Bernays Project: Text No. 23

Comments on Ludwig Wittgenstein's *Remarks on the Foundations of Mathematics* (1959)

Betrachtungen zu Ludwig Wittgensteins *Bemerkungen über die Grundlagen der Mathematik*

(*Ratio* 2, pp. 1–18; repr. in [?], pp. ■ –■
repr. in *Abhandlungen*, pp. 119–141)

Translation by: ■ *Ratio: Benecerraf/Putnam*■
Revised by: *Erich Reck*
Final revision by: *CMU*

I

The following comments are concerned with a book that is the second part of the posthumous publications of selected fragments from Wittgenstein in which he sets forth his later philosophy.[1] The necessity of making a selection and the fragmentary character noticeable at points are not overly problematic, since in his publications Wittgenstein refrains from a systematic presentation anyway and expresses his thoughts in separate paragraphs—jumping frequently from one theme to another. In fairness to the author it has to be admitted, however, that he would doubtlessly have made extensive changes in the arrangement and selection of the material had he been able to complete the work himself. The editors of the book have, by the way, greatly facilitated an overview over the book by providing a very detailed table of

[1]The book was originally published in German, with English translation attached. All pages and numbers quoted refer to [?].

contents and an index. The preface provides information about the origins of the different parts I–V.

Compared with the viewpoint of the *Tractatus*, which considerably influenced the initially rather extreme doctrine of the Vienna Circle, Wittgenstein's later philosophy represents a rectification and clarification in essential respects. In particular, the very schematic conception of the structure of scientific language—especially of the composition of statements out of atomic propositions—is here dropped. What remains, however, is the negative attitude towards speculative thinking and the constant tendency to disillusionize.

Thus Wittgenstein himself says, evidently with his own philosophy in mind (p. 63, No. 18): "Finitism and behaviorism are quite similar trends. Both say, but surely, all we have here is .... Both deny the existence of something, both with a view to escaping from a confusion. What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test." Later on he explains (p. 174, No. 16): "It is my task, not to attack Russell's logic from *within*, but from without. That is to say: not to attack it mathematically—otherwise I should be doing mathematics—but its position, its office. My task is, not to talk about (e. g.) Gödel's proof, but to pass it by."

As one can see, a certain jocularity of expression is not missing in Wittgenstein; and in the numerous parts written in dialogue form he often enjoys acting the rogue.

On the other hand, he does not lack *esprit de finesse*, and his remarks contain many implicit suggestions, in addition to what is explicitly stated.

Throughout, however, two problematic tendencies play a role. The first is to explain away the actual role of thinking—of reflective intending—along behavioristic lines. It is true that David Pole, in his interesting account and exposition of Wittgenstein's later philosophy,[2] denies that Wittgenstein is a supporter of behaviorism. And this contention is justified insofar as Wittgenstein certainly does not deny the existence of mental experiences of feeling, perceiving and imagining. Still, with regard to thinking his attitude is behavioristic after all. In this connection he tends towards a short circuit everywhere. Images and perceptions are, in each case, supposed to be followed immediately by behavior. "We do it like this," that is usually the last word of explanation—or else he appeals to some need as an anthropological

[2] *Vide* [**?**].

fact. Thought, as such, is left out. Along these lines, it is characteristic that a "proof" is conceived of as a "picture" or "paradigm;" and although Wittgenstein is critical of the method of formalizing proofs he keeps using the formal method of proof in Russell's system as an example. Instances of mathematical proofs proper, which are neither just calculations nor result merely from exhibiting a figure or a formal procedure, do not occur at all in this book on the foundations of mathematics, a book a major part of which addresses the question as to what proofs really are; and that in spite of the fact that the author has evidently concerned himself with many mathematical proofs.

One passage may be mentioned as characteristic for Wittgenstein's behavioristic attitude, and as an illustration of what is meant here by a short circuit. Having rejected as unsatisfactory various attempts to characterize inference, he continues (p. 8, No. 17): "This is why it is necessary to look and see how we carry out inferences in the practice of language; what kind of procedure in the language-game inferring is. For example: a regulation says: 'All who are taller than five foot six are to join the ... section.' A clerk reads out the men's names and their heights. Another allots them to such-and-such sections. 'N. N. five foot nine.' 'So N. N. to the ... section.' That is inference." It is evident here that Wittgenstein is satisfied only with the characterization of an inference in which one passes directly from a linguistic specification of the premises to an action; one in which, therefore, the specifically reflective element is eliminated. Language, too, appears under the aspect of behavior ("language-game").

The other problematic tendency has its source in the program—already present in Wittgenstein's earlier philosophy—of separating strictly the linguistic and the factual, a separation also present in Carnap's *Logical Syntax of Language*. That this separation should have been retained in the new version of Wittgenstein's doctrine does not go without saying because here the approach, compared with the earlier one, is in many respects less rigid. Some signs of change are, in fact, apparent, as for instance on p. 119, No. 18: "It is clear that mathematics as a technique for transforming signs for the purpose of prediction has nothing to do with grammar." Elsewhere (p. 125, No. 42), he even speaks of the "synthetic character of mathematical propositions." As he puts it: "It might perhaps be said that the synthetic character of propositions of mathematics appears most obviously in the unpredictable occurrence of the prime numbers. But their being synthetic (in this sense) does not make them any the less *a priori* .... The distribution of primes would be an ideal example of what could be called synthetic *a priori*, for one can say that it is at any rate not discoverable by an analysis of the concept of a prime number." As we can see, Wittgenstein turns here from the Vienna Circle concept of "analyticity" back to a conception that is more Kantian.

