Introduction to Categorical Logic

[DRAFT: October 31, 2009]

Steven Awodey Andrej Bauer

October 31, 2009

Contents

Chapter 2

First-Order Logic

Having considered type theories, we now move on to first-order logic. This is the usual logic with propositional connectives $\wedge$, $\Rightarrow$, and $\vee$, and quantifiers $\forall$ and $\exists$. The general approach to studying logic via category theory is similar to the approach taken for type theory—we specify certain categorical structures and show how to use them to model first-order logic, or a suitable fragment of it. Here adjoint functors play an imporant role, as the basic logical operations are recognized as adjoints. We also show that semantics is "functorial", which means that models of a theory are functors that preserve suitable categorical structure. Finally, we construct classifying categories, which are the counterparts of the syntactic categories of type theories and algebraic theories that we have already met.

Let us demonstrate our approach with an example. In section **??** we considered models of algebraic theories in categories with finite products. Recall that e.g. a group is a structure of the form:

$$e : 1 \to G \;, \qquad\qquad m : G \times G \to G \;, \qquad\qquad i : G \to G \;.$$

for which, moreover, certain diagrams built from these basic arrows must commute. We can express some properties of groups in terms of equations, for example commutativity

$$x \cdot y = y \cdot x \;.$$

As we saw, such equations can be interpreted in any category with finite products. This provides a large scope for categorical semantics of algebraic theories.

However, there are also many significant properties of algebraic structures which cannot be expressed with equations. Consider the statement that a group (G, e, m, i) has no non-trivial roots of unity,

$$\forall\, x : G \,.\, (x \cdot x = e \Rightarrow x = e) \;. \tag{2.1}$$

This is a first-order logic statement which cannot be rewritten as a system of equations. To see what its categorical interpretation ought to be, we look at its usual set-theoretic interpretation. Each of subformulas, $x \cdot x = e$ and $x = e$, determines a subset of G:

$$\begin{tikzcd}
\{x \in G \mid x \cdot x = e\} \arrow[rr, tail, dotted] \arrow[dr, tail, "i"'] & & \{x \in G \mid x = e\} \arrow[dl, tail, "j"] \\
& G &
\end{tikzcd}$$

The implication $x \cdot x = e \Rightarrow x = e$ holds when $\{x \in G \mid x \cdot x = e\}$ is contained in $\{x \in G \mid x = e\}$. In categorical language we can say that the inclusion i factors through the inclusion j. Observe also that such a factorization is necessarily a mono and is unique, if it exists. The defining formulas of the subsets $\{x \in G \mid x \cdot x = e\}$ and $\{x \in G \mid x = e\}$ are equations, and so the subsets themselves can be constructed as equalizers (as above, interpreting $\cdot$ as m):

$$\begin{tikzcd}
\{x \in G \mid x \cdot x = e\} \arrow[r, tail] & G \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[rr, bend right, "e \circ !_G"'] & G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
\{x \in G \mid x = e\} \arrow[r, tail] & G \arrow[r, shift left, "1_G"] \arrow[r, shift right, "e \circ !_G"'] & G
\end{tikzcd}$$

Note that the second equalizer is in fact isomorphic to the morphism $e : 1 \to G$.

In sum, we can interpret condition (2.1) in any category with products and equalizers, i.e. in any category with finite limits.[1] This allows us to define the notion of a group without roots of unit in any category $\mathcal{C}$ with finite limits as an object G with morphisms $e : 1 \to G$, $m : G \times G \to G$ and $i : G \to G$ such that (G, e, m, i) is a group in $\mathcal{C}$, and the equalizer of $m \circ \langle 1_G, 1_G \rangle$ and $e \circ !_G$ factors through $e : 1 \to G$.

The aim of this chapter is to analyze how such examples can be treated in general. We want to relate first-order logic and fragments of it to categorical structures that are suitable for the interpretation of the logic. The general outline will be as follows:

1. A language $\mathcal{L}$ for a first-order theory consists, as usual, of some basic relation, function, and constant symbols, say $\mathcal{L} = (R, f, c)$.

2. An $\mathcal{L}$-structure in a category $\mathcal{C}$ with finite limits is an interpretation of $\mathcal{L}$ in $\mathcal{C}$ as an object M equipped with corresponding relations and operations (of appropriate arities) and "points", e.g.

$$R^M \rightarrowtail M \times \cdots \times M$$
$$f^M : M \times \cdots \times M \to M$$
$$c^M : 1 \to M.$$

---

[1] We are *not* claiming that finite limits suffice for an interpretation of arbitrary formulas built from universal quantifiers and implications. The formula at hand has a very special form $\forall x \,.\, (\varphi(x) \Longrightarrow \psi(x))$, where $\varphi(x)$ and $\psi(x)$ do not contain further $\forall$ or $\Longrightarrow$.

3. Formulas $\varphi(x_1, \ldots, x_n)$ in (some fragment of) first-order logic will be interpreted as "generalized subsets", i.e. subobjects,

$$[\![\varphi(x_1, \ldots, x_n)]\!] \rightarrowtail M \times \cdots \times M.$$

   The interpretation makes use of categorical operations in $\mathcal{C}$ corresponding to the logical ones appearing in the formula $\varphi(x_1, \ldots, x_n)$.

4. A theory $\mathbb{T}$ in (a fragment of) first-order logic will consist of a set of (binary) sequents,

$$\varphi(x_1, \ldots, x_n) \vdash \psi(x_1, \ldots, x_n).$$

5. A model of the theory is then an interpretation M in which the corresponding subobjects satisfy all the sequents of $\mathbb{T}$, in the sense that

$$[\![\varphi(x_1, \ldots, x_n)]\!] \leq [\![\psi(x_1, \ldots, x_n)]\!] \qquad \text{in } \mathsf{Sub}(M^n).$$

6. We shall give a deductive calculus for such sequents, prove that it is sound with respect to categorical models, and then use it to construct a classifying category $\mathcal{C}_{\mathbb{T}}$, with the expected universal property: models of $\mathbb{T}$ correspond to (structure-preserving) functors on $\mathcal{C}_{\mathbb{T}}$.

7. Completeness of the calculus in general follows from classification, and more specialized completeness results from embedding theorems applied to the classifying category.

2.1 Theories in first-order logic

A *first-order theory* $\mathbb{T}$ consists of an underlying *type theory* and a set of formulas in a *fragment of first-order logic*. Recall from Chapter **??** that a type theory is given by a set of basic types, a set of basic constants together with their types, rules for forming types, rules and axioms for deriving typing judgments

$$x_1 : A_1, \ldots, x_n : A_n \mid t : B \ ,$$

expressing that term t has type B in typing context $x_1 : A_1, \ldots, x_n : A_n$, and a set of axioms and rules of inference which tell us which equations between terms

$$x_1 : A_1, \ldots, x_n : A_n \mid t = u : B \ .$$

are valid. This part of the theory $\mathbb{T}$ may be regarded as providing the underlying structure, on top of which the logical formulas are defined.

A fragment of first-order logic is given by a set of *basic relation symbols* together with a specification of which first-order operations are being considered in building formulas.

Each basic relation symbol has a *signature* $(A_1, \dots, A_n)$, which specifies the types of its arguments. The *arity* of a relation symbol is the number of arguments it takes. The judgment[2]

$$x_1 : A_1, \dots, x_n : A_n \mid \phi \ \mathtt{pred}$$

states that ϕ is a well-formed formula in typing context $x_1 : A_1, \dots, x_n : A_n$. For each basic relation symbol R with signature $(A_1, \dots, A_n)$ there is an inference rule

$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \dots, t_n) \ \mathtt{pred}}$$

Depending on what fragment of first-order logic is involved, there may be other rules for forming logical formulas. For example, if equality is present, then for each type A there is a rule

$$\frac{\Gamma \mid t : A \qquad \Gamma \mid u : A}{\Gamma \mid t =_A u \ \mathtt{pred}}$$

and if conjunction is present, then there is a rule

$$\frac{\Gamma \mid \varphi \ \mathtt{pred} \qquad \Gamma \mid \psi \ \mathtt{pred}}{\Gamma \mid \varphi \wedge \psi \ \mathtt{pred}}$$

Other such rules will be given when we come to the study of particular logical operations.

The basic logical judgment of a first-order theory is *logical entailment* between formulas,

$$x_1 : A_1, \dots, x_n : A_n \mid \varphi_1, \dots, \varphi_m \vdash \psi$$

which states that in the typing context $x_1 : A_1, \dots, x_n : A_n$, the hypotheses $\varphi_1, \dots, \varphi_m$ entail ψ. It is understood that the terms appearing in the formulas are well-typed in the typing context, and that formulas $\varphi_1, \dots, \varphi_m, \psi$ are part of the fragment of the logic of $\mathbb{T}$. When the fragment contains conjunction $\wedge$ it is convenient to restrict attention to *binary* sequents,

$$x_1 : A_1, \dots, x_n : A_n \mid \varphi \vdash \psi,$$

by replacing $\varphi_1, \dots, \varphi_m$ with $\varphi_1 \wedge \ldots \wedge \varphi_m$. When the fragment contains equality, we may replace the type-theoretic equality judgments

$$x_1 : A_1, \dots, x_n : A_n \mid t = u : B$$

with the logical statements

$$x_1 : A_1, \dots, x_n : A_n \mid \cdot \vdash t =_B u \ .$$

The subscript at the equality sign indicates the type at which the equality is taken. In a theory $\mathbb{T}$ there are basic entailments, or axioms, which together with the inference rules for the opertations involved can be used for deriving valid judgments, as usual.

