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# On the current question of method in Hilbert's proof theory (1938/1941)

## Über die aktuelle Methodenfrage der Hilbertschen Beweistheorie

(Manuscript from the *Nachlass*)

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This report on the current situation of Hilbert's proof theory also includes some theoretical considerations. I remark at the outset that, concerning the the existing situation, the views presented here cannot claim to represent without qualification the standpoint of Hilbert's school.

This combination of principled observations with the exposition of the current situation in proof theory is suggested by this situation itself. As you may know, proof theory has recently suffered from a crisis of sorts, and some have already declared the Hilbertian enterprise a failure. This assessment of the situation is based on the circumstance that the program of proof theory as Hilbert proposed it in his publications from 1922–1927 is, to all appearances, in need of revision, namely, in respect to the methodological standpoint to be assumed.

Technically speaking, the issue is that one needs stronger methods of inference for metamathematical reasoning than those with which Hilbert originally thought he could get by, in the sense of his "finitistic attitude." This need was already felt in connection with a problem that was thought to have been solved, namely the demonstration of the consistency of the full arithmetical formalism.

In connection herewith it also became clear that the finitistic standpoint as intended by Hilbert is not—as it first had seemed—equivalent with

Brouwer's intuitionism. Gödel succeeded in showing that in the formalism of number theory, with help of a rather simple interpretation, all modes of inference of classical mathematics can be transformed into intuitionistically admissable ones. Hence from the standpoint of intuitionism the consistency of the number theoretical formalism follows directly.

Here by the formalism of number theory we mean that formal deductive system that is obtained from the logical calculus of first order (called "predicate calculus" or "restricted functional calculus"), the axioms of equality, the number theoretical axioms:

$$a' \neq 0, \quad a' = b' \;\rightarrow\; a = b \quad (a' \text{ denotes the number succeeding } a)$$

as well as the schema of complete induction and the elementary recursive definitions. (The notion of the least number of a certain property, which occurs in number theoretical deductions, can be avoided in the investigation into consistency by the elimination procedure for the notion "that, which.")

This formalism already slightly exceeds what is absolutely necessary to formalize number theoretical proofs. In fact, as Skolem was first to show, for this purpose a more restricted formalism of "recursive number theory" suffices, and which is still capable of a direct finitary interpretation.

The formalism of number theory considered here differs from recursive number theory as well as from intuitionistic number theory by the unrestricted employment of the notions "all" and "there is."

However, for the inferences representable in the formalism of number theory, an agreement can be reached between the proponent of the usual mathematical standpoint (who regards all these modes of inference as legitimate) and the intuitionist (who does not in general acknowledge the principle of excluded middle). This can be accomplished as follows: the former has to declare that a proposition of the form "there is a $x$ such that $\mathfrak{A}(x)$ holds" should merely be another way of expressing that, in any case, the opposite of $\mathfrak{A}(x)$ does not hold for all $x$. Likewise, a proposition "$\mathfrak{A}$ or $\mathfrak{B}$" should say nothing other than that the opposite of $\mathfrak{A}$ and the opposite of $\mathfrak{B}$ do not both hold. With this interpretation of the existential judgement and the disjunction, the intuitionist must acknowledge as legitimate all modes of inference in the mentioned domain of classical mathematics—at least, if he accepts the rules of intuitionistic inference devised by Heyting.

Now, this discovery that the intuitionistic modes of inference in number theory are so close to the "classical" ones results, on the one hand, immediately in a demonstration of the consistency of the number theoretical formalism from the standpoint of intuitionism. On the other hand, this discovery shows that the intuitionistic standpoint differs essentially from the finitistic. In particular, one will note the following difference as to general propositions (propositions of general form): while intuitionism only contests the application of the law of the excluded middle to such general propositions, the finitist standpoint avoids, in principle, the negation of general propositions, as well as their employment as premisses in conditional sentences.

The negation of a proposition has a finitary meaning only if it is equivalent to a claim with positive content. Thus, e. g., the negative proposition "the numeral $\mathfrak{a}$ is not identical with the numeral $\mathfrak{b}$" denotes the same as the positive claim that the numeral $\mathfrak{a}$ is different from the numeral $\mathfrak{b}$. And a condition or an assumption is finitary only if it has as its content either an intuitively determined configuration or an intuitively determined operation (respectively, the result of such operations). Thus, e. g., the assumption that Fermat's last theorem is true is not finitary. The assumption, however, that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{n}$ are four positive integers (numerals) such that $\mathfrak{n} > 2$ and $\mathfrak{a}^{\mathfrak{n}} + \mathfrak{b}^{\mathfrak{n}} = \mathfrak{c}^{\mathfrak{n}}$—i. e., the assumption that the four numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{n}$ provide a counterexample to Fermat's last theorem—is finitary. Furthermore, the assumption that this theorem is deducible in the formalism of number theory is finitary in the following sense: one assumes as given a figure of formulae—with a terminal formula representing Fermat's theorem—having the properties of a deduction within the formalism of number theory. The assumption, however, that some intuitively compelling proof of Fermat's last theorem is given is not finitary.

