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Problems of theoretical logic (1927)

Probleme der theoretischen Logik

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The topic of the lecture and its title were chosen in the spirit of Hilbert. What is here called theoretical logic is usually referred to as symbolic logic, mathematical logic, algebra of logic, or logical calculus. The purpose of the following remarks is to present this research area in a way that justifies calling it theoretical logic.

Mathematical logic is in general not very popular. It is most often regarded as a mere triviality that neither supports effectively practical inference nor contributes significantly to our logical insights.

To begin with, the charge of triviality is only justified with regard to the initial treatment of mathematical logic. The main emphasis was initially put on the formal analogy to algebra, and the pursuit of the latter was often considered as an end in itself. But this was the state of affairs decades ago, and today the problems of mathematical logic are inseparably intertwined with the questions concerning the foundations of the exact sciences, so that one can no longer speak of a trivial character.

Secondly, concerning the application to practical inference, it has to be mentioned first that a symbolic calculus promises advantages only to someone who has sufficient practice in using it. But, in addition, one has to consider that—in contrast to most kinds of symbolisms which serve, after all, the purpose of abbreviating and contracting operations—it is the primary task of the logical calculus to decompose the inferences into their ultimate constituents and to make outwardly evident each individual step and bring it thereby into focus. The main interest connected with the application of the logical calculus is consequently not one of technique, but of theory and principle. This leads me to the third charge; namely that mathematical logic does not significantly further our logical insights. This opinion is connected with the view of logic expressed by Kant in the second preface to the *Critique of Pure Reason*, where he says: "It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine."[a]

It is my intention to show that this standpoint is erroneous. To be sure, Aristotle's formulation of the ultimate principles of inference and their immediate consequences constitutes one of the most significant intellectual accomplishments; it is also one of the very few accomplishments which belong to the permanently secured part of the realm of philosophical knowledge. This fact will continue to receive its full due. But this does not prevent us from ascertaining that traditional logic, in posing its problems, is essentially open-ended, and in arranging its facts it is insufficiently adapted to the needs of either a systematic overview or of methodological and critical epistemological insights. Only the newer logic, as it has developed under the name of algebra of logic or mathematical logic, has introduced such concept formations and such an approach to formal logic as makes it possible to satisfy these needs of systematics and of philosophy.

The realm of logical laws, the world of abstract relations, has only thereby been revealed to us in its formal structure, and the relationship of mathematics and logic has been illuminated in a new way. I will try briefly to give an idea of this transformation and of the results it has brought to light.

In doing so I will not be concerned with presenting the historical development or the various forms in which mathematical logic has been pursued. Instead, I want to choose a presentation of the new logic that best facilitates relating and comparing it to traditional logic. As for logical symbols, I shall use the symbolism Hilbert employs now in his lectures and publications.

Traditional logic subdivides its problems into the investigation of concept formation, of judgment, and of inference. It is not advantageous to begin with concept formation, because its essential forms are not elementary but are already based on judgments. Let us begin, therefore, with judgment.

[a] [**?**], p. 17.

Here, the newer logic immediately introduces an essentially new vantage point, replacing classifications by the search for elementary logical operations. One does not speak of the categorical or the hypothetical or the negative judgment, but of the categorical or hypothetical connection, of negation as a logical operation. In the same way, one does not classify judgments into universal and particular ones but introduces logical operators for universality and particularity.

This approach is more appropriate than that of classification for the following reason. In judgments different logical processes generally occur in combination, so that a unique corresponding classification is not possible at all.

First let us consider the *categorical* relationship, i. e. that of subject and predicate. We have here an object and a proposition about it. The symbolic representation for this is

$$P(x),$$

to be read as:

"x has the property P."

The relation of the predicate to an object is here explicitly brought out by the variable. This is merely a clearer kind of notation; however, the remark that *several objects* can be subjects of a proposition is crucial. In that case one speaks of a *relation* between several objects. The notation for this is

$$R(x, y),\ \text{ or } R(x, y, z),\ \text{ etc.}$$

Cases and prepositions are used in ordinary language to indicate the different members of relations.