We shall consider several standard fragments of first-order logic, determined by selecting a subset of logical connectives and quantifiers. These are as follows:

---

[2] We follow type-theoretic practice here by adding the tag `pred` to turn what would otherwise be an exhibited formula in context into a judgement concerning the formula.

1. *Full first-order logic* is built from logical operations
$$= \quad \top \quad \bot \quad \neg \quad \wedge \quad \vee \quad \Rightarrow \quad \forall \quad \exists\,.$$
2. *Cartesian logic* is the fragment built from
$$= \quad \top \quad \wedge\,.$$
3. *Regular logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists\,.$$
4. *Coherent logic* is the fragment built from
$$= \quad \top \quad \wedge \quad \exists \quad \bot \quad \vee\,.$$
5. A *geometric formula* is a formula of the form
$$\forall\, x : A\,.\,(\varphi \Longrightarrow \psi)\ ,$$
where φ and ψ are coherent formulas.

The names for these fragments come from the names of various categorical structures in which they are interpreted.

The well-formed terms and formulas of a first-order theory $\mathbb{T}$ constitute its *language*. It may seem that we are doing things backwards, because we should have spoken of first-order languages before we spoke of first-order theories. While this is possible for simple theories, it becomes difficult to do when we consider more complicated theories in which types and logical formulas are intertwined. In such cases the typing judgments and logical entailments may be given by a mutual recursive definition. In order to find out whether a given term is well-formed, we might have to prove a logical statement. In everyday mathematics this occurs all the time, for example, to show that the term $\int_0^\infty f$ denotes a real number, it may be necessary to prove that $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and that the integral has a finite value. This is why it does not always make sense to strictly differentiate a language from a theory.[3]

In order to focus on the logical part of first-order theories, we are going to limit attention to only two very simple kinds of type theory. A *single-sorted* first-order theory has as its underlying type theory a single type A, and for each $k \in \mathbb{N}$ a set of basic k-ary function symbols. The rules for typing judgments are:

1. Variables in contexts:
$$\frac{}{x_1 : A, \ldots, x_n : A \mid x_i : A}$$

---

[3] However, it *does* make sense to distinguish syntax from theory. Rules of substitution and the behaviour of free and bound variables are syntactic considerations, for example.

2. For each basic function symbol f of arity k, there is an inference rule
$$\frac{\Gamma \mid t_1 : A \qquad \cdots \qquad \Gamma \mid t_n : A}{\Gamma \mid f(t_1, \ldots, t_n) : A}$$

This much is essentially an algebraic theory. In addition, however, a single-sorted first-order theory may contain relation symbols, formulas, axioms, and rules of inference which an algebraic theory does not.

A slight generalization of a single-sorted theory is a *multi-sorted* one. Its underlying type theory is given by a set of types, and a set of basic function symbols. Each function symbol f has a *signature* $(A_1, \ldots, A_n; B)$, where n is the arity of f and $A_1, \ldots, A_n, B$ are types. The rules for typing judgments are:

1. Variables in contexts:
$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \mid x_i : A_i}$$

2. For each basic function symbol f with signature $(A_1, \ldots, A_n; B)$, there is an inference rule
$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid f(t_1, \ldots, t_n) : B}$$

We often write suggestively $f : A_1 \times \cdots \times A_n \to B$ to indicate that $(A_1, \ldots, A_n; B)$ is the signature of f. However, this does not mean that $A_1 \times \cdots \times A_n \to B$ is a type! A multi-sorted first-order theory does *not* have any type forming operations, such as $\times$ and $\to$.

2.2 Predicates as subobjects

Formulas of first-order logic will be interpreted as "generalized subsets", i.e. subobjects. We therefore need to review some of the basic theory of these.

Let A be an object in a category $\mathcal{C}$. If $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are monos into A, we say that i is smaller than j, and write $i \leq j$, when there exists a morphism $k : I \to J$ such that the following diagram commutes:

$$\begin{tikzcd}
I \arrow[rr, dotted, "k"] \arrow[dr, tail, "i"'] & & J \arrow[dl, tail, "j"] \\
& A &
\end{tikzcd}$$

If such a k exists then it, too, is monic, since i is, and it is unique, since j is monic. The class $\mathsf{Mono}(A)$ of all monos into A is this preordered by this relation $\leq$, it is the same as the slice category $\mathsf{Mono}(\mathcal{C})/A$ of all monos in $\mathcal{C}$, sliced over the object A. Let $\mathsf{Sub}(A)$ be the poset reflection of this preorder. Thus the elements of $\mathsf{Sub}(A)$ are equivalence classes of monos into A, where monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent when $i \leq j$ and $j \leq i$ (note that then $I \cong J$). The induced relation $\leq$ on $\mathsf{Sub}(A)$ is then a partial order.

We have to be a bit careful with the formation of $\mathsf{Sub}(A)$, since it is defined as a quotient of a *class* $\mathsf{Mono}(A)$. In many particular cases the general construction by quotients can be avoided. If we can demonstrate that the preorder $\mathsf{Mono}(A)$ is equivalent, as a category, to a poset P then we can simply take $\mathsf{Sub}(A) = P$. At any rate, we usually require that $\mathsf{Sub}(A)$ is small.

Definition 2.2.1 A category $\mathcal{C}$ is *well-powered* when, for all $A \in \mathcal{C}$, there is at most a *set* of subobjects of A, so that the category $\mathsf{Mono}(A)$ is equivalent to a small poset. In other words, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ is a small category.

We shall often speak of subobjects as if they were monos rather than equivalence classes of monos. It is understood that we mean the subobjects represented by monos and not the monos themselves. Sometimes we refer to a mono $i : I \rightarrowtail A$ by its domain I only, even though the object I itself does not determine the morphism i. Hopefully this will not cause confusion, as it is always going to be clear which mono is meant to go along with the object I.

The assignment $A \mapsto \mathsf{Sub}(A)$ is the object part of the *subobject functor*

$$\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Poset} .$$

The morphism part of Sub is pullback. More precisely, given a morphism $f : A \to B$, let $\mathsf{Sub}(f) = f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ be the monotone map which maps the subobject $[i : I \rightarrowtail B]$ to the subobject $[f^*i : f^*I \rightarrowtail A]$, where $f^*i : f^*I \rightarrowtail A$ is a pullback of i along f:

$$\begin{tikzcd}
f^*I \arrow[r] \arrow[d, tail, "f^*i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & I \arrow[d, tail, "i"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Recall that a pullback of a mono is again mono, so this definition makes sense. We need to verify (1) that if two monos $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ are equivalent, then their pullbacks are so as well; and (2) that $\mathsf{Sub}(1_A) = 1_{\mathsf{Sub}(A)}$ and $\mathsf{Sub}(g \circ f) = \mathsf{Sub}(f) \circ \mathsf{Sub}(g)$. These all follow easily from the fact that pullback is a functor $\mathcal{C}/B \to \mathcal{C}/A$, which reduces to the familiar "2-pullbacks" lemma:

Lemma 2.2.2 *Suppose both squares in the following diagram are pullbacks:*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Then the outer rectangle is a pullback diagram as well. Moreover, if the outer rectangle and the right square are pullbacks, then so is the left square.

Proof. This is left as an exercise in diagram chasing. ■

Of course, pullbacks are really only determined up to isomorphism, but this does not cause any problems because isomorphic monos represent the same subobject.

In the semantics to be given below, a formula

$$x : A \mid \varphi \ \texttt{pred}$$

will be interpreted as a subobject

$$[\![x : A \mid \varphi]\!] \rightarrowtail [\![A]\!].$$

Thus $\mathsf{Sub}(A)$ can be regarded as the poset of "predicates" on A, generalizing the powerset of a set A. Logical operations on formulas then correspond to operations on the $\mathsf{Sub}(A)$. Therefore, the structure of $\mathsf{Sub}(A)$ determines which logical connectives can be interpreted. If $\mathsf{Sub}(A)$ is a Heyting algebra, then we can interpret the full intuitionistic propositional calculus (cf. Subsection **??**), but if $\mathsf{Sub}(A)$ only has binary meets then all that can be interpreted are $\top$ and $\wedge$. We will work out details of different operations in the following sections, but one common aspect that we require is the "stability" of the interpretation of the logical operations, in a sense that we now make clear.