Negations, and hence the negations of general propositions in particular, can of course be eliminated in intuitionism. By an arbitrary choice of an elementary false proposition, e. g., $0 = 1$, one is able to interpret the negation $\overline{\mathfrak{A}}$ of a proposition $\mathfrak{A}$ by $\mathfrak{A} \rightarrow 0 = 1$ ("the assumption $\mathfrak{A}$ results in $0 = 1$"). With this interpretation, the intuitionistic modes of inference which employ negation transform themselves into intuitionistically admissable inferences.

But the elimination of negation thus gained is only apparent, in that we find ourselves forced to operate with hypothetical conditional sentences. That is, implications $\mathfrak{A} \rightarrow \mathfrak{B}$ occur, which are to be interpreted in a hypothetical sense: "Suppose $\mathfrak{A}$ held, then $\mathfrak{B}$ would result." In fact, such indirect arguments are used not only for elementary propositions $\mathfrak{A}$—for which they are admissable in finitary reasoning as well—but in an essential way also for general sentences and for implications with general (or even more logically complex) sentences as premisses.

In any case, the use of the notion "absurdity" for arbitrary propositions remains essential for intuitionistic reflections.

Now, considering the fact that the finitistic standpoint has proven too narrow for proof theory, the following question occurs: Is it necessary to take over all the methodological presuppositions of intuitionism?

At the moment, we can give at least a partial answer to this question. For Gentzen has provided a consistency proof for the formalism of number theory, whose methodological requirements constitute a kind of intermediate step between the finitistic standpoint and the standpoint of intuitionism.

It is advisable to refer to the newer version of the proof, also given by Gentzen. For, in comparison to the version first published, it has not only the advantage that here the proof idea is made perspicuous, but also that certain methodological complications of the first proof become unnecessary.

Recently, Gentzen's newer proof has again been simplified by Kalmár, and it turns out that in particular one can dispense with Gentzen's transformation of the number theoretical formalism into a certain equivalent calculus.

Let me briefly sketch the logical form of Gentzen's consistency proof, with an eye toward how the finitistic standpoint is exceeded (with certain insignificant deviations from Gentzen's presentation).

According to a remark already used in previous consistency proofs, asserting the consistency of the formalism of number theory comes to the same thing as asserting that in this formalism the formula $0 = 1$—which we indicate by "$\mathfrak{f}$"—is not deducible. That is the same as to assert that each deduction within this formalism has a terminal formula different from $\mathfrak{f}$.

One can see directly that this [latter] assertion is true for those deductions in which neither complete induction nor the rules for "all" and "there is" are employed—which we call, for short, "elementary deductions."

For the general demonstration "ordinal numbers" are employed, taken from the domain of Cantor's first und second number class (these are those below Cantor's first $\epsilon$-number). The introduction of these numbers can be made in an independent way, i. e., without recourse to Cantor's theory: the respective ordinal numbers can be characterized as certain (finite) figures, for which one can define, intuitively, a "smaller than" relation—with the properties of a well-ordering—in such a way, that for two different ordinal numbers it is always decidable which one of the two is the smaller one.

One then assigns an ordinal number to each deduction of the number theoretical formalism by a simple rule of computation. Based on this assignment, one can determine for each non-elementary deduction another deduction with the same terminal formula but a smaller ordinal number. This results in the following: if each deduction with an ordinal number smaller than a certain ordinal number $\alpha$ has a terminal formula different from $\mathfrak{f}$, then the same is true of each deduction with the ordinal number $\alpha$.

So far the proof remains within the framework of finitary reasoning. Now, to get from this consequence to the result that, generally, each deduction in the number theoretical formalism has a terminal formula different from $\mathfrak{f}$—which is the assertion to be proved—it is still necessary to justify the following principle of inference: "If a proposition $\mathfrak{B}(\alpha)$ about an ordinal number $\alpha$ holds for 0 (the least of the ordinal numbers), and if one can determine for each ordinal number $\alpha$ a smaller ordinal number $\beta$ such that, whenever $\mathfrak{B}(\beta)$ holds, also $\mathfrak{B}(\alpha)$ holds, then $\mathfrak{B}(\alpha)$ holds for each ordinal number $\alpha$." This mode of inference is in turn taken from the principle: "If a proposition $\mathfrak{B}(\alpha)$ about an ordinal number $\alpha$ holds for 0, and if it holds for the ordinal number $\alpha$ whenever it holds for each smaller ordinal , then it holds for each ordinal number."

This principle of inference is a kind of generalisation of complete induction. In set theory, a generalized induction of this kind is called "transfinite induction," because it extends to transfinite ordinal numbers. For our purposes, however, this expression is not appropriate. For we employ the word "finite" in a methodological sense, and the difference between ordinary induction (inference from $n$ to $n+1$) and transfinite induction does not coincide at all with the difference between finitistic and non-finitistic modes of inference. In general, an ordinary induction is finitistic only if the predicate (and whether it holds for a number) is elementary. On the other hand, there are (according to the usual terminology) transfinite inductions, which are still of a finitistic character.