By taking relations into account, logic is extended in an essential way when compared with its traditional form. I shall speak about the significance of this extension when discussing the theory of inference.

The forms of universality and particularity are based on the categorical relationship. Universality is represented symbolically by

$$(x)P(x)$$

"all x have the property P."

The variable x appears here as a "bound variable;" the proposition does not depend on x—in the same way as the value of an integral does not depend on the variable of integration.

We sharpen the particular judgment first by replacing the somewhat indefinite proposition, "some x have the property P," with the existential judgment:

"there is an x with the property P,"

written symbolically:

$$(Ex)P(x).$$

By adding *negation*, the four types of judgment are obtained which are denoted in Aristotelian logic by the letters "a, e, i, o."

If we represent negation by putting a bar over the expression to be negated, then we obtain the following representations for the four types of judgment:

$$\begin{array}{rrl} \text{a}: & (x) & P(x) \\ \text{e}: & (x) & \overline{P(x)} \\ \text{i}: & (Ex) & P(x) \\ \text{o}: & (Ex) & \overline{P(x)}. \end{array}$$

Already here, in the theory of "opposition," it proves useful for the comprehension of matters to separate the operations; thus we recognize, for example, that the difference between contradictory and contrary opposition lies in the fact that in the former case the whole proposition, e. g., $(x)P(x)$, is negated, whereas in the latter case only the predicate $P(x)$ is negated.

Let us now turn to the *hypothetical relationship.*

$$A \rightarrow B \qquad \text{"if } A\text{, then } B\text{."}$$

This includes a connection of *two* propositions (predications). So the members of this connection already have the form of propositions, and the hypothetical relationship applies to these propositions as *undivided units.* The latter already holds also for the negation $\overline{A}$.

There are still other such propositional connections, in particular:

the fact that A and B *both hold*: $A \,\&\, B$,

and further, the *disjunctive connection*; there we have to distinguish between the exclusive "or," in the sense of the Latin "aut-aut," and the "or" in the sense of "vel." In accordance with Russell's notation this latter connection is represented by $A \vee B$.

In ordinary language, such connections are expressed with the help of conjunctions.

A consideration analogous to that used in the theory of opposition suggests itself here, namely to combine the binary propositional connections with negation in one of two ways, either by negating the individual members of the connection or by negating the latter as a whole. And now, let us see what dependency relations result.

To indicate that two connections have materially the same meaning (are "equivalent"), I will write "eq" between them (though, "eq" is not a sign of our logical symbolism).

In particular the following connections and equivalences result:

$$
\begin{array}{rcl}
\overline{A} \,\&\, \overline{B} & : & \text{“neither } A \text{ nor } B\text{”} \\
\overline{A \,\&\, B} & : & \text{“}A \text{ and } B \text{ exclude each other”} \\
\overline{A \,\&\, B} & \text{eq} & \overline{A} \vee \overline{B} \\
 & \text{eq} & A \rightarrow \overline{B} \\
 & \text{eq} & B \rightarrow \overline{A} \\
\overline{A} \rightarrow B & \text{eq} & A \vee B \\
\overline{\overline{B}} & \text{eq} & B
\end{array}
$$

(double negation is equivalent to affirmation).

From this it furthermore follows:

$$
\begin{array}{rcl}
A \rightarrow B & \text{eq} & \overline{A \,\&\, \overline{B}} \\
 & \text{eq} & \overline{A} \vee B \\
\overline{A \vee B} & \text{eq} & \overline{\overline{A} \rightarrow B} \\
 & \text{eq} & \overline{A} \,\&\, \overline{B}.
\end{array}
$$