A certain rapprochement to Kant's conception can also be found in Wittgenstein's view that mathematics first determines the character or "creates the forms of what we call facts" (see p. 173, No. 15). Along these lines, Wittgenstein strongly opposes the view that the propositions of mathematics have the same function as empirical propositions. At the same time, he emphasizes on a number of occasions that the applicability of mathematics, in particular of arithmetic, depends on empirical conditions; e. g., on p. 14, No. 37 he says: "This is how our children learn sums; for one makes them put down three beans and then another three beans and then count what is there. If the result at one time were five, at another seven ..., then the first thing we said would be that beans were no good for teaching sums. But if the same thing happened with sticks, fingers, lines and most other things, that would be the end of all sums.—'But shouldn't we then still have $2 + 2 = 4$?'—This sentence would have become unusable."

Nevertheless, statements like the following remain important for Wittgenstein's conception (p. 160, No. 2): "If you know a mathematical proposition, that's not to say that you yet know *anything*." He repeats this twice, at short intervals, and adds: "I. e., the mathematical proposition is only to supply a framework for a description." In the manner of Wittgenstein one could ask back here: "Why is the person in question *supposed* to still know nothing? What need is expressed by this 'supposed to'?" It appears that only a philosophical preconception leads to this requirement, the view, namely, that there can exist only *one* kind of factuality: that of concrete reality. This view corresponds to a kind of nominalism that also plays a role elsewhere in discussions on the philosophy of mathematics. In order to justify such a nominalism Wittgenstein would, at the very least, have to go back further than he does in this book. In any case, he cannot appeal to our actual attitudes here. And indeed, he attacks our tendency to regard arithmetic, say, "as the natural history of the domain of numbers" (see p. 117, No. 13, and p. 116, No. 11). Then again, he is not fully definite on this point. He asks himself (p. 142, No. 16) whether it already constitutes "mathematical alchemy" to claim that mathematical propositions are regarded as statements about mathematical objects. But he also notes: "In a certain sense it is not possible to appeal to the meaning of signs in mathematics, just because it is only mathematics that gives them their meaning. What is typical of the phenomenon I am talking about is that a *mysteriousness* about some mathematical concept is not *straight away* interpreted as an erroneous conception, as a mistake of ideas; but rather as something that is at any rate not to be despised, is perhaps even rather to be respected. All that I can do, is to show an easy escape from this obscurity and this glitter of the concepts. Strangely, it can be said that there is so to speak a solid core to all these glistening concept-formations. And I should like to say that that is what makes them into mathematical productions."

One may doubt whether Wittgenstein has indeed succeeded in exhibiting "an easy escape from this obscurity;" one may even be inclined to think that the obscurity and the "mysteriousness" really have their origin in a philosophical conception, or in the philosophical language used by Wittgenstein.

His fundamental separation of the sphere of mathematics from the sphere of the factual comes up in several passages in the book. In this connection, Wittgenstein often speaks with a matter-of-factness that contrasts strangely with his readiness to doubt so much of what is generally accepted. A passage on p. 26, No. 80 is typical for this; he says: "But of course you can't get to know any property of the material by imagining." Again on p. 29, No. 98, we can read: "I can calculate in the imagination, but not experiment." From the point of view of common experience, all of this is certainly not obvious. An engineer or technician has, no doubt, just as lively a mental image of materials as a mathematician has of geometrical curves; and the mental image which any one of us may have of a thick iron rod is doubtlessly such as to make it clear that the rod could not be bent by a light pressure of the hands. Moreover, in the case of technical invention a major role is definitely played by experimenting in the imagination. It seems that Wittgenstein simply, without critical reflection, uses a philosophical schema which distinguishes the *a priori* from the empirical. To what extent and in which sense this distinction—so important particularly in the Kantian philosophy—is justified will not be discussed here; but its introduction, particularly at the present moment, should not be taken very lightly. With regard to the *a priori*, Wittgenstein's viewpoint differs from Kant's, incidentally, insofar as it includes the principles of general mechanics in the sphere of the empirical. Thus he argues, e. g. (p. II 4, No. 4): "Why are the Newtonian laws not axioms of mathematics? Because we could quite well imagine things being otherwise ... To say of a proposition: 'This could be imagined otherwise' ... ascribes the role of an empirical proposition to it." The notion of "being able to imagine otherwise," also used by Kant, has the unfortunate difficulty of being ambiguous; the impossibility of imagining something may be meant in various senses. This difficulty occurs particularly in the case of geometry, as we will discuss later.

The tendency of Wittgenstein, mentioned earlier, to recognize only one kind of factuality becomes evident not only with regard to mathematics, but also with respect to any phenomenological consideration. Thus he discusses the proposition that white is lighter than black (p. 30, No. 105), and explains it by saying that black serves us as a paradigm for what is dark, and white as a paradigm for what is light, which makes the statement one without content. In his opinion statements about differences in brightness have content only when they refer to specific visually given objects; and for the sake of clarity one should not even talk about differences in the brightness of colors. This attitude obviously precludes a descriptive theory of colors.

Actually, phenomenological considerations should be congenial to Wittgenstein, one might think. This is suggested by the fact that he likes to draw examples, for the purpose of comparison, from the field of art. It is only his philosophical program, then, that prevents the development of an explicitly phenomenological viewpoint.

This aspect is an example of how Wittgenstein's methodology is aimed at eliminating a very great deal. He sees himself in the part of the free thinker who combats superstition. However, the latter's goal is freedom of the mind, whereas it is exactly the mental that Wittgenstein restricts in many ways—by means of a mental asceticism in the service of an irrationality whose goal is quite indeterminate.