Stability under pullbacks

Let us consider the interpretation of substitution of terms for variables. There are two kinds of substitution, into a term, and into a formula. We may substitute a term $x : A \mid t : B$ for a variable y in a term $y : B \mid u : C$ to obtain a new term $x : A \mid u[t/y] : C$. If t and u are interpreted as morphisms

$$\begin{tikzcd} {[\![A]\!]} \arrow[r, "{[\![t]\!]}"] & {[\![B]\!]} \arrow[r, "{[\![u]\!]}"] & {[\![C]\!]} \end{tikzcd}$$

then $u[t/y]$ is interpreted as their composition:

$$[\![x : A \mid u[t/y] : C]\!] = [\![y : B \mid u : C]\!] \circ [\![x : A \mid t : B]\!] .$$

Thus, *substitution into a term is composition.*

The second kind of substitution occurs when we substitute a term $x : A \mid t : B$ for a variable y in a formula $y : B \mid \varphi$ to obtain a new formula $x : A \mid \varphi[t/y]$. If t is interpreted as a morphism $[\![t]\!] : [\![A]\!] \to [\![B]\!]$ and φ is interpreted as a subobject $[\![\varphi]\!] \rightarrowtail [\![B]\!]$ then the interpretation of $\varphi[t/y]$ is the pullback of $[\![\varphi]\!]$ along $[\![t]\!]$:

$$\begin{tikzcd} {[\![\varphi[t/y]]\!] = [\![t]\!]^*[\![\varphi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\varphi]\!]} \arrow[d, tail] \\ {[\![A]\!]} \arrow[r, "{[\![t]\!]}"'] & {[\![B]\!]} \end{tikzcd}$$

Thus, *substitution into a formula is pullback*,

$$[\![x : A \mid \varphi[t/y]\,]\!] = [\![x : A \mid t : B]\!]^{*}[\![y : B \mid \varphi]\!].$$

Note that the latter fact that substitution into a formula is interpreted as pullback generalizes the familiar situation of set-theoretic semantics,

$$x \in \{x \mid \varphi(f(x))\} \iff f(x) \in \{y \mid \varphi(y)\} \iff x \in f^{-1}(\{y \mid \varphi(y)\}).$$

Now, because substitution respects the syntactical, logical operations,

$$(\varphi \wedge \psi)[t/x] = \varphi[t/x] \wedge \psi[t/x],$$

the categorical structures used to interpret the various logical operations must also behave well with respect to the interpretation of substitution, i.e. pullback. We say that a categorical property or structure is *stable (under pullbacks)* if it is preserved by pullbacks.

For example, a category $\mathcal{C}$ has *stable meets* if each poset $\mathsf{Sub}(A)$ has binary meets, and the pullback of a meet $I \wedge J \rightarrowtail A$ along any map $f : B \to A$ is the meet $f^{*}I \wedge f^{*}J \rightarrowtail A$ of the respective pullbacks

$$f^{*}(I \wedge J) = f^{*}I \wedge f^{*}J.$$

This means that the meet operation,

$$\wedge : \mathsf{Sub}(A) \times \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A)$$

is natural in A.

Exercise 2.2.3 Show that any category $\mathcal{C}$ with finite limits has stable meets in the foregoing sense. Show, moreover, that each poset $\mathsf{Sub}(A)$ has all finite meets (i.e. including the "empty meet" 1), and that these are also stable under pullbacks. Conclude that $\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Posets}$ factors through the subcategory of $\wedge$-semi-lattices.

Generalized elements

In any category, we sometimes consider arbitrary arrows $x : X \to C$ as *generalized elements* of C, thinking thereby of variable elements or parameters. With respect to a subobject $U \rightarrowtail C$, such an element is said to be *in the subobject*, writtten

$$x \in_C U,$$

if it factors through (any mono representing) the subobject,

$$\begin{tikzcd} & U \arrow[d, tail] \\ X \arrow[ru, dotted] \arrow[r, "x"'] & C \end{tikzcd}$$

which, observe, it then does uniquely. The following "generalized element semantics" can be a useful technique for "externalizing" the operations on subobjects into statements about generalized elements.

Proposition 2.2.4 *Let C be any object in a category $\mathcal{C}$ with finite limits.*

1. *for the top element $1 \in \mathsf{Sub}(C)$ and any $U \in \mathsf{Sub}(C)$,*
$$U = 1 \iff x \in_C U \text{ for all } x : X \to C.$$
2. *for any $U, V \in \mathsf{Sub}(C)$,*
$$U \leq V \iff x \in_C U \text{ implies } x \in_C V, \text{ for all } x : X \to C.$$
3. *for any $U, V \in \mathsf{Sub}(C)$, and for all $x : X \to C$,*
$$x \in_C U \wedge V \iff x \in_C U \text{ and } x \in_C V.$$
4. *for the subobject $\Delta = [\langle 1_C, 1_C \rangle] \in \mathsf{Sub}(C \times C)$, and for all $x, y : X \to C$,*
$$\langle x, y \rangle \in \Delta \iff x = y.$$
5. *for the equalizer $E_{(f,g)} \rightarrowtail A$ of a pair of arrows $f, g : A \rightrightarrows B$, and for all $x : X \to A$,*
$$x \in_A E_{(f,g)} \iff fx = gx.$$
6. *for the pullback $f^*U \rightarrowtail A$ of a subobject $U \rightarrowtail B$ along any arrow $f : A \to B$, and for all $x : X \to A$,*
$$x \in_A f^*U \iff fx \in_B U.$$

Exercise 2.2.5 Prove the proposition.

2.3 Cartesian logic

As a first example we look at the logic of *cartesian categories*, which are categories with finite limits, to be called *cartesian logic*. This is a logic of formulas over a multi-sorted type theory with unit type `1`. (See section **??** for multi-sorted type theories and the axioms for the unit type). The logical operations are $=$, $\top$, and $\wedge$.

Formation rules for cartesian logic

Given a basic language consisting of a stock of relation and function symbols (with arities), the terms are built up as usual from the basic function symbols and variables (we take "constants" to be 0-ary function symbols). The rules for constructing formulas are as follows:

1. The 0-ary relation symbol $\top$ is a formula:

$$\frac{}{\Gamma \mid \top \ \mathtt{pred}}$$

2. For each basic relation symbol R with signature $(A_1, \ldots, A_n)$ there is a rule

$$\frac{\Gamma \mid t_1 : A_1 \qquad \cdots \qquad \Gamma \mid t_n : A_n}{\Gamma \mid R(t_1, \ldots, t_n)\ \texttt{pred}}$$

3. For each type A, there is a rule

$$\frac{\Gamma \mid s : A \qquad \Gamma \mid t : A}{\Gamma \mid s =_A t\ \texttt{pred}}$$

4. Conjunction:

$$\frac{\Gamma \mid \varphi\ \texttt{pred} \qquad \Gamma \mid \psi\ \texttt{pred}}{\Gamma \mid \varphi \wedge \psi\ \texttt{pred}}$$

5. Weakening:

$$\frac{\Gamma \mid \varphi\ \texttt{pred}}{\Gamma, x : A \mid \varphi\ \texttt{pred}}$$

Observe that, as usual, there is then a derived operation of substitution of terms for variables into formulas, defined by structural recursion on the above specification of formulas:

Substitution:

$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \varphi\ \texttt{pred}}{\Gamma \mid \varphi[t/x]\ \texttt{pred}}$$

Inference rules for cartesian logic

Although we do not yet need them, we state the rules of inference here, too, for the convenience of having the entire specification of cartesian logic in one place. As already mentioned, we can conveniently state this deductive calculus entirely in terms of *binary* sequents,

$$\Gamma \mid \psi \vdash \varphi.$$

We omit mention of the context Γ when it is the same in the premisses and conclusion of a rule.

1. Weakening:

$$\frac{\Gamma \mid \psi \vdash \varphi}{\Gamma, x : A \mid \psi \vdash \varphi}$$

2. Substitution:

$$\frac{\Gamma \mid t : A \qquad \Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi[t/x] \vdash \varphi[t/x]}$$

3. Identity:

$$\frac{}{\varphi \vdash \varphi}$$

4. Cut:
$$\frac{\psi \vdash \theta \qquad \theta \vdash \varphi}{\psi \vdash \varphi}$$

5. Equality:
$$\frac{}{\psi \vdash t =_A t} \qquad \frac{\psi \vdash t =_A u \qquad \psi \vdash \varphi[t/z]}{\psi \vdash \varphi[u/z]}$$

6. Truth:
$$\frac{}{\psi \vdash \top}$$

7. Conjunction:
$$\frac{\psi \vdash \varphi \qquad \psi \vdash \psi}{\psi \vdash \varphi \wedge \psi} \qquad \frac{\psi \vdash \varphi \wedge \psi}{\psi \vdash \psi} \qquad \frac{\psi \vdash \varphi \wedge \psi}{\psi \vdash \varphi}$$

Exercise 2.3.1 Derive symmetry and transitivity of equality:

$$\frac{\Gamma \mid \Psi \vdash t =_A u}{\Gamma \mid \Psi \vdash u =_A t} \qquad \frac{\Gamma \mid \Psi \vdash t =_A u \qquad \Gamma \mid \Psi \vdash u =_A v}{\Gamma \mid \Psi \vdash t =_A v}$$

Example 2.3.2 The theory of a poset is a cartesian theory. There is one basic sort $\mathtt{P}$ and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$. The axioms are the familiar axioms for reflexivity, transitivity, and antisymmetry:

$$\begin{aligned} x : \mathtt{P} &\mid \cdot \vdash x \leq x \\ x : \mathtt{P}, y : \mathtt{P}, z : \mathtt{P} &\mid x \leq y,\ y \leq z \vdash x \leq z \\ x : \mathtt{P}, y : \mathtt{P} &\mid x \leq y,\ y \leq x \vdash x =_{\mathtt{P}} y \end{aligned}$$

There are also many examples, such as ordered groups, ordered fields, etc., that are posets with further algebraic structure.