On the basis of these equivalences it is possible to express some of the logical connections

$$\overline{}\,,\ \rightarrow,\ \&,\ \vee$$

by means of others. In fact, according to the above equivalences one can express

$$
\begin{array}{ccl}
\rightarrow & \text{by} & \vee \text{ and } \overline{} \\
\vee & \text{by} & \& \text{ and } \overline{} \\
\& & \text{by} & \rightarrow \text{ and } \overline{}
\end{array}
$$

so that each of

$$\begin{array}{rl} & \& \text{ and } \overline{} \\ \text{resp.} & \vee \text{ and } \overline{} \\ \text{resp.} & \rightarrow \text{ and } \overline{} \end{array}$$

alone suffice as basic connections. One can even get along with a single basic connection, but, to be sure, not with one of those for which we already have a sign. If we introduce for the connection of mutual exclusion $\overline{A \,\&\, B}$ the sign $A|B$ then the following equivalences hold:

$$\begin{array}{rll} A|A & \text{eq} & \overline{A} \\ A|\overline{B} & \text{eq} & \overline{A \,\&\, \overline{B}} \\ & \text{eq} & A \rightarrow B. \end{array}$$

This shows that with the aid of this connection one can represent negation as well as $\rightarrow$ and, consequently, the remaining connections. Just like the relation of mutual exclusion also the connection

$$\text{"neither — nor"} \quad \overline{A} \,\&\, \overline{B}$$

can be taken as the only basic connection. If for this connection we write

$$A \parallel B,$$

then we have

$$\begin{array}{rll} A \parallel A & \text{eq} & \overline{A} \\ \overline{A} \parallel \overline{B} & \text{eq} & A \,\&\, B, \end{array}$$

thus, negation as well as & is expressible by means of this connection.

These reflections already border somewhat on the trivial. Nevertheless, it is remarkable that the discovery of such a simple fact as that of reducing all propositional connections to a single one was reserved for the 20th century. The equivalences between propositional connections were not at all systematically investigated in the old logic.[1] There one finds only occasional remarks like, for example, that of the equivalence of

[1]Today these historical remarks stand in need of correction. In the first place, the reducibility of all propositional connections to a single one was already discovered in the 19th century by Charles S. Peirce—to be sure, a fact which became more generally known only with the publication of his collected works in 1933. Further, it is not correct that the equivalences between propositional connectives were not considered systematically in the old logic—to be sure, not in Aristotelian logic, but in other Greek schools of philosophy. (On this topic see the book *Formal Logic* (*vide* [**?**]).)

Remark: This footnote, as well as the next three, are subsequent additions occasioned by the republication of this lecture.

$$A \to \overline{B} \quad \text{with} \quad B \to \overline{A}$$

on which the inference by "contraposition" is based. The systematic search for equivalences is, however, all the more rewarding as one reaches here a self-contained and entirely surveyable part of logic, the so-called *propositional calculus*. I will explain in some detail the value of this calculus for reasoning.

Let us reflect on what the sense of equivalence is. When I say

$$\overline{A \mathbin{\&} B} \quad \text{eq} \quad \overline{A} \vee \overline{B},$$

I do not claim that the two complex propositions have the same sense but only that they *have the same truth value*. That is, no matter how the individual propositions A, B are chosen, $\overline{A} \vee \overline{B}$ and $\overline{A \mathbin{\&} B}$ are always simultaneously true or false, and consequently these two expressions can represent each other with respect to truth.

Indeed, any proposition connecting A and B can be viewed as a mathematical function assigning to each pair of propositions A, B one of the values "true" or "false." The actual content of the propositions A, B does not matter at all. Rather, what matters is only whether A is true or false and whether B is true or false. So we are dealing with *truth functions*: To a pair of truth values another truth value is assigned.

Each such function can be given by a schema in such a way that the four possible connections of two truth values (corresponding to the propositions A, B) are represented by four cells, and in each of these the corresponding truth value of the function ("true" or "false") is written down.

The schemata for $A \mathbin{\&} B, A \vee B, A \to B$ are specified here.