Yet this tendency is by no means as extreme in the later philosophy of Wittgenstein's as it was in the earlier form. One may already gather from the passages quoted above that he was probably on the way to giving mental contents more of their due.

A related fact may be that, in contrast to the simply assertoric form of philosophical statements in the *Tractatus*, a largely aporetical attitude prevails in the present book. With respect to philosophical pedagogics this presents a danger, however, especially as Wittgenstein's philosophy is exerting such a strong attraction on younger minds. The old Greek observation that philosophical contemplation often begins in philosophical wonder[3] misleads many philosophers today into believing that the cultivation of aston-

[3] θαυμάζειν.

ishment is in itself a philosophical achievement. One may surely have ones doubts about the soundness of a method which trains young philosophers in wondering, as it were. Wondering is heuristically fruitful only when it is the expression of an instinct for research. Clearly it cannot be demanded of any philosophy to make comprehensible everything that is astonishing. But perhaps it is characteristic for the various philosophical viewpoints what they accept as ultimate that which is astonishing. In Wittgenstein's philosophy it is, as far as epistemological questions are concerned, sociological facts. A few quotations may serve as illustrations of this point (p. 13, No. 35): "... how does it come about that all men ... accept these patterns as proofs of these propositions?—It is true, there is a great—and interesting—agreement here." (p. 20, No. 63): "... it is a peculiar procedure: I *go through* the proof and then accept its result.—I mean: this is simply what we *do*. This is use and custom among us, or a fact of our natural history." (p. 23, No. 74): "If you talk about *essence*—, you are merely noting a convention. But here one would like to retort: there is no greater difference than that between a proposition about the depth of the essence and one about—a mere convention. But what if I reply: to the *depth* that we see in the essence there corresponds the *deep* need for the convention." (p. 122, No. 30): "Do not look at the proof as a procedure that *compels* you, but as one that *guides* you ... But how does it come about that it guides *each one* of us in such a way that we agree in the influence it has on us? Well, how does it come about that we agree in *counting*? 'That is just how we are trained' one may say, 'and the agreement produced in this way is carried further by the proofs.'"

II

So much for a general characterization of Wittgenstein's observations. But their content is by no means exhausted by the general philosophical aspects that have been mentioned; various specific questions of a basic philosophical nature are also discussed in detail. In what follows, we shall deal with their principal aspects.

Let us begin with a question that is connected with a problem previously touched on, namely the distinction between the *a priori* and the empirical: the question of geometrical axioms. Wittgenstein does not deal specifically with geometrical axioms as such. Instead, he raises the general question as to how far the axioms of an axiomatized mathematical system should be self-evident; and he takes as his example the parallel axiom. Let us quote a few sentences from his discussion of this subject (p. 113, No. 2ff): "What do we say when we are presented with such an axiom, e. g., the parallel axiom? Has experience shown us that this is how it is? ... Experience plays a part; but not the one we would *immediately expect*. For we haven't made experiments and found that in reality only *one* straight line through a given point fails to intersect another. And yet the proposition is evident.—Suppose I now say: it is quite indifferent why it is evident. It is enough that we accept it. All that is important is how we use it .... When the words for e. g. the parallel axiom are given ... the kind of use this proposition has and hence its sense are as yet quite undetermined. And when we say that it is evident, this means that we have already chosen a definite kind of employment for the proposition without realizing it. The proposition is not a mathematical axiom if we do not employ it precisely *for this purpose*. The fact, that is, that here we do not make experiments, but accept the self-evidence, is enough to fix the employment. For we are not so naive as to make the self-evidence count in place of the experiment. It is not our finding the proposition self-evidently true, but our making the self-evidence count, that makes it into a mathematical proposition."

In discussing these remarks, it must first be realized that we need to distinguish two things: whether we recognize an axiom as geometrically valid, or whether we choose it as an axiom. The latter is, of course, not determined by the wording of the proposition. But here we are concerned merely with a technical question concerning the deductive arrangement of propositions. What interests Wittgenstein, on the other hand, is surely the recognition of the proposition as geometrically valid. It is along these lines that Wittgenstein's assertion ("that the recognition is not determined by the words") must be considered; and its correctness is at the very least not immediately evident. He says simply: "For we have not made experiments." Admittedly, there has been no experimenting in connection with the formulation of the parallel axiom considered by him; this formulation does not lend itself to this purpose. However, within the framework provided by the other geometrical axioms the parallel axiom is equivalent to any one of the following statements of metrical geometry: "In a triangle the sum of the angles is equal to two right angles. In a quadrilateral in which three angles are right angles the fourth angle is also a right angle. Six congruent equilateral triangles with a common vertex P (lying consecutively side by side) exactly fill up the neighborhood of point P." Such propositions—in which, it should be noted, there is no mention of the infinite extendibility of a straight line—can definitely be tested by experiment. And as is well known, Gauss did in fact check experimentally the proposition about the sum of the angles of a triangle, thereby making use of the assumption of the linear propagation of light, to be sure. In addition, this is not the only possibility for an experiment. Hugo Dingler, in particular, has shown that for the concepts of straight line, plane, and right angle there exists a natural and, as it were, compulsory kind of experimental realization. By means of such an experimental realization of geometrical concepts, statements like the second one above, especially, can be experimentally tested with great accuracy. Moreover, in a less accurate way they are checked by us all the time implicitly in the normal practice of drawing figures. Our instinctive estimations of lengths and of the sizes of angles, too, can be regarded as the result of manifold experiences; and propositions such as those mentioned above must, after all, agree with those instinctive estimations.