Example 2.3.3 An *equivalence relation* in a cartesian category is a model of the corresponding theory with one basic sort $\mathtt{A}$ and one binary relation symbol $\sim$ with signature $(\mathtt{A}, \mathtt{A})$. The axioms are the familiar axioms for reflexivity, symmetry, and transitivity:

$$\begin{aligned} x : \mathtt{A} &\mid \cdot \vdash x \sim x \\ x : \mathtt{A}, y : \mathtt{A} &\mid x \leq y \vdash y \leq x \\ x : \mathtt{A}, y : \mathtt{A}, z : \mathtt{A} &\mid x \leq y,\ y \leq z \vdash x \leq z \end{aligned}$$

Before we embark on the semantics of cartesian logic, we note a couple of useful propositions regarding cartesian categories.

Proposition 2.3.4 *If a category $\mathcal{C}$ has pullbacks then, for every $A \in \mathcal{C}$, $\mathsf{Sub}(A)$ has finite limits. Moreover, these are stable under pullback.*

Proof. The poset $\mathsf{Sub}(A)$ has finite limits if it has a top object and binary meets. The top object of $\mathsf{Sub}(A)$ is the subobject $[1_A : A \to A]$. The meet of subobjects $i : I \rightarrowtail A$ and $j : J \rightarrowtail A$ is the subobject $i \wedge j = i \circ (i^* j) = j \circ (j^* i) : I \wedge J \rightarrowtail A$ obtained by pullback, as in the following diagram:

$$\begin{tikzcd}
I \wedge J \arrow[r, tail, "j^* i"] \arrow[d, tail, "i^* j"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, tail, "j"] \\
I \arrow[r, tail, "i"'] & A
\end{tikzcd}$$

It is easy to verify that $I \wedge J$ is the infimum of I and J. Finally, stability follows from a familiar diagram chase on a cube of pullbacks. ■

Proposition 2.3.5 *If a category has finite products and pullbacks of monos along monos then it has all finite limits.*

Proof. It is sufficient to show that the category has equalizers. To construct the equalizer of parallel arrows $f : A \to B$ and $g : A \to B$, first observe that the arrows

$$A \xrightarrow{\langle 1_A, f\rangle} A \times B \qquad\qquad A \xrightarrow{\langle 1_A, g\rangle} A \times B$$

are monos because the projection $\pi_0 : A \times B \to A$ is their left inverse. Therefore, we may construct the pullback

$$\begin{tikzcd}
P \arrow[r, tail, "p"] \arrow[d, tail, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\langle 1_A, f\rangle"] \\
A \arrow[r, tail, "\langle 1_A, g\rangle"'] & A \times B
\end{tikzcd}$$

The morphisms p and q coincide because $\langle 1_A, f\rangle$ and $\langle 1_A, g\rangle$ have a common left inverse π_0:

$$p = 1_A \circ p = \pi_0 \circ \langle 1_A, f\rangle \circ p = \pi_0 \circ \langle 1_A, f\rangle \circ q = 1_A \circ q = q \ .$$

Let us show that $p : P \to A$ is the equalizer of f and g. First, p equalizes f and g,

$$f \circ p = \pi_1 \circ \langle 1_A, f\rangle \circ p = \pi_1 \circ \langle 1_A, g\rangle \circ q = g \circ q = g \circ p \ .$$

If $k : K \to A$ also equalizes f and g then

$$\langle 1_A, f\rangle \circ k = \langle k, f \circ k\rangle = \langle k, g \circ k\rangle = \langle 1_A, g\rangle \circ k \ ,$$

therefore by the universal property of the constructed pullback there exists a unique factorization $\overline{k} : K \to P$ such that $k = p \circ \overline{k}$, as required. ■

We now explain how cartesian logic is interpreted in a cartesian category $\mathcal{C}$ (i.e. $\mathcal{C}$ is finitely complete). Let $\mathbb{T}$ be a multi-sorted cartesian theory. Recall that the type theory of $\mathbb{T}$ is specified by a set of sorts (types) and a set of basic function symbols together with their signatures, while the logic is given by a set of basic relation symbols with their signatures, and a set of axioms in the form of logical entailments,

$$\Gamma \mid \psi \vdash \varphi .$$

An interpretation of $\mathbb{T}$ in $\mathcal{C}$ is given by the following data, where Γ stands for a typing context $x_1 : A_1, \ldots, x_n : A_n$, and ψ and ψ are formulas:

1. A sort A is interpreted as an object $[\![A]\!]$.
2. The unit sort $\mathtt{1}$ is interpreted as the terminal object $\mathtt{1}$.
3. A typing context $x_1 : A_1, \ldots, x_n : A_n$ is interpreted as the product $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$. The empty context is interpreted as the terminal object $\mathtt{1}$.
4. A basic function symbol f with signature $(A_1, \ldots, A_m; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots [\![A_m]\!] \to [\![B]\!]$.
5. A term in a context $\Gamma \mid t : B$ is interpreted as a morphism $[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$, as follows:
    - (a) A variable $x_0 : A_1, \ldots, x_n : A_n \mid x_i : A_i$ is interpreted as the i-th projection $\pi_i : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![A_i]\!]$.
    - (b) The interpretation of $\Gamma \mid * : \mathtt{1}$ is the unique morphism $!_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to \mathtt{1}$.
    - (c) A composite term $\Gamma \mid f(t_1, \ldots, t_m) : B$, where f is a basic function symbol with signature $(A_1, \ldots, A_m; B)$, is interpreted as the composition

      $$[\![\Gamma]\!] \xrightarrow{\langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle} [\![A_1]\!] \times \cdots \times [\![A_m]\!] \xrightarrow{[\![f]\!]} [\![B]\!]$$

      Here $[\![t_i]\!]$ is shorthand for $[\![\Gamma \mid t_i : A_i]\!]$.
6. A basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_n]\!])$.
7. A formula in a context $\Gamma \mid \varphi$ is interpreted as a subobject $[\![\Gamma \mid \varphi]\!] \in \mathsf{Sub}([\![\Gamma]\!])$. The details are given below.
8. A logical entailment $\Gamma \mid \psi \vdash \varphi$ is interpreted as an inequality $[\![\psi]\!] \leq [\![\varphi]\!]$ in $\mathsf{Sub}([\![\Gamma]\!])$.

It remains to explain how formulas are interpreted as subobjects. The logical constant $\top$ is interpreted as the maximal subobject:

$$[\![\Gamma \mid \top]\!] = [1_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]] \ .$$

An atomic formula $\Gamma \mid R(t_1, \dots, t_m)$, where R is a basic relation symbol with signature $(A_1, \dots, A_m)$ is interpreted as the left-hand side of the pullback pullback

$$\begin{tikzcd}
\llbracket R(t_1, \dots, t_m) \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & \llbracket R \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle"'] & \llbracket A_1 \rrbracket \times \cdots \times \llbracket A_m \rrbracket
\end{tikzcd}$$

An equation $\Gamma \mid t =_A u$ `pred` is interpreted as the subobject represented by the equalizer of $\llbracket \Gamma \mid t : A \rrbracket$ and $\llbracket \Gamma \mid u : A \rrbracket$:

$$\begin{tikzcd}
\llbracket \Gamma \mid t =_A u \rrbracket \arrow[r, tail] & \llbracket \Gamma \rrbracket \arrow[r, shift left, "\llbracket t \rrbracket"] \arrow[r, shift right, "\llbracket u \rrbracket"'] & \llbracket A \rrbracket
\end{tikzcd}$$

By Proposition 2.3.4, each $\mathsf{Sub}(A)$ is a poset with binary meets. Thus we interpret a conjunction $\Gamma \mid \varphi \wedge \psi$ `pred` as the infimum of subobjects

$$\llbracket \Gamma \mid \varphi \wedge \psi \rrbracket = \llbracket \Gamma \mid \varphi \rrbracket \wedge \llbracket \Gamma \mid \psi \rrbracket \ .$$

A formula formed by weakening is interpreted as pullback along a projection:

$$\begin{tikzcd}
\llbracket \Gamma, x : A \mid \varphi \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & \llbracket \Gamma \mid \varphi \rrbracket \arrow[d, tail, "i"] \\
\llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \arrow[r, "\pi"'] & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

This pullback can be computed and the interpretation of $\llbracket \Gamma, x : A \mid \varphi \rrbracket$ turns out to be the subobject

$$\begin{tikzcd}
\llbracket \Gamma \mid \varphi \rrbracket \times \llbracket A \rrbracket \arrow[r, tail, "i \times 1_A"] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$

This concludes the description of an interpretation of cartesian theory $\mathbb{T}$ in a cartesian category $\mathcal{C}$.