$A \mathbin{\&} B$:

B \ A	true	false
true	true	false
false	false	false

$A \vee B$:

B \ A	true	false
true	true	true
false	true	false

$$A \to B : \qquad \begin{array}{c|c|c|} & \multicolumn{2}{c}{\overbrace{\text{true} \qquad \text{false}}^{A}} \\ \overbrace{\phantom{\text{true}}}^{B} & \multicolumn{2}{c}{} \\ \cline{2-3} \text{true} & \text{true} & \text{true} \\ \cline{2-3} \text{false} & \text{false} & \text{true} \\ \cline{2-3} \end{array}$$

One can easily calculate that there are exactly 16 different such functions. The number of different functions of n truth values

$$A_1, A_2, \ldots, A_n$$

is, correspondingly, $2^{(2^n)}$.

To each function of two or more truth values corresponds a class of substitutable[b] propositions of connections. Among these one class is distinguished, namely the class formed by those connections that are always true.

These connections represent all logical statements that hold generally and in which individual propositions occur only as undivided units. We will call the expressions representing statements that hold generally *valid formulas.*[c]

We master propositional logic if we know the valid formulas (among the propositional connections), or if we can decide for a given propositional connection whether or not it is valid. After all, the task for reasoning in propositional logic can be formulated as follows:

Certain connections

$$V_1, V_2, \ldots, V_k$$

are given; they are built up from elementary propositions $A, B, \ldots$, and represent true statements for a certain interpretation of the elementary propositions. The question is whether another given connection W of these elementary propositions follows logically whenever $V_1, V_2, \ldots, V_k$ hold, indeed without considering the more precise content of the propositions $A, B, \ldots$.

The answer to this question is "yes" if and only if

$$(V_1 \ \& \ V_2 \ \& \ \ldots \ \& \ V_k) \to D,$$

expressed in terms of $A, B, \ldots$, represents a valid formula.

The decision concerning the validity of a propositional connection can in principle always be reached by trying out all relevant truth values. The

[b] *Vide* [**?**], pp. 47–48: "Let us say briefly that two propositional connections are 'substitutable' for each other if they represent the same truth value."

[c] *Vide* ■ for the distinction between "to hold generally" and "to be valid."

method of considering equivalences, however, provides a more convenient procedure. That is to say, by means of equivalent transformations each formula can be put into a certain *normal form* in which only the logical symbols $\&, \vee, {}^{-}$ occur, and from this normal form one can read off directly whether or not the formula is valid.

The rules of transformation are also very simple. In particular, one can calculate with & and $\vee$ in full analogy to $+$ and $\cdot$ in algebra. Indeed, matters are here even simpler, as & and $\vee$ can be treated in a completely symmetrical way.

In considering equivalences we entered, as already mentioned, the domain of inferences. But here we carried out the inferences in a naive way, as it were, on the basis of the meaning of the logical connections, and we turned the task of making inferences into a decision problem.

But for logic there remains the task of *systematically* presenting the rules of inference.

Aristotelian logic lays down the following principles of inference:

1. Rule of categorical inference: the *dictum de omni et nullo*: what holds universally, holds in each particular instance.

2. Rule of hypothetical inference: if the antecedent is given, then the consequent is given, i. e. if A and if $A \rightarrow B$, then B.

3. Laws of negation: law of contradiction and law of excluded middle: A and $\overline{A}$ can not both hold, and, at least one of the two propositions must hold.

4. Rule of disjunctive inference: if at least one of A or B holds and if $A \rightarrow C$ as well as $B \rightarrow C$, then C holds.

One can say that each of these laws represents the implicit definition of a logical process: 1. of universality, 2. of the hypothetical connection, 3. of negation, 4. of disjunction ($\vee$).

These laws indeed contain the essence of what is expressed when inferences are being made. But for a complete analysis of inferences this does not suffice. For this we demand that nothing needs to be reflected upon, once the principles of inference have been spelled out. The rules of inference must be constituted in such a way that they eliminate logical thinking. Otherwise we would have to have again logical reasoning which specify how to apply those rules.

This demand to exorcise the mind can indeed be met. The development of the theory of inference obtained in this way is analogous to the axiomatic development of a theory. Certain logical laws written down as formulas correspond here to the axioms, and operating on formulas externally according to fixed rules leading from the initial formulas to further ones corresponds to the material inference that usually leads from axioms to theorems.

Each formula that can be derived in such a way represents a valid logical proposition.