It cannot be upheld, therefore, that our experience plays no role in the acceptance of propositions as geometrically valid. But Wittgenstein does not mean that either, as becomes clear from what follows immediately after the passage quoted (p. 114, Nos. 4 and 5): "Does experience tell us that a straight line is possible between any two points? ... It might be said: *imagination* tells us. And the germ of truth is here; only one must understand it right. *Before* the proposition the concept is still pliable. But might not experience force us to reject the axiom?! Yes. And nevertheless it does not play the role of an empirical proposition. ... Why are the Newtonian laws not axioms of mathematics? Because we could quite well imagine things being otherwise. ... Something is an axiom, *not* because we accept it as extremely probable, nay certain, but because we assign it a particular function, and one that conflicts with that of an empirical proposition. ... The axiom, I should like to say, is a different part of speech." Further on (p. 124, No. 35), he says: "What about e. g. the fundamental laws of mechanics? If you understand them you must know how experience supports them. It is otherwise with the propositions of pure mathematics."

In support of these remarks, it must certainly be conceded that experience alone does not force the theoretical acceptance of a proposition. An exact theoretical approach must always go beyond the facts of experience in its conception.

Nevertheless, the view that in this respect there exists a sharp dividing line between mathematical propositions and the principles of mechanics is by no means justified. In particular, the last quoted assertion that, in order to understand the basic laws of mechanics, the experience on which they are based must be known can hardly be upheld. Of course, when mechanics is taught at the university it is desirable that the empirical starting points be made clear. But this is not done with a view towards the theoretical and practical manipulation of the laws, but for being epistemologically alert and with an eye to the possibilities of eventually necessary modifications of the theory. An engineer or productive technician who wants to become skilled in mechanics and capable of handling its laws does not have to concern himself with how we came upon these laws. With respect to these laws applies, moreover, the same as what Wittgenstein so frequently emphasizes with respect to mathematical laws: that the facts of experience relevant for the empirical motivation of these propositions by no means make up the content of what is asserted in the laws. What is important instead for learning to handle the mechanical laws is to become familiar with the concepts involved and to make them intuitive to oneself in some way. This kind of acquisition is not only practically, but also theoretically significant: the theory is fully assimilated only in the process of rationally shaping and extending it, to which it is subsequently subjected. With regard to mechanics, most philosophers and many of us mathematicians have little to say in this connection, not having acquired mechanics in the said manner.—What distinguishes the case of geometry from that of mechanics is the (philosophically in a sense accidental) circumstance that the acquisition of the world of concepts and of corresponding intuitions is for the most part already completed in an (at least for us) unconscious stage of mental development.

Ernst Mach's opposition to a rational foundation of mechanics has its justification insofar as such a foundation endeavors to pass over the role of experience in arriving at the principles of mechanics. We must keep in mind that the concepts and principles of mechanics comprise, as it were, an extract of experience. On the other hand, it would be unjustified to simply reject all efforts at constructing mechanics rationally on the basis of this criticism.

What is special about geometry is the phenomenological character of its laws, and hence the significant role played by intuition. Wittgenstein points to this aspect only in passing: "Imagination tells us. And the germ of truth is here; only one must understand it right" (p. 8). The term "imagination" is very general, and what he says at the end of the second sentence is a qualification which shows that the author feels the topic of intuition to be rather tricky. Indeed, it is very difficult to characterize the epistemological role of intuition in a satisfactory way. The sharp opposition between intuition and concepts as it occurs in Kant's philosophy does not, on closer inspection, appear to be justified. When considering geometrical thinking in particular it is difficult to separate sharply the part played by intuition from that played by the conceptual; since we find here a formation of concepts that is in a certain sense guided by intuition—one that, in the sharpness of its intentions, goes beyond what is intuitive in the strict sense, but also cannot be understood adequately if it is considered apart from intuition. What is strange is that Wittgenstein assigns no specific epistemological role to intuition in spite of the fact that his thinking is dominated by the visual. For him a proof is always a picture. At one time he gives a mere figure as an example of a geometrical proof. It is also striking that he never talks about the intuitive evidence of topological facts, such as the fact that the surface of a sphere divides (the rest of) space into an interior and an exterior part, in such a way that a curve which connects an inside point with an outside point always passes through a point on the surface of the sphere.

Questions concerning the foundations of geometry and its axioms belong primarily to the domain of general epistemology. What is called research on the foundations of mathematics in the narrower sense today is directly mainly at the foundations of arithmetic. Here one tends to eliminate, as much as possible, what is special about geometry by separating the latter into an arithmetical and a physical side. We shall leave aside the question of whether this procedure is justified; that question is not discussed by Wittgenstein. In contrast, he deals in great detail with basic questions concerning arithmetic. Let us now take a closer look at his remarks concerning this area of inquiry.

The viewpoint from which Wittgenstein looks at arithmetic is not the usual one of a mathematician. More than with arithmetic itself, Wittgenstein is concerned with theories of the foundations of arithmetic (in particular with Russell's theory). With regard to the theory of numbers, especially, his examples seldom go beyond the numerical. An uninformed reader might well conclude that the theory of numbers consists almost entirely of numerical equations—which, actually, are normally not regarded as propositions to be proved, but as simple statements. Wittgenstein's treatment is more mathematical in the sections where he discusses questions of set theory, such as concerning denumerability and non-denumerability, as well as concerning the theory of Dedekind cuts.

Throughout, Wittgenstein advocates the standpoint of strict finitism. In so doing he considers the various types of problems concerning the infinite that there are from a finitist viewpoint, in particular the problems of the *tertium non datum* and of impredicative definitions. The quite forceful and vivid account he provides in this connection is well suited for introducing the finitist's position to those still unfamiliar with it. However, it hardly contributes anything essentially new to the debate; and those who hold the position of classical mathematics in a deliberate way will scarcely be convinced by it.

