was to confine himself to intuitive and combinatorial considerations; his "finitist point of view" was restricted to these methods.

In this framework, the theory was developed up to a certain point. Several mathematicians have contributed to it: Ackermann, von Neumann, Skolem, Herbrand, Gödel, Gentzen.

Nonetheless, these investigations have remained within a relatively restricted domain. In fact, they did not even reach a proof of the consistency of the axiomatic theory of integers. It is known that the symbolic representation of this theory is obtained by adding to the ordinary logical calculus formalizations of Peano's axioms and the recursive definitions of sum $(a+b)$ and product $(a \cdot b)$.

Light was shed on this situation by a general theorem of Gödel, according to which a proof of the consistency of a formalized theory cannot be represented by means of the formalism considered. From this theorem, the following more special proposition follows: it is impossible to prove by elementary combinatorial methods the consistency of a formalized theory which can express every elementary combinatorial proof of an arithmetical proposition.

Now it seems that this proposition applies to the formalism of the axiomatic theory of numbers. At least, no attempt made up to now has given us any example of an elementary combinatorial proof which cannot be expressed in this formalism, and the methods by which one can, in the cases considered, translate a proof into the aforementioned formalism, seem to suffice in general.

Assuming that this is so,[2] we arrive at the conclusion that means more powerful than elementary combinatorial methods are necessary to prove the consistency of the axiomatic theory of numbers. A new discovery of Gödel and Gentzen leads us to such a more powerful method. They have shown (independently of one another) that the consistency of intuitionist arithmetic implies the consistency of the axiomatic theory of numbers. This result was obtained by using Heyting's formalization of intuitionist arithmetic and logic. The argument is conducted by elementary methods, in a rather simple

[2]In trying to demonstrate the possibility of translating each elementary combinatorial proof of an arithmetical proposition into the formalism of the axiomatic theory of numbers, we are confronted with the difficulty of delimiting precisely the domain of elementary combinatorial methods.

manner. In order to conclude from this result that the axiomatic theory of numbers is consistent, it suffices to assume the consistency of intuitionist arithmetic.

This proof of the consistency of axiomatic number theory shows us, among other things, that intuitionism, by its abstract arguments, goes essentially beyond elementary combinatorial methods.

The question that now arises is whether the strengthening of the method of proof theory obtained by admitting the abstract arguments of intuitionism would put us into a position to prove the consistency of analysis. The answer would be very important and even decisive for proof theory, and even, it seems to me, for the role which is to be attributed to intuitionistic methods.

Research in the foundations of mathematics is still developing. Several basic questions are open, and we do not know what we shall discover in this domain. But these investigations excite our curiosity by their changing perspectives, and that is a sentiment which is not aroused to the same degree by the more classical parts of science, which have attained greater perfection.

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