Here it is once again advisable to separate out *propositional logic*, which rests on the principles 2, 3, and 4. We then only need the following rules: the elementary propositions are represented by variables,

$$X, Y, \ldots$$

The first rule now states: any propositional connection can be substituted for such variables (substitution rule).

The second rule is the inference schema

$$\frac{\begin{array}{c}\mathfrak{S}\\ \mathfrak{S} \to \mathfrak{T}\end{array}}{\mathfrak{T}}$$

according to which the formula $\mathfrak{T}$ is obtained from two formulas $\mathfrak{S}$, $\mathfrak{S} \to \mathfrak{T}$.

The choice of the initial formulas can be made in quite different ways. One has taken great pains, in particular, to get by with the smallest possible number of axioms, and in this respect the limit of what is possible has indeed been reached. The purpose of logical investigations is better served, however, when we separate, as in the axiomatics for geometry, various *groups of axioms* from one another, such that each group gives expression to the role of one logical operation. The following list then emerges:

I	Axioms of implication
IIa)	Axioms for &
IIb)	Axioms for $\vee$
III	Axioms of negation.

Through application of the rules, this system of axioms generates *all* valid formulas of propositional logic.[2][d] This *completeness* of the axiom system can be characterized even more sharply by the following facts: if we add any underivable formula to the axioms, then we can deduce with the help of the rules arbitrary propositional formulas.

The division of the axioms into groups has a particular advantage, as it allows one to separate out *positive logic*. We understand this to be the system of those propositional connections that are valid without assuming that an opposite exists.[e] For example:

$$(A \mathbin{\&} B) \to A$$
$$(A \mathbin{\&} (A \to B)) \to B.$$

The system of these formulas presents itself in our axiomatics as the totality of those formulas that are derivable without using axiom group III. This system is far less perspicuous than the full system of valid formulas. Also, no decision procedure is known by which one can determine, in accordance with a definite rule, whether a formula belongs to this system.[3] It does not hold, for instance, that every formula expressible in terms of $\to, \&, \vee$, which is valid and therefore derivable on the basis of I–III, is already derivable from I–II. One can rigorously prove that this is not the case.

An example is provided by the formula

$$A \vee (A \to B).$$

Representing $\to$ by $\vee$ and $^{-}$ this formula turns into

$$A \vee (\overline{A} \vee B),$$

[2]We refer here only to those formulas that can be built up with the operations $\to, \&, \vee$ and with negation. If further operation symbols are added, then they can be introduced by replacement rules. To be sure, one need not distinguish the four mentioned operations in this particular way.

[3]In the meantime, decision procedures for positive logic were given by Gerhard Gentzen and Mordechaj Wajsberg.

[d]Assuming the axioms as in [?], p. 65, as already formulated in the early twenties. For the completeness, cf. the Habilitationsschrift■ [?] of Bernays, written in 1918.

[e]*Vide* [?], p. 67: "Die '*positive Logik*' ..., d. h. die Formalisierung derjenigen logischen Schlüsse, welche unabhängig sind von der Voraussetzung, daß zu jeder Aussage ein Gegenteil existiert."

and this representation allows one immediately to recognize the formula as valid. However, it can be shown that the formula is not derivable within positive logic, i. e., on the basis of axioms I–II. Hence, it does not represent a law of positive logic.

We recognize here quite clearly that negation plays the role of an *ideal element* whose introduction aims at rounding off the logical system to a totality with a simpler structure, just as the system of real numbers is extended to a more perspicuous totality by the introduction of imaginary numbers, and just as the ordinary plane is completed to a manifold with a simpler projective structure by the addition of points at infinity. Thus this method of ideal elements, fundamental to science, is already encountered here in logic, even if we are usually not aware of its significance.

A special part of positive logic is constituted by the doctrine of *chain inferences*, which was already discussed in Aristotelian logic. In this area there are also natural problems and simple results, not known to traditional logic and again requiring that specifically mathematical considerations be brought to bear. I have in mind Paul Hertz's investigations of sentence-systems.[f] —

The axiomatization we have considered up to now refers to those inferences depending solely on the rules for hypothetical and disjunctive inference and for negation. Now we still have the task of incorporating *categorical reasoning* into our axiomatization. How this is done I will describe here only briefly.