Let us discuss a few points in more detail. Wittgenstein deals with the question of whether in the infinite expansion of π a certain sequence of numbers ϕ such as, say, "777," ever occurs. Along Brouwer's lines, he draws attention to the possibility that this question may not as yet have a definite answer. Along these lines he then says (p. 138, No. 9): "However queer it sounds, the further expansion of an irrational number is a further development of mathematics." This formulation is obviously ambiguous. If it merely means that any determination of a not yet calculated decimal place of an irrational number is a contribution to the development of mathematics, then every mathematician will agree with it. But since the statement is said to be "queer sounding," something else is most likely meant. Perhaps it is that the course of the development of mathematics at a given time is undecided, and that this undecidedness can have to do with the continuation of the expansion of an irrational number given by a definition; so that the decision as to what digit is to be put at the ten-thousandth decimal place of π would be a contribution to the direction of the history of thought. But such a view is not appropriate even according to Wittgenstein's own position, for he says (p. 138, No. 9): "The question … changes its status when it becomes decidable." Now, it is a fact that the digits in the decimal expansion of π are decidable up to any chosen decimal place. Hence the suggestion about the further development of mathematics does not contribute anything to our understanding of the situation in the case of the expansion of π. One can even say the following: Suppose we maintained firmly that the question of the occurrence of the sequence of numbers ϕ is undecidable, then this would imply that the figure ϕ occurs nowhere in the expansion of π; for if it did, and if k was the decimal place that the last digit of ϕ had on its first occurrence in the decimal expansion of π, then the question whether the figure ϕ occurs before the $(k + 1)$th place would be a decidable question and could be answered positively; thus the initial question would be decidable, too. (Incidentally, this argument does not require the principle of *tertium non datur*.)

Further on in the text, Wittgenstein comes back repeatedly to the example of the decimal expansion of π. At one point in particular (p. 185, No. 34) we find an assertion that is characteristic for his position: "Suppose that people go on and on calculating the expansion of π. So God, who knows everything, knows whether they will have reached a '777' by the end of the world. But can his *omniscience* decide whether they *would* have reached it after the end of the world? It cannot. ... Even for him the mere rule of expansion cannot decide anything that it does not decide for us."

That is certainly not convincing. If we concede the idea of a divine omniscience at all, then we would certainly ascribe to it the ability to survey at *one* glance a totality every single element of which is in principle accessible to us. Here we must pay special attention to the double role the recursive definition plays for the decimal expansion: as the definitory determination of decimal fractions, on the other hand; and as the means for the "effective" calculation of decimal places, on the other hand. If we here take "effective" in the usual sense, then it is true that even a divine intelligence can *effectively* calculate nothing other than what we are able to effectively calculate ourselves (no more than it world be capable of carrying out the trisection of an angle with ruler and compass, or of deriving Gödel's underivable proposition in the corresponding formal system). But it is not to be ruled out that this divine intelligence would be able to survey in some other (not humanly effective) manner all the possible calculation results of the application of a recursive definition.

In his criticism of the theory of Dedekind cuts, Wittgenstein's main argument is that in this theory an extensional approach is mixed up with an intensional approach. This criticism is, in fact, appropriate with respect to certain versions of the theory, namely those in which the goal is to create the appearance of a stronger constructive character of the procedure than is actually achieved. If one wants to introduce the cuts not as mere sets of numbers, but as defining arithmetical laws for such sets, then either one has to use a very vague concept of "law," thus gaining little; or, if ones aim is to clarify that concept, one is confronted with the difficulty which Hermann Weyl has termed the vicious circle in the foundation of analysis. This difficulty was sensed instinctively by a number of mathematicians for a while, who consequently advocated a restriction of the procedure of analysis. Such a criticism of impredicative formations of concepts plays a considerable role in discussions on the foundations of mathematics even today. However, the difficulties disappear if an extensional standpoint is maintained consistently. Moreover, Dedekind's conception can certainly be understood in that sense, and was probably meant that way by Dedekind himself. All that is required is that one accepts, besides the concept of number itself, also the concept of a set of natural numbers (and, consequently, the concept of a set of fractions) as an intuitively significant concept that is not in need of a reduction. This does bring with it a certain moderation with respect to the goal of arithmetizing analysis, and thus geometry too. But—as one could ask in a Wittgensteinian manner—must geometry be arithmetized completely anyway? Scientists are often very dogmatic in their attempts at reductions. They are often inclined to treat such an attempt as completely successful even if it does not succeed in the manner intended, but only to a certain extent or within a certain degree of approximation. Confronted with such attitudes, considerations of the kind suggested in Wittgenstein's book can be very valuable.

Wittgenstein's more detailed discussion of Dedekind's proof procedure is not satisfactory. Some of his objections can be disposed of simply by giving a clearer account of Dedekind's line of thought.