As was explained in the previous section, the operation of substitution of terms into formulas is then interpreted as pullback:

Lemma 2.3.6 *Let the formula* $\Gamma, x : A \mid \varphi$ *and the term* $\Gamma \mid t : A$ *be given. Then the substituted formula* $\Gamma \mid \varphi[t/x]$ *is interpreted as the pullback indicated in the following diagram:*

$$\begin{tikzcd}
\llbracket \Gamma \mid \varphi[t/x] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & \llbracket \Gamma, x : A \mid \varphi \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle 1_\Gamma, \llbracket t \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$

Proof. A simple induction on the structure of φ. ■

Exercise 2.3.7 Prove this.

When we deal with many interpretations at once we name them M, N, $\dots$, and subscript the semantic brackets accordingly, $[\![\Gamma]\!]_M$, $[\![\Gamma]\!]_N$, $\dots$

If $\Gamma \mid \psi \vdash \psi$ is a logical entailment in $\mathbb{T}$ such that $[\![\Gamma \mid \psi]\!]_M \leq [\![\Gamma \mid \varphi]\!]_M$ holds in an interpretation M, then we say that M *satisfies* or *models* $\Gamma \mid \psi \vdash \varphi$ and write

$$M \models \ \Gamma \mid \psi \vdash \varphi\ .$$

An interpretation M is a *model* of $\mathbb{T}$ if it satisfies all the axioms of $\mathbb{T}$.

Theorem 2.3.8 (Soundness of cartesian logic) *If a cartesian theory* $\mathbb{T}$ *proves an entailment*

$$\Gamma \mid \psi \vdash \varphi$$

then every model M *of* $\mathbb{T}$ *satisfies the entailment:*

$$M \models \ \Gamma \mid \psi \vdash \varphi\ .$$

Proof. The proof proceeds by induction on the proof of the entailment. In the following we often omit the typing context Γ to simplify notation, and all inequalities are interpreted in $\mathsf{Sub}([\![\Gamma]\!])$. We consider all possible last steps in the proof of the entailment:

1. Weakening: if $[\![\Gamma \mid \psi]\!] \leq [\![\Gamma \mid \varphi]\!]$ in $\mathsf{Sub}([\![\Gamma]\!])$ then
$$[\![\Gamma, x : A \mid \psi]\!] = [\![\Gamma \mid \psi]\!] \times A \leq [\![\Gamma \mid \varphi]\!] \times A = [\![\Gamma, x : A \mid \varphi]\!] \quad \text{in } \mathsf{Sub}([\![\Gamma, x : A]\!]).$$

2. Substitution: by lemma 2.3.6, substitution is interpreted by pullback so that $[\![\varphi[t/x]]\!] = \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!]$ and $[\![\psi[t/x]]\!] = \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!]$. Because
$$\langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* : \mathsf{Sub}([\![\psi]\!]) \to \mathsf{Sub}([\![\psi]\!] \times [\![A]\!])$$
is a functor it is a monotone map, therefore $[\![\psi]\!] \leq [\![\varphi]\!]$ implies
$$\langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\psi]\!] \leq \langle 1_{[\![\psi]\!]}, [\![t]\!]\rangle^* [\![\varphi]\!]\ .$$

3. Identity: trivially
$$[\![\varphi]\!] \leq [\![\varphi]\!]\ .$$

4. Cut: if $[\![\psi]\!] \leq [\![\theta]\!]$ and $[\![\theta]\!] \leq [\![\varphi]\!]$ then clearly $[\![\psi]\!] \leq [\![\varphi]\!]$, since $\mathsf{Sub}([\![\Gamma, x : A]\!])$ is a poset.

5. Truth: trivially $[\![\psi]\!] \leq [\![\top]\!]$.

6. The rules for conjunction clearly hold because by the definition of infimum $\llbracket \Psi \rrbracket \leq \llbracket \varphi \wedge \psi \rrbracket$ if, and only if, $\llbracket \Psi \rrbracket \leq \llbracket \varphi \rrbracket$ and $\llbracket \Psi \rrbracket \leq \llbracket \psi \rrbracket$.

7. Equality: the axiom $t =_A t$ is satisfied because an equalizer of $\llbracket t \rrbracket$ with itself is the maximal subobject:
$$\llbracket \psi \rrbracket \leq [1_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \to \llbracket \Gamma \rrbracket] = \llbracket t =_A t \rrbracket \ .$$
For the other axiom, suppose $\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket$ and $\llbracket \psi \rrbracket \leq \llbracket \varphi[t/z] \rrbracket$. It suffices to show $\llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket$ for then
$$\llbracket \psi \rrbracket \leq \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket \leq \llbracket \varphi[u/z] \rrbracket \ .$$
The interpretation of $P = \llbracket t =_A u \rrbracket \wedge \llbracket \varphi[t/z] \rrbracket$ is obtained by two successive pullbacks, as in the following diagram:
$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \varphi[t/z] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \varphi \rrbracket \arrow[d, tail] \\
\llbracket t =_A u \rrbracket \arrow[r, tail, "e"'] & \llbracket \Gamma \rrbracket \arrow[r, "\langle 1_\Gamma, \llbracket t \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$
Here e is the equalizer of $\llbracket u \rrbracket$ and $\llbracket t \rrbracket$. Observe that e equalizes $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle$ and $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle$ as well:
$$\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle \circ e = \langle e, \llbracket t \rrbracket \circ e \rangle = \langle e, \llbracket u \rrbracket \circ e \rangle = \langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle \circ e \ .$$
Therefore, if we replace $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle$ with $\langle 1_{\llbracket \Gamma \rrbracket}, \llbracket u \rrbracket \rangle$ in the above diagram, the outer rectangle still commutes. By the universal property of the pullback
$$\begin{tikzcd}
\llbracket \Gamma \mid \varphi[u/z] \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma, z : A \mid \varphi \rrbracket \arrow[d, tail] \\
\llbracket \Gamma \rrbracket \arrow[r, "\langle 1_\Gamma, \llbracket u \rrbracket \rangle"'] & \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket
\end{tikzcd}$$
it follows that P factors through $\llbracket \varphi[u/z] \rrbracket$, as required.

■

Example 2.3.9 Recall the cartesian theory of posets (example 2.3.2). There is one basic sort `P` and one binary relation symbol $\leq$ with signature $(\mathtt{P}, \mathtt{P})$ and the axioms of reflexivity, transitivity, and antisymmetry. A poset in a cartesian category $\mathcal{C}$ is thus given by an object P, which is the interpretation of the sort `P`, and a subobject $r : R \rightarrowtail P \times P$, which the interpretation of $\leq$, such that the axioms are satisfied. As an example we spell out when the reflexivity axiom is satisfied. The interpretation of $x : P \mid x \leq x$ is obtained by the following pullback:

$$\begin{tikzcd}
\llbracket x \leq x \rrbracket \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
P \arrow[r, tail, "\delta_P"'] & P \times P
\end{tikzcd}$$

where $\delta_P = \langle 1_P, 1_P \rangle$ is the diagonal. The first axiom is satisfied when $\llbracket x \leq x \rrbracket = P$, which happens if, and only if, δ_P factors through r. Therefore, reflexivity can be expressed as follows: there exists a "reflexivity" morphism $\rho : P \to R$ such that $r \circ \rho = \delta_P$. Equivalently, morphisms $\pi_0 \circ r$ and $\pi_1 \circ r$ have a common right inverse ρ.