In addition to the *dictum de omni et nullo* we also need its converse: "what holds in each particular instance, also holds generally." Furthermore, we have to take into account the particular judgments. Thus, analogously:

> "If a proposition $A(x)$ is true of some object x, then there is an object of which it is true, and vice versa."

Thus we obtain four principles of reasoning that are represented in the axiomatics by two new initial formulas and two rules. A substitution rule for the individual variables $x, y, \ldots$ is also added.

Moreover, the substitution rule concerning propositional variables $X, Y, \ldots$ has to be extended in such a way now that the formulas of propositional logic can be applied also to expressions containing individual variables.

[f] *Vide* [**?**], p. 84, but also [**?**]■ , p. 300.

Let us now see, how the typical Aristotelian inferences are worked out from this standpoint. For that it is necessary to first say something about the interpretation of the universal judgment “all S are P.”

According to the Aristotelian view, such a judgment presupposes that there are certain objects with property S, and it is then claimed that all these objects have property P. This interpretation of the universal judgment, to which Franz Brentano in particular objected from the side of philosophy, is admittedly quite correct. But it is suited neither for the purposes of theoretical science nor for the formalization of logic, since the implicit presupposition brings with it unnecessary complications. Therefore we shall restrict the content of the judgment, “all S are P,” to the assertion, “an object having property S has also property P.”

Accordingly, such a judgment is simultaneously universal and hypothetical. It is represented in the form

$$(x)(S(x) \to P(x)).$$

The so-called categorical inferences thus contain a combination of categorical and hypothetical inferences. I will illustrate this by a classical example:

“All men are mortal, Cajus is a man, therefore Cajus is mortal.”

If we represent “x is human” and “x is mortal” in our notation by $H(x)$ and $Mrt(x)$ respectively, then the premises are

$$(x)(H(x) \to Mrt(x)),$$
$$H(\text{Cajus}),$$

and the conclusion is: $Mrt(\text{Cajus})$.

The derivation proceeds, first, according to the inference from the general to the particular, by deducing from

$$(x)(H(x) \to Mrt(x))$$

the formula

$$H(\text{Cajus}) \to Mrt(\text{Cajus}).$$

And this proposition together with

$$H(\text{Cajus})$$

yields according to the schema of the hypothetical inference:

$$Mrt(\text{Cajus}).$$

It is characteristic for this representation of the inference that one refrains from giving a quantitative interpretation of the categorical judgment (in the sense of subsumption). Here one recognizes particularly clearly that mathematical logic does not depend in the least upon being a logic of extensions.

Our rules and initial formulas permit us now to derive all the familiar Aristotelian inferences as long as they agree with our interpretation of the universal judgment—that leaves just 15. In doing so one realizes that there are actually only very few genuinely different kinds of inferences. Furthermore, one gets the impression that the underlying problem is delimited in a rather arbitrary way.

A more general problem, which is also solved in mathematical logic, consists in finding a decision procedure that allows one to determine whether a predicate formula is valid or not. In this way, one masters reasoning in the domain of predicates, just as one masters propositional logic with the decision procedure mentioned earlier.

But our rules of inference extend much farther. The actual wealth of logical connections is revealed only when we consider *relations* (predicates with several subjects). Only then does it become possible to capture *mathematical proofs* in a fully logical way.

However, one is here induced to add various *extensions* which are suggested by ordinary language.

The first extension consists in introducing a formal expression for "x is the same object as y," or "an object different from y." For this purpose the "*identity* of x and y" has to be formally represented as a particular relation, the properties of which are to be formulated as axioms.

Second, we need a symbolic representation of the logical relation we express linguistically with the aid of the genitive or the relative pronoun in such phrases as "the son of Mr. X" or "the object that." This relation forms the basis of the *concept of a function* in mathematics. The essential point here is that an object uniquely having a certain property or satisfying a certain relation to particular objects, is characterized by this property or relation.