In Wittgenstein's discussion of denumerability and non-denumerability, the reader has to bear in mind that by a cardinal number he always means a finite cardinal number, and by a series always one of the order type of the natural numbers. His polemics against the theorem stating the non-denumerability of the totality of real numbers is unsatisfactory primarily insofar as the analogy between the concepts "non-denumerable" and "infinite" is not exhibited clearly. Corresponding to the way in which "infinitness of a totality G" can be defined as the property that to any finite number of things in G one can always find a further thing in it, the non-denumerability of a totality G is defined as the property that to every denumerable sub-totality one can always find an element of G not yet contained in the sub-totality. Understood in that sense, the non-denumerability of the totality of real numbers is demonstrated by means of the diagonal procedure; and there is nothing foisted in here, as would appear to be the case according to Wittgenstein's argument. The theorem of the non-denumerability of the totality of real numbers is, as such, independent of the comparison of transfinite cardinal numbers. Besides—and this is often neglected—, for that theorem there also exist other, more geometrical proofs than the one involving the diagonal procedure. In fact, from the point of view of geometry we can call this a rather gross fact. It is strange, also, to find the author raising a question like the following: "For how do we make use of the proposition: 'There is no greatest cardinal number.'? ... First and foremost, notice that we ask the question at all; this points to the fact that the answer is not ready to hand" (p. 57, No. 5). One should think that one needs not search long for an answer here. Our entire analysis, with all its applications in physics and technology, rests on the infinity of the number series. Probability theory and statistics, too, make constant implicit use of that infinity. Wittgenstein acts as if mathematics existed almost solely for the purposes of housekeeping.

The finitist and constructive attitude taken on the whole by Wittgenstein concerning the problems of the foundations of mathematics conforms to general tendencies in his philosophy. It cannot be said, however, that he finds confirmation for his position in the foundational situation in mathematics. All he shows is how this position is to be applied when dealing with the questions under dispute. In general, it is characteristic for the situation regarding the foundational problems that the results obtained so far do not favor either of the two main philosophical views opposing each other—the finitist-constructive view and the "Platonist"-existential view. Each of the two sides can advance arguments against the other. Yet, the existential conception has the advantage that it enables us to appreciate investigations aimed at the establishment of constructive methods (just as in geometry the investigation of constructions with ruler and compass has significance even for a mathematician who admits other methods of construction), while for a strict constructivist a large part of classical mathematics simply falls by the wayside.

Wittgenstein's observations concerning the foundational issues of the role of formalization, the reduction of number theory to logic, and the question of consistency are to some degree independent of partisanship in the above mentioned opposition. His views here show more independence, hence these considerations are of greater interest.

With regard to the question of consistency, in particular, he asserts what has meanwhile also been stressed by various other theorists in the field of foundational studies: that within the framework of a formal system a contradiction should not be seen so exclusively as objectionable, and that a formal system in itself can still be of interest even if it leads to a contradiction. It should be noted, on the other hand, that in the earlier systems of Frege and Russell the contradiction arises already within a few steps, as it were directly from the basic structure of the system. In addition, much of what Wittgenstein says in this connection overshoots the mark by a long way. Unsatisfactory, in particular, is his frequently used example of the derivability of contradictions by admitting division by zero. (One need only consider the justification for the rule of reduction in order to see that it is not applicable in the case of the factor zero.)

In any case, Wittgenstein acknowledges the importance of demonstrating consistency. However, it is doubtful whether he is sufficiently aware of the role played by the requirement of consistency in proof-theoretic investigations. Thus his discussion of Gödel's theorem of non-derivability and its proof, in particular, suffers from the defect that Gödel's quite explicit premise concerning the consistency of the formal system under consideration is ignored. A fitting comparison, drawn by Wittgenstein in connection with Gödel's theorem, is that between a proof of formal unprovability, on the one hand, and a proof of the impossibility of a certain construction with ruler and compass, on the other. Such a proof, says Wittgenstein, contains an element of prediction. But the remark which follows is strange (p. 52, No. 14): "A contradiction is unusable as such a prediction." As a matter of fact, such impossibility proofs usually proceed via the derivation of a contradiction.

In his remarks on the theory of numbers, Wittgenstein shows a noticeable reserve towards Frege's and Russell's foundation of number theory, such as was not present in the earlier stages of his philosophy. Thus on one occasion (p. 67, No. 4) he says: "… the logical calculus is only—frills tacked on to the arithmetical calculus." This thought has perhaps never been formulated as strikingly as here. It might be good, then, to reflect on the sense in which the claim holds true. There is no denying that the attempt to incorporate arithmetical and, in particular, numerical propositions into logistic has been successful. That is to say, it has proved possible to formulate these propositions in purely logical terms and, on the basis of this formulation, to prove them within the framework of logistic. It is open to question, however, whether this result should be regarded as yielding a proper philosophical understanding of arithmetical propositions. If we consider, e. g., the logistic proof of an equation such as $3 + 7 = 10$, we can see that within the proof we have to carry out quite the same comparative verification that occurs in our usual counting. This necessity comes to the fore particularly clearly in the formalized version logic; but it is also present if we interpret the content of the formula logically. The logical definition of three-numberedness, for example, is structurally so constituted that it contains within itself, as it were, the element of three-numberedness. For the three-numberedness of a predicate P (or of the class that forms the extension of P) is defined in terms of the condition that there exist things x, y, z having the property P and differing from each other pairwise, and further that everything having the property P is identical with x or y or z. Now, the conclusion that for a three-numbered predicate P and a seven-numbered predicate Q, in the case where these predicates do not apply to anything jointly, the disjunction $P \vee Q$ is a ten-numbered predicate requires for its justification just the kind of comparison that is used in elementary calculation—only that here an additional logical apparatus (the "frills") comes into play as well. When this is clearly realized, it appears that the proposition in predicate logic is valid because $3 + 7 = 10$ holds, not vice versa.

In spite of the possibility of incorporating it into logistic, arithmetic constitutes thus the more abstract (the "purer") schema; and this seems paradoxical only because of the traditional, but on closer examination unjustified, view according to which logical generality is the highest generality in every respect.