Since the definition of a poset in a cartesian category is thus stated entirely in terms of finite limits, and these are computed pointwise in functor categories $\mathsf{Set}^{\mathbb{C}}$, it follows that a poset P in $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor $P : \mathbb{C} \to \mathsf{Poset}$. Indeed, as was the case for algebraic theories, we have an equivalence (and isomorphism, actually) of categories,

$$\mathsf{Poset}(\mathsf{Set}^{\mathbb{C}}) \;\cong\; \mathsf{Poset}(\mathsf{Set})^{\mathbb{C}} \;\cong\; \mathsf{Poset}^{\mathbb{C}}.$$

2.3.1 Subset types

Let us consider whether the theory of a category is a cartesian theory. We begin by expressing the definition of a category so that it can be interpreted in any cartesian category $\mathcal{C}$. An *internal category* in $\mathcal{C}$ consists of an *object of morphisms* C_1, an *object of objects* C_0, and *domain*, *codomain*, and *identity* morphisms,

$$\mathsf{dom} : C_1 \to C_0 \;, \qquad \mathsf{cod} : C_1 \to C_0 \;, \qquad \mathsf{id} : C_0 \to C_1 \;.$$

There is also a *composition* morphism $c : C_2 \to C_1$, where C_2 is obtained by the pullback

$$\begin{tikzcd}
C_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\mathsf{dom}"] \\
C_1 \arrow[r, "\mathsf{cod}"'] & C_0
\end{tikzcd}$$

The following equations must hold:

$$\begin{gathered}
\mathsf{dom} \circ i = 1_{C_0} = \mathsf{cod} \circ i \;, \\
\mathsf{cod} \circ p_1 = \mathsf{cod} \circ c \;, \qquad \mathsf{dom} \circ p_0 = \mathsf{dom} \circ c \;. \\
c \circ \langle 1_{C_1}, i \circ \mathsf{dom} \rangle = 1_{C_1} = c \circ \langle i \circ \mathsf{cod}, 1_{C_1} \rangle \;,
\end{gathered}$$

The first two equations state that the domain and codomain of an identity morphism 1_A are both A. The second equation states that $\mathsf{cod}\,(f \circ g) = \mathsf{cod}\, f$ and the third one that

$\mathsf{dom}\,(f \circ g) = \mathsf{dom}\, g$. The fourth equation states that $f \circ 1_{\mathsf{dom}\, f} = f = 1_{\mathsf{cod}\, f} \circ f$. It remains to express associativity of composition. For this purpose we construct the pullback

$$\begin{tikzcd}
C_3 \arrow[r, "q_2"] \arrow[d, "q_{01}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\mathsf{dom}"] \\
C_2 \arrow[r, "\mathsf{cod} \circ p_1"'] & C_0
\end{tikzcd}$$

The object C_3 can be thought of as the set of triples of morphisms (f, g, h) such that $\mathsf{cod}\, f = \mathsf{dom}\, g$ and $\mathsf{cod}\, g = \mathsf{dom}\, h$. We denote $q_0 = p_0 \circ q_{01}$ and $q_1 = p_1 \circ q_{01}$. The morphisms $q_0, q_1, q_2 : C_3 \to C_1$ are like three projections which select the first, second, and third element of a triple, respectively. With this notation we can write $q_{01} = \langle q_0, q_1 \rangle_{C_2}$ because q_{01} is the unique morphism such that $p_0 \circ q_{01} = q_0$ and $p_1 \circ q_{01} = q_1$. The subscript C_2 reminds us that the "pair" $\langle q_0, q_1 \rangle_{C_2}$ is obtained by the universal property of the pullback C_2.

Morphisms $c \circ q_{01} : C_3 \to C_1$ and $q_2 : C_3 \to C_1$ factor through the pullback C_2 because

$$\mathsf{cod} \circ c \circ q_{01} = \mathsf{cod} \circ p_1 \circ q_0 = \mathsf{dom} \circ q_2 \ .$$

Thus let $r : C_3 \to C_2$ be the unique factorization for which $p_0 \circ r = c \circ q_{01}$ and $p_1 \circ r = q_2$. Because p_0 and p_1 are like projections from C_2 to C_1, morphism r can be thought of as a pair of morphisms, so we write $r = \langle c \circ q_{01}, q_2 \rangle_{C_2}$. Morphism $c \circ \langle c \circ q_{01}, q_2 \rangle_{C_2} : C_3 \to C_1$ corresponds to the operations $\langle f, g, h \rangle \mapsto (f, g) \circ h$, whereas the morphism corresponding to $\langle f, g, h \rangle \mapsto f \circ (g \circ h)$ is obtained in a similar way and is equal to

$$c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} : C_3 \to C_1 \ .$$

Thus associativity is expressed by the equation

$$c \circ \langle c \circ \langle q_0, q_1 \rangle_{C_2}, q_2 \rangle_{C_2} = c \circ \langle q_0, c \circ \langle q_1, q_2 \rangle_{C_2} \rangle_{C_2} \ .$$

Example 2.3.10 An internal category in Set is a small category.

We have successfully formulated the theory of a category so that it makes sense in any cartesian category. In fact, the definition of an internal category refers only to certain pullbacks, hence the notion of an internal category makes sense in any category with pullbacks. However, if we try to formulate it as a multi-sorted cartesian theory, there is a problem. Obviously, there ought to be a basic sort of objects $\mathtt{C}_0$ and a basic sort of morphisms $\mathtt{C}_1$. There are also basic function symbols with signatures

$$\mathtt{dom} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{cod} : (\mathtt{C}_1; \mathtt{C}_0) \qquad \mathtt{id} : (\mathtt{C}_0, \mathtt{C}_1) \ .$$

However, it is not clear what the signature for composition should be. It is not $(\mathtt{C}_1, \mathtt{C}_1; \mathtt{C}_1)$ because composition is undefined for non-composable pairs of morphisms. We might be tempted to postulate another basic sort $\mathtt{C}_2$ but then we would have no way of stating that $\mathtt{C}_2$ is the pullback of $\mathtt{dom}$ and $\mathtt{cod}$. And even if we somehow axiomatized the fact that $\mathtt{C}_2$ is a pullback, we would then still have to formalize the object $\mathtt{C}_3$ of composable triples, $\mathtt{C}_4$ of composable quadruples, and so on. What we lack is the ability to define the type $\mathtt{C}_2$ as a *subset type* of $\mathtt{C}_1 \times \mathtt{C}_1$.

In order to remedy the situation we need to use a richer type theory, namely one that allows *simple subset types.* We explain what these are. The formation rule for simple subset types is

$$\frac{x : A \mid \varphi \ \mathtt{pred}}{\{x : A \mid \varphi\} \ \mathtt{type}}$$

We can think of $\{x : A \mid \varphi\}$ as the subset of all those $x : A$ that satisfy φ. Note that we did not allow an arbitrary context Γ to be present. This means that we cannot define subset types that depend on parameters, which why they are called "simple".

Inference rules for subset types are as follows:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \mathtt{in}_\varphi\, t : A} \qquad \frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \varphi[t/x]} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}}$$

$$\begin{array}{c} \Gamma, x : A \mid \Psi, \varphi \vdash \theta \\ \hline\hline \Gamma, y : \{x : A \mid \varphi\} \mid \Psi[\mathtt{in}_\varphi\, y/x] \vdash \theta[\mathtt{in}_\varphi\, y/x] \end{array}$$

The first rule states that a term t of subset type $\{x : A \mid \varphi\}$ can be converted to a term $\mathtt{in}_\varphi\, t$ of type A. We can think of the constant $\mathtt{in}_\varphi$ as the *inclusion* $\mathtt{in}_\varphi : \{x : A \mid \varphi\} \to A$. The second rule states that every term of a subset type $\{x : A \mid \varphi\}$ satisfies the defining predicate φ. The third rule states that a term t of type A which satisfies φ can be converted to a term $\mathtt{rs}_\varphi\, t$ of type $\{x : A \mid \varphi\}$. A good way to think of the constant $\mathtt{rs}_\varphi$ is as a partially defined *restriction*, or a type-casting operations, $\mathtt{rs}_\varphi : A \rightharpoonup \{x : A \mid \varphi\}$.[4] The last rule tells us how to replace a variable x of type A and an assumption φ about it with a variable y of type $\{x : A \mid \varphi\}$ and remove the assumption. Note that this is a two-way rule.

There are two more axioms that relate inclusions and restrictions:

$$\frac{\Gamma \mid t : \{x : A \mid \varphi\}}{\Gamma \mid \cdot \vdash \mathtt{rs}_\varphi\,(\mathtt{in}_\varphi\, t) = t} \qquad\qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \cdot \vdash \varphi[t/x]}{\Gamma \mid \cdot \vdash \mathtt{in}_\varphi\,(\mathtt{rs}_\varphi\, t) = t\ .}$$

In an informal discussion it is customary for the inclusions and restrictions to be omitted, or at least for the subscript φ to be missing.[5]

Exercise 2.3.11 Suppose $x : A \mid \psi$ and $x : A \mid \varphi$ are formulas. Show that

$$x : A \mid \psi \vdash \varphi$$

---

[4] Inclusions and restrictions are like type-casting operations in some programming languages. For example in Java, an inclusion corresponds to an (implicit) type cast from a class to its superclass, whereas a restriction corresponds to a type cast from a class to a subclass. Must I write that Java is a registered trademark of Sun Microsystems?

[5] Strictly speaking, even the notation $\mathtt{in}_\varphi\, t$ is imprecise because it does not indiciate that ϕ stands in the context $x : A$. The correct notation would be $\mathtt{in}_{(x:A|\varphi)}\, t$, where x is bound in the subscript. A similar remark holds for $\mathtt{rs}_\varphi\, t$.

is provable if, and only if, $\{x : A \mid \psi\}$ factors through $\{x : A \mid \varphi\}$, which means that there exists a term k,

$$y : \{x : A \mid \psi\} \mid k : \{x : A \mid \varphi\} \ ,$$

such that

$$y : \{x : A \mid \psi\} \mid \cdot \vdash \mathtt{in}_\psi\, y =_A \mathtt{in}_\varphi\, k$$

is provable. Show also that k is determined uniquely up to provable equality.