What matters for us here is not so much to fix the exact limit up to which inductions are finitistic. Rather, it is to make clear to ourselves, from the intuitive standpoint, upon what the legitimacy of the principle of inference under consideration rests, and in what way it constitutes a proper generalisation of the ordinary induction.

Let us recall how the finitistic motivation for ordinary induction proceeds: we have the assumption that $\mathfrak{A}(0)$ holds and that we can infer $\mathfrak{A}(\mathfrak{n}+1)$ from $\mathfrak{A}(\mathfrak{n})$. Because we can arrive at each finite number by an iterated progression of 1 starting at 0, we can likewise infer from $\mathfrak{A}(0)$ that $\mathfrak{A}(\mathfrak{n})$ holds for each finite number $n$.

Now, the ordering of the ordinal numbers under consideration is analogous to that of the ordinary number series. This holds insofar as the former also has the property of a well-ordering—every initial segment has an element which immediately succeeds it—and, even more, the order type of this well-ordering can be reduced, in a recursive manner, to the natural order of the number series. Thereby an intuitive kind of "running through" is made possible. With reference to this, Cantorian set theory speaks of "counting beyond the infinite."

This counting beyond the infinite of course does not mean operating with a representation of the actual infinite. Rather, it means the transition from a progressive process to its metamathematical consideration. This transition is of the kind that already takes place in ordinary induction, with which we go beyond the particular propositions $\mathfrak{A}(0)$, $\mathfrak{A}(1)$, $\mathfrak{A}(2)$, $\ldots$, by means of the general metamathematical observation that we can arrive at the proposition $\mathfrak{A}(\mathfrak{n})$ for all $\mathfrak{n}$.

While running through the order type under consideration, superposed inductions occur. That is, we obtain higher inductions from ordinary induction by employing metamathematical considerations to the processes of iterating inductions. Now, to this superposition of inductions corresponds, as the logical form of expressing it, a superposition of conditional sentences in which general sentences enter as premisses. But these are always those general sentences which are seen to be true by means of the already-mentioned metamathematical considerations, so that here the conditional form has the meaning of anticipating one stage in a progressive process of inference.

Hence, the use of the principle of transfinite induction under consideration amounts to an extension of the methodological framework of proof theory, if not a complete acceptance of the intuitionistic modes of inference. The procedure of this extension can also be generalized. For it is possible to comprehend intuitively the "running through" of types of well-ordering even higher than those employed in Gentzen's consistency proof (the ordinal numbers below Cantor's first $\epsilon$-number), and therewith to intuitively justify the principle of transfinite induction related to this well-order type.

At the moment, there is no way to determine whether such a higher induction principle, taken as an additional means (i. e., added to the finitistic methods), suffices for a consistency proof of analysis.

According to Gödel's general theorem on formally underivable sentences, the induction principle in question—which would in any case be expressible as a theorem about a certain well-order of the ordinary numbers—would have to be such that its proof cannot be formalizable within the framework of analysis. At first, it seems impossible to satisfy this requirement; for the general theory of well-orderings of the number series, including the general theorem on transfinite induction, can be developed in the formalism of analysis. However, one has to keep in mind that the general theorem of transfinite induction does not determine whether a certain defined ordering of the number series is a well-ordering; and the higher principle of induction in question could amount to just such an assertion.

In any case, in view of these considerations it does not seem expedient to fix the methodological framework for proof theoretical investigations in advance. The expectation that the finitary standpoint (in its original sense) would suffice for the whole of proof theory was aroused by the fact that the problems of proof theory can already be formulated from this standpoint. But there is no simple relationship between the ability to express and to prove sentences, and therefore neither is there one between the ability to formulate and to solve problems.

But now the question arises, what, then, is the characterization of the methodological limitation of proof theory, if not the demand for the elementary evidence that distinguishes the finitistic standpoint. The answer is that the tendency toward methodological limitation remains essentially the same, except that, if we want to retain the possibility of extending the methodological framework, then we must avoid using the concepts of evidence and security in a sense that is overly absolute. On the other hand, we thereby gain the principal advantage of not having to stigmatize the usual methods of analysis as unjustified or doubtful.

The distinctive general aspect of the methodological attitude of Hilbert's proof theory can be recognized in the value it places on restriction to the arithmetical mode of thinking, in a strict sense, while the usual methods of analysis and set theory are, in an essential respect, inspired by geometrical ideas, especially that of a point mannifold, and draw their intuitive evidence therefrom. In fact, one can say—and this surely is the main point of the finitistic and intuitionistic critiques of the usual procedures in mathematics—that the arithmetization of geometry in analysis and set theory is not without remainder.

The methodological orientation of Hilbert's proof theory can contribute to a forceful development of the specifically arithmetical mode of thinking and to bringing out more clearly the stages in the formation of arithmetical concepts.

Moreover, concerning the achievements of proof theory it should be emphasized that the proof of consistency of the formalism of number theory in no way represents the only progress that metamathematical investigations of recent years has to show. Especially with regard to questions of decidability and the effective calculability of functions, remarkable results have been achieved through the investigations of Gödel, Church, Turing, Kleene, and Rosser. Metamathematics today is already such that its appreciation is independent of ones position on philosophical questions of foundational research.