It might be good to look at the matter from yet another side as well. According to Frege, a cardinal number is to be defined as the property of a predicate. This view is already problematic with respect to the normal use of the number concept; for in many contexts in which a number is determined the specification of a predicate of which it is the property appears to be highly forced. It should be noted, in particular, that numbers occur not only in statements, but also in directions and commands—for example, when a housewife says to an errand-boy: "Fetch me ten apples." Moreover, the theoretical elaboration of this view is not without its complications either. In general, a definite number does not belong to a predicate as such, but only relative to a domain of objects, a universe of discourse (even apart from the many cases of extra-scientific predicates to which no determinate number can be ascribed at all). Thus it would be more appropriate to characterize a number as a relation between a predicate and a domain of individuals. To be sure, in Frege's theory this complication does not occur because he presupposes what might be called an absolute domain of individuals. But as we know now, it it is precisely this approach that leads to the contradiction noted by Russell. Apart from that, the Fregean conception of the theory of predicates, according to which the courses of values of predicates are treated as things on the same level as ordinary individuals, already constitutes a clear deviation from customary logic, understood as the theoretical construction of a general framework. The idea of such a framework has retained its methodological importance, and the question as to its most appropriate form is still one of the main problems in foundational research. However, with regard to such a framework one can speak of a "logic" only in an extended sense. Logic in its usual sense, in which it merely means the specification of the general rules for deductive reasoning, must be distinguished from it.

Yet Wittgenstein's criticism of the incorporation of arithmetic into logic is not advanced in the sense that he acknowledges arithmetical propositions as stating facts that are *sui generis*. Instead, his tendency is to deny that such propositions express facts at all. He even declares it to be the "curse of the invasion of mathematics by mathematical logic that now any proposition can be represented in a mathematical symbolism, and this makes us feel obliged to understand it. Although of course this method of writing is nothing but the translation of vague ordinary prose" (p. 155, No. 46). In fact, he recognizes calculating only as an acquired skill with practical utility. More particularly, he seeks to explain away what seems factual about arithmetic as definitional. He asks, for instance (p. 33, No. 112): "What am I calling 'the multiplication 13×13'? Only the correct pattern of multiplication, at the end of which comes 169? Or a 'wrong multiplication' too?" Elsewhere, too, he raises the question as to what it is that we "call calculating" (p. 97, No. 73). And on p. 92, No. 58 he argues: "Suppose it were said: 'By calculating we get acquainted with the properties of numbers.' But do the properties of numbers *exist* outside the calculating?" The tendency is, apparently, to take correct additions and multiplications as defining calculating, thus to characterize them as "correct" in a trivial sense. But this doesn't work out in the end, i. e., one cannot express in this way the general facts that hold in terms of the arithmetic relations of numbers. Let us take, say, the associativity of addition. It is certainly possible to fix by definition the addition of single digits. But then the strange fact remains that the addition $3+(7+8)$ gives the same result as $(3+7)+8$, and the same holds whatever numbers replace $3, 7, 8$. With respect to possible definitions the number-theoretic expressions are, so to speak, over-determined. It is actually on this kind of over-determinateness that many of the checks available in calculating are based.

Occasionally Wittgenstein raises the question as to whether the result of a calculation carried out in the decimal system also holds for the comparison of numbers by means of their direct representation as sequences of strokes. The answer to this question is to be found in the usual mathematical justification of the method of calculating with decadic figures. But Wittgenstein does touch upon something fundamental here: the proofs for the justification of the decadic rules of calculation rest, if they are given in a finitist way, upon the assumption that every number that can be formed decadically can also be produced in the direct stroke notation, and that the operations of concatenation etc., as well as of comparison can always be performed with such stroke sequences. What this shows is that even finitistic number theory is not in the full sense "concrete," but uses idealizations.

The previously mentioned statements in which Wittgenstein speaks of the synthetic character of mathematics are in apparent contrast with his tendency to regard numerical calculation as merely definitional, as well as with his denial that arithmetical propositions are factual in the first place. Note in this connection the following passage (p. 160, No. 3): "How can you say that '... 625 ...' and '... 25 × 25 ...' say the same thing?—Only through our arithmetic do they *become one*." What is meant here is closely related to what Kant had in mind in his argument against the view that $7 + 5 = 12$ is a mere analytical proposition. Kant contends there that the concept 12 "is by no means already thought in merely thinking this union of 7 and 5," and he adds: "That 7 should be added to 5, I have indeed already thought in the concept of a sum $= 7 + 5$, but not that this sum is equivalent to the number 12" (*Critique of Pure Reason*, B 14ff.). In modern terminology, this Kantian argument could be expressed as follows: The concept "$7 + 5$" is an individual concept (to use Carnap's terminology) expressible by means of the description $\iota_x\ (x = 7{+}5)$, and this concept is different from the concept "12;" the only reason why this is not obvious is that we involuntarily carry out the addition of the small numbers 7 and 5 directly. We have here the case, in the new logic often discussed following the example of Frege, of two terms with a different "sense" but the same "meaning" (called "denotation" by A. Church); and to determine the synthetic or analytic character of a judgment one must, of course, always consider the sense, not the meaning. The Kantian thesis that mathematics is synthetic does, incidentally, not stand in conflict with what the Russellian school maintains when it declares the propositions of arithmetic to be analytic. For we have here two entirely different concepts of the analytic—a fact which, in recent times, has been pointed out especially by E. W. Beth.[4]

Another intrinsic tension is to be found in Wittgenstein's position with respect to logistic. On the one hand, he often tends towards regarding proofs as formalized. Thus we can read on p. 93, No. 64: "Suppose I were to set someone the problem: 'Find a proof of the proposition ...'—The answer would surely be to show me certain signs." The distinctive and indispensable role of everyday language relative to that of a formalized language is not given prominence in his remarks. He often speaks of "the language game,"

[4] *Vide* [**?**].

and does not restrict the use of this expression to an artificial formal language, for which alone it is really appropriate. Indeed, our natural language does not have the character of a game at all; it is part of us, almost in the same way in which our limbs are. Apparently Wittgenstein is here still under the sway of the idea of a scientific language that encompasses all scientific thought. In contrast with this stand his highly critical remarks about mathematical logic. Apart from the one already quoted concerning "the curse of the invasion of mathematics by mathematical logic," the following is especially noteworthy (p. 156, No. 48): "'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts. Of course in this it has only continued to build on the Aristotelian logic."