Example 2.3.12 We are now able to formulate the theory of a category as a cartesian theory whose underlying type theory has product types and subset types. The basic types are the type of objects $\mathtt{C}_0$ and the type of morphisms $\mathtt{C}_1$. We define the type $\mathtt{C}_2$ to be

$$\mathtt{C}_2 \equiv \{p : C_1 \times C_1 \mid \mathtt{cod}(\mathtt{fst}\, p) = \mathtt{dom}(\mathtt{snd}\, p)\} \ .$$

The basic function symbols and their signatures are:

$$\mathtt{dom} : \mathtt{C}_1 \longrightarrow \mathtt{C}_0 \ , \qquad \mathtt{cod} : \mathtt{C}_1 \longrightarrow \mathtt{C}_0 \ , \qquad \mathtt{id} : \mathtt{C}_0 \longrightarrow \mathtt{C}_1 \ , \qquad \mathtt{c} : \mathtt{C}_2 \longrightarrow \mathtt{C}_1 \ .$$

The axioms are:

$$\begin{aligned}
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{dom}(\mathtt{id}(a)) = a \\
a : \mathtt{C}_0 \mid \cdot &\vdash \mathtt{cod}(\mathtt{id}(a)) = a \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{dom}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = f \\
f : \mathtt{C}_1, g : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g) &\vdash \mathtt{cod}(\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)) = g \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle \mathtt{id}(\mathtt{dom}(f)), f\rangle) = f \\
f : \mathtt{C}_1 \mid \cdot &\vdash \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{id}(\mathtt{cod}(f))\rangle) = f
\end{aligned}$$

Lastly, the associativity axiom is

$$\begin{aligned}
&f : \mathtt{C}_1, g : \mathtt{C}_1, h : \mathtt{C}_1 \mid \mathtt{cod}(f) = \mathtt{dom}(g), \mathtt{cod}(g) = \mathtt{dom}(h) \vdash \\
&\qquad\qquad \mathtt{c}(\mathtt{rs}\,\langle \mathtt{c}(\mathtt{rs}\,\langle f, g\rangle), h\rangle) = \mathtt{c}(\mathtt{rs}\,\langle f, \mathtt{c}(\mathtt{rs}\,\langle g, h\rangle)\rangle) \ .
\end{aligned}$$

This notation is quite unreadable. If we write $g \circ f$ instead of $\mathtt{c}(\mathtt{rs}\,\langle f, g\rangle)$ then the axioms take on a more familiar form. For example, associativity is just $h \circ (g \circ f) = (h \circ g) \circ f$. However, we need to remember that we may form the term $g \circ f$ only if we first prove $\mathtt{dom}(g) = \mathtt{cod}(f)$.

A subset type $\{x : A \mid \varphi\}$ is interpreted as the domain of a monomorphism representing $x : A \mid \varphi$:

$$\begin{tikzcd}
\llbracket \{x : A \mid \varphi\} \rrbracket \arrow[r, tail, "\llbracket x : A \mid \varphi \rrbracket"] & \llbracket A \rrbracket
\end{tikzcd}$$

Some care must be taken here because monos representing a given subobject are only determined up to isomorphism. We assume that a suitable canonical choice of monos can be made.

An inclusion $\Gamma \mid \mathtt{in}_\varphi\, t : A$ is interpreted as the composition

$$\begin{tikzcd}
[\![\Gamma]\!] \arrow[r, "{[\![t]\!]}"] & {[\![\{x : A \mid \varphi\}]\!]} \arrow[r, tail, "{[\![x : A \mid \varphi]\!]}"] & {[\![A]\!]}
\end{tikzcd}$$

A restriction $\Gamma \mid \mathtt{rs}_\varphi\, t : \{x : A \mid \varphi\}$ is interpreted as the unique $\overline{[\![t]\!]}$ which makes the following diagram commute:

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "\overline{[\![t]\!]}"] \arrow[dr, "{[\![t]\!]}"'] & {[\![x : A \mid \varphi]\!]} \arrow[d, tail] \\
 & {[\![A]\!]}
\end{tikzcd}$$

Exercise 2.3.13 Formulate and prove a soundness theorem for subset types. Pay attention to the interpretation of restrictions, where you need to show unique existence of $\overline{[\![t]\!]}$.

2.4 Quantifiers

The categorical semantics of quantification is one of the central features of the subject, and quite possibly one of the nicest contributions of categorical logic to the field of logic. You might expect that the quantifiers $\forall$ and $\exists$ are “just a big conjunction and disjunction”, respectively. In fact the Polish school of algebraic logicians worked to realize this point of view—but categorical logic shows how quantifiers are treated algebraically as adjoint functors to give a much more satisfactory theory.

2.5 Quantifiers as adjoints

Let us first recall the rules of inference for quantifiers. The formation rules are:

$$\frac{\Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid (\forall\, x : A \,.\, \varphi) \ \mathtt{pred}} \qquad\qquad \frac{\Gamma, x : A \mid \varphi \ \mathtt{pred}}{\Gamma \mid (\exists\, x : A \,.\, \varphi) \ \mathtt{pred}}$$

The variable x is bound in $\forall\, x : A \,.\, \varphi$ and $\exists\, x : A \,.\, \varphi$. If x and y are distinct variables and x does not occur freely in the term t then substitution of t for y commutes with quantification over x:

$$(\exists\, x : A \,.\, \varphi)[t/y] = \exists\, x : A \,.\, (\varphi[t/y]) \ ,$$
$$(\forall\, x : A \,.\, \varphi)[t/y] = \forall\, x : A \,.\, (\varphi[t/y]) \ .$$

For each quantifier we have a two-way rule of inference:

$$\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall\, x : A \,.\, \varphi} \qquad\qquad \frac{\Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid (\exists\, x : A \,.\, \varphi) \vdash \vartheta}$$

Note that these rules implicitly impose the usual condition that x must not occur freely in ψ and ϑ, because ψ and ϑ are supposed to be well formed in context Γ, which does not contain x.

Exercise 2.5.1 A common way of stating the inference rules for quantifiers is as follows. For the universal quantifier, the introduction and elimination rules are

$$\frac{\Gamma, x : A \mid \psi \vdash \varphi}{\Gamma \mid \psi \vdash \forall x : A \,.\, \varphi} \qquad\qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \forall x : A \,.\, \varphi}{\Gamma \mid \psi \vdash \varphi[t/x]}$$

The introduction rule for existential quantifier is

$$\frac{\Gamma \mid t : A \qquad \Gamma \mid \psi \vdash \varphi[t/x]}{\Gamma \mid \psi \vdash \exists x : A \,.\, \varphi}$$

and the elimination rule is

$$\frac{\Gamma \mid \psi \vdash \exists x : A \,.\, \varphi \qquad \Gamma, x : A \mid \varphi \vdash \vartheta}{\Gamma \mid \psi \vdash \vartheta}$$

Note that these rules implicitly impose a requirement that x does not occur in Γ and that it does not occur freely in ψ because the context $\Gamma, x : A$ must be well formed and the hypotheses ψ must be well formed in context Γ. Show that these rules can be derived from the ones above, and vice versa. Of course, you may also use the inference rules for cartesian logic, cf. page 15.

In order to discover what the semantics of existential quantifier ought to be, we look at the following instance of the two-way rule for quantifiers:

$$\begin{array}{c} y : B, x : A \mid \varphi \vdash \vartheta \\ \hline\hline y : B \mid \exists x : A \,.\, \varphi \vdash \vartheta \end{array} \tag{2.2}$$

First observe that this rule implicitly requires

$$y : B, x : A \mid \varphi \ \mathtt{pred} \qquad y : B \mid \vartheta \ \mathtt{pred} \qquad y : B \mid (\exists x : A \,.\, \varphi) \ \mathtt{pred}$$

Therefore the interpretations of φ, ϑ, and $\exists x : A \,.\, \varphi$ are subobjects

$$\begin{aligned} [\![y : B, x : A \mid \varphi]\!] &\in \mathsf{Sub}([\![B]\!] \times [\![A]\!]) \,, \\ [\![y : B \mid \vartheta]\!] &\in \mathsf{Sub}([\![B]\!]) \,, \\ [\![y : B \mid \exists x : A \,.\, \varphi]\!] &\in \mathsf{Sub}([\![B]\!]) \,. \end{aligned}$$

In fact, ϑ appears twice, once in the context $y : B$ and once in the context $y : B, x : A$. The later instance is obtained from the former by a weakening rule

$$\frac{y : B \mid \vartheta \ \mathtt{pred}}{y : B, x : A \mid \vartheta \ \mathtt{pred}}$$

The interpretation of weakening is pullback along a projection, cf. page 19, as in the following pullback diagram:

$$\begin{tikzcd}
\llbracket y : B, x : A \mid \vartheta \rrbracket \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket y : B \mid \vartheta \rrbracket \arrow[d, tail] \\
\llbracket B \rrbracket \times \llbracket A \rrbracket \arrow[r, "\pi"'] & \llbracket B \rrbracket
\end{tikzcd}$$