The most significant extension, however, is brought about by the circumstance that we are led to consider predicates and relations themselves as objects, just as we do in ordinary language when we say, for example, "patience is a virtue." We can state properties of predicates and relations, and furthermore, relations between predicates and also between relations. Likewise, the forms of universality and particularity can be applied with respect to predicates and relations. In this way we arrive at a logic of *second order*; for its formal implementation the laws of categorical reasoning have to be extended appropriately to the domain of predicates and relations.

The solution of the decision problem—which, incidentally, is here automatically subsumed under a more general problem—presents an enormous task for this enlarged range of logical relations resulting from the inclusion of relations and the other extensions mentioned. Its solution would mean that we have a method that permits us, at least in principle, to decide for any given mathematical proposition whether or not it is provable from a given list of axioms. As a matter of fact, we are far from having a solution of this problem. Nevertheless, several important results of a very general character have been obtained in this area through the investigations of Löwenheim and Behmann; in particular, the decision problem for *predicate logic* in the case of second-order logic has also been completely solved.[4]

Here we see that the traditional theory of inference comprises only a tiny part of what really belongs to the domain of logical inference.

As yet I have not even mentioned *concept formation*. And, for lack of time, I cannot consider it in detail. I will only say this much: a truly penetrating logical analysis of concept formation becomes possible only on the basis of the theory of relations. Only by means of this theory does one realize what complex combinations of logical expressions (relations, existential propositions, etc.) are concealed by short expressions of ordinary language. Such an analysis of concept formation has already been pursued quite far, especially by Bertrand Russell, and it has led to knowledge about general logical processes of concept formation. The methodological understanding of science is being advanced considerably through their clarification.

I now come to the end of my remarks. I have tried to show that logic, that is to say the correct old logic as it was always intended, obtains its genuine rounding off, its proper development and systematic completion, only through its mathematical treatment. The mathematical mode of consideration is not introduced here artificially, but rather arises in an entirely natural

[4]Notice that one speaks here of "predicate logic" in the sense of the distinction between predicates and relations. Thus, what is meant here by "predicate logic" is what is currently usually called the logic of monadic predicates. The logic of polyadic predicates is already generally undecidable for the first order case, as was shown by Alonzo Church.

way, in the further pursuit of problems.

The resistance to mathematical logic is widespread, particularly among philosophers; apart from the reasons mentioned at the beginning, there is also a principled reason for this. Many approve of having mathematics absorbed into logic. But here one realizes the opposite, namely, that the system of logic is absorbed into mathematics. With respect to the mathematical formalism logic appears here as a specific interpretation and application, perfectly resembling the relation between, for example, the theory of electricity and mathematical analysis, when the former is treated according to Maxwell's theory.

That does not contradict the generality of logic, but rather the view that this generality is super-ordinate to that of mathematics. Logic is about certain contents that find application to any subject matter whatsoever, insofar as it is reasoned about. Mathematics, on the other hand, is about the most general laws of any sort of combination. This is also a kind of highest generality, namely, in the direction towards the *formal*. Just as every argument, including the mathematical ones, is subordinate to the laws of logic, each structure, each manifold however primitive—and thus also the manifold given by the combination of statements or parts of statements—must be subject to mathematical laws.

If we wanted a logic free of mathematics, no theory at all would be left, but only pure reflection on the most simple connections of meaning. Such purely material considerations—which can be comprised under the name "philosophical logic"—are, in fact, indispensable and decisive as a starting point for the logical theory, just as the purely physical considerations serving as the starting point for a physical theory constitute the intellectual basis for that theory. But such considerations do not fully constitute the theory itself. Its development requires the formalism of mathematics. The exact systematic theory of a subject is surely the mathematical treatment, and it is in this sense that Hilbert's dictum holds: ■ "Anything at all that can be the object of scientific thought, as soon as it is ripe for the formation of a theory ... will be part of mathematics."■ [g] Even logic can not escape this fate.

[g] *Vide* [**?**], p. ■ .