We can get clearer about the idea that seems to underlie this criticism if we keep in mind the following: the logical calculus was intended, by various of its founders, as a realization of the Leibnizian idea of a *characteristica universalis*. With regard to Aristotle, Wittgenstein's remark, if looked at more closely, is not a criticism; since all Aristotle wanted to do with his logic was to fix the usual forms of logical argument and to test their legitimacy. The task of a *characteristica universalis*, on the other hand, was to be much more comprehensive; it was to establish a conceptual world that would make it possible to understand all real connections. With respect to an undertaking aimed at that goal it cannot be taken for granted, however, that the grammatical structures of our language are to function also as the basic framework for the theory; since the categories of that grammar have a character that is at least partially anthropomorphic. At the same time it should be emphasized that, besides our usual logic, nothing even approaching its value has been devised in philosophy so far. What Hegel, in particular, put in place of Aristotelian logic when he rejected it consists of a mere comparing of universals in terms of analogies and associations, without any clear regulative procedure. This method can certainly not pass as even an approximate fulfillment of the Leibnizian idea.

Unfortunately, from Wittgenstein we do not get any guidance for how to replace conventional logic by something philosophically more efficient either. Most likely, he considered the analysis of the structures of reality to be a misguided project; his goal was, after all, not to find a procedure that is somehow determinate. The "logical compulsion," the "inexorability of logic," the "hardness of the logical must" are a constant stumbling-block for him and, again and again, a cause for consternation. Perhaps he does not always bear in mind that all these terms merely have the character of popular comparisons, and are inappropriate in many respects. The strictness of the logical and the exact does not constrain our freedom. Indeed, it is our very freedom that enables our intention to be precise in thought while confronted with a world of imprecision and inexactitude. Wittgenstein speaks of the "must of kinematics" as being "much harder that the causal must" (p. 37, No. 121). Is it not an aspect of freedom that we can conceive of virtual motions that are subject merely to kinematic laws, apart from the real, causally determined motions, and that we can compare the former with the latter?

Enlightened humanity has sought liberation in rational determination when confronted with the dominance of the merely authoritative. But at present awareness of this fact has for the most part been lost, and for many the validity of science appears to be an oppressive authority.

In Wittgenstein's case, it is certainly not this aspect that evokes his critical attitude towards scientific objectivity. Nevertheless, his tendency is to declare the intersubjective unanimity in the field of mathematics to be an heteronomous one. Our agreement, he believes, is to be explained by the fact that we are in the first place "trained" together in basic techniques, and that the agreement thus created is then continued through the proofs (cf. the quotation on p. 195). That this kind of explanation is inadequate will occur to anybody not blinded by the apparent originality of the point. Already the possibility of our calculating techniques, with their manifold possibilities of decomposing a problem into simpler parts made possible by the validity of the laws of arithmetic, cannot be regarded as a consequence of agreement (cf. the remark three pages earlier). Furthermore, when we think of the enormously rich and systematic formations of concepts in, e. g., the theory of functions—where one can say of the theorems obtained at each stage what Wittgenstein once said: "We rest, or lean, on them" (p. 124, No. 35)—we see that the position mentioned above doesn't in any way explain why these conceptual edifices are not continually collapsing. Considering Wittgenstein's point of view, it is in fact not surprising that he does not feel the contradiction to be something strange; but what does not become clear in his account is that contradictions in mathematics are to be found only in quite peripheral extrapolations and nowhere else. In this respect one can say that Wittgenstein's philosophy does not make the fact of mathematics intelligible at all.

But what is the source of Wittgenstein's initial conviction that in the domain of mathematics there is no proper objectual knowledge, that everything consists instead of techniques, measuring devices, and customary attitudes? He must think: “There is nothing here to which knowledge could be directed.” This is connected with the fact, already mentioned above, that he does not recognize any role for phenomenology. What probably provokes his opposition to it are phrases such as when one talks about the “essence” of a color; where the word “essence” suggests the idea of hidden properties of the color, whereas colors as such are nothing but what is evident in their manifest properties and relations. But this does not prevent such properties and relations from being the content of objective statements; colors are, after all, not nothing. And even if we do not adopt the pretensions of Husserl’s philosophy with regard to the “intuition of essences,” this does not preclude the possibility of an objective phenomenology. The fact that phenomenological investigations in the domain of colors and sounds are still in their infancy is surely connected with the fact that they have no great importance for theoretical physics; since in physics we are induced, already at an early stage, to eliminate colors and sounds as qualities. Mathematics, on the other hand, can be regarded as the theoretical phenomenology of structures. Indeed, what contrasts phenomenologically with the qualitative is not the quantitative, as traditional philosophy teaches, but the structural, which consists of the forms of juxtaposition, succession, and composition, together with all the corresponding concepts and laws.

Such a conception of mathematics leaves ones position with respect to the problems of the foundations of mathematics still largely undetermined. But it can open the door, for someone starting with Wittgenstein’s views, for a viewpoint that does greater justice to the peculiar character and the significance of the mathematical.