Thus we have

$$\llbracket y : B, x : A \mid \vartheta \rrbracket = \pi^* \llbracket y : B \mid \vartheta \rrbracket = \llbracket y : B \mid \vartheta \rrbracket \times \llbracket A \rrbracket \ .$$

We want to interpret existential quantification $\exists x : A$ as a suitable functor $\exists_A : \mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) \to \mathsf{Sub}(\llbracket B \rrbracket)$ so that

$$\llbracket y : B \mid \exists\, x : A \,.\, \varphi \rrbracket = \exists_A \llbracket y : B, x : A \mid \varphi \rrbracket \ .$$

The two-way rule (2.2) is then interpreted as a two-way inequality rule

$$\begin{array}{c} \llbracket y : B, x : A \mid \varphi \rrbracket \leq \pi^* \llbracket y : B \mid \vartheta \rrbracket \\ \hline\hline \exists_A \llbracket y : B, x : A \mid \varphi \rrbracket \leq \llbracket y : B \mid \vartheta \rrbracket \end{array}$$

If we replace the interpretations of φ and ϑ by general subobjects $S \in \mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket)$ and $T \in \mathsf{Sub}(\llbracket B \rrbracket)$, we obtain

$$\begin{array}{c} S \leq \pi^* T \\ \hline\hline \exists_A S \leq T \end{array}$$

This is nothing but an adjunction between $\exists_A$ and π^*! Therefore, *existential quantification is left-adjoint to weakening*:

$$\exists_A \dashv \pi^* \ .$$

A dual argument shows that *universal quantification is right-adjoint to weakening*:

$$\pi^* \dashv \forall_A \ .$$

Thus, in sum, we have shown that the rules of inference require the quantifiers to be interpreted as operations that are adjoints to the interpretation of weakening, i.e. pullback π^* along the projection $\pi : \llbracket B \rrbracket \times \llbracket A \rrbracket \to \llbracket B \rrbracket$.

$$\begin{tikzcd}
\mathsf{Sub}(\llbracket B \rrbracket \times \llbracket A \rrbracket) \arrow[d, bend right=60, "\exists"'] \arrow[d, bend left=60, "\forall"] \\
\mathsf{Sub}(\llbracket B \rrbracket) \arrow[u, "\dashv\ \pi^*\ \dashv" description]
\end{tikzcd}$$

Let us see how this works for the usual interpretation in Set. A predicate $y : B, x : A \mid \varphi$ corresponds to a subset $\varphi \subseteq B \times A$, and $y : B \mid \vartheta$ corresponds to a subset $\vartheta \subseteq B$. Weakening of ϑ is the subset $\pi^* \vartheta = \vartheta \times A \subseteq B \times A$. Then we have

$$\exists_A \varphi = \big\{ y \in B \;\big|\; \exists\, x : A \,.\, \langle x, y \rangle \in \varphi \big\} \subseteq B \;,$$
$$\forall_A \varphi = \big\{ y \in B \;\big|\; \forall\, x : A \,.\, \langle x, y \rangle \in \varphi \big\} \subseteq B \;.$$

A moment's thought convinces us that with this interpretation we really have

$$\frac{\varphi \subseteq \vartheta \times A}{\exists_A \varphi \subseteq \vartheta} \qquad\qquad \frac{\vartheta \times A \subseteq \varphi}{\vartheta \subseteq \forall_A \varphi}$$

The unit of the adjunction $\exists_A \dashv \pi^*$ amounts to the inequality

$$\varphi \subseteq (\exists_A \varphi) \times A \;, \tag{2.3}$$

and the universal property of the unit says that $\exists_A \varphi$ is the smallest set satisfying (2.3). Similarly, the counit of the adjunction $\pi^* \dashv \forall_A$ is just the inequality

$$(\forall_A \varphi) \times A \subseteq \varphi \;, \tag{2.4}$$

and the universal property of the counit says that $\forall_A \varphi$ is the largest set satisfying (2.4). Figure 2.1 shows the geometric meaning of existential and universal quantification.

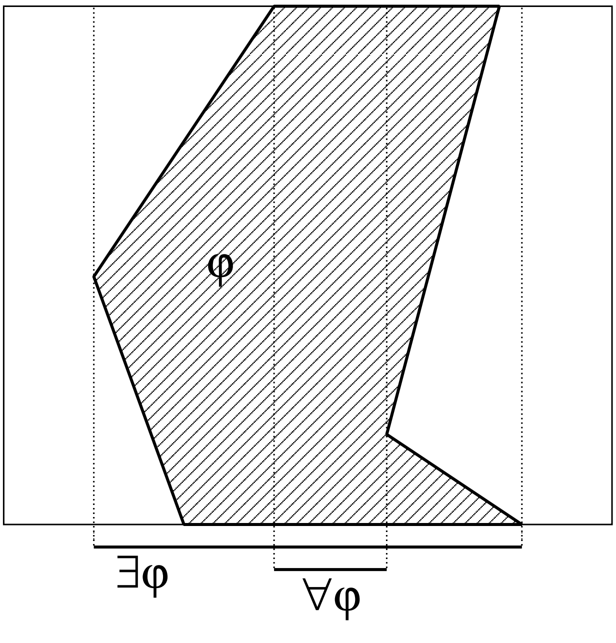

Figure 2.1: $\exists \varphi$ and $\forall \varphi$

Exercise 2.5.2 What do the universal properties of the counit of $\exists_A \dashv \pi^*$ and the unit of $\pi^* \dashv \forall_A$ say?

The weakening functor π^* is a special case of a pullback functor $f^* : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$ for a morphism $f : B \to A$. This gives us the idea that we may regard the left and the right adjoint to f^* as a kind of generalized existential and universal quantifier.

We may also be tempted to *define* quantifiers as left and right adjoints to pullback functors. However there is a bit more to quantifiers than that—we are still missing the important *Beck-Chevalley condition.*

2.5.1 Beck-Chevalley condition

Recall that quantification commutes with substitution, as long as no variables are captured by the quantifier. Thus if $\Gamma \mid t : B$ and $\Gamma, y : B, x : A \mid \varphi$ `pred` then

$$(\exists\, x : A\,.\,\varphi)[t/y] = \exists\, x : A\,.\,(\varphi[t/y]) \ .$$
$$(\forall\, x : A\,.\,\varphi)[t/y] = \forall\, x : A\,.\,(\varphi[t/y]) \ .$$

If semantics of quantifiers is to be sound, the interpretation of these equations must be valid. Because substitution of a term in a formula is interpreted as pullback this means that quantifiers must be *stable* under pullbacks. This is known as the *Beck-Chevalley condition.*

Definition 2.5.3 A family of functors $F_f : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ parametrized by morphisms $f : A \to B$ is said to satisfy the *Beck-Chevalley condition* when for every pullback on the left-hand side, the right-hand square commutes:

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sub}(D) \arrow[d, "F_k"'] & \mathsf{Sub}(C) \arrow[l, "h^*"'] \arrow[d, "F_g"] \\ \mathsf{Sub}(A) & \mathsf{Sub}(B) \arrow[l, "f^*"] \end{tikzcd}$$

Definition 2.5.4 A cartesian category $\mathcal{C}$ has *existential quantifiers* if, for every $f : A \to B$, the left adjoint $\exists_f \dashv f^*$ exists and it satisfies the Beck-Chevalley condition. Similarly, $\mathcal{C}$ has *universal quantifiers* if the right adjoints $f^* \dashv \forall_f$ exist and they satisfy the Beck-Chevalley condition.

To convince ourselves that Beck-Chevalley condition is what we want, we spell it out in the case of a substitution into an existentially quantified formula. In order to keep the notation simple we omit the semantic brackets $[\![-]\!]$. Suppose we have a term $\Gamma \mid t : B$ and a formula $\Gamma, y : B, x : A \mid \varphi$ `pred`. The diagram

$$\begin{tikzcd} \Gamma \times A \arrow[r, "{\langle \pi_0, t \circ \pi_0, \pi_1 \rangle}"] \arrow[d, "\pi_0^{\Gamma,A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \times B \times A \arrow[d, "\pi_0^{\Gamma,B,A}"] \\ \Gamma \arrow[r, "{\langle 1_\Gamma, t \rangle}"'] & \Gamma \times B \end{tikzcd}$$

is a pullback. By Beck-Chevalley condition for $\exists$, the following square commutes:

$$\begin{tikzcd}
\mathsf{Sub}(\Gamma \times A) \arrow[d, "\exists_A^{\Gamma,A}"'] & \mathsf{Sub}(\Gamma \times B \times A) \arrow[l, "\langle \pi_0, t \circ \pi_0, \pi_1 \rangle^*"'] \arrow[d, "\exists_A^{\Gamma,B,A}"] \\
\mathsf{Sub}(\Gamma) & \mathsf{Sub}(\Gamma \times B) \arrow[l, "\langle 1_\Gamma, t \rangle^*"]
\end{tikzcd}$$

Therefore, for $\Gamma, y : B, x : A \mid \varphi$ `pred`,

$$[\![(\exists\, x : A\,.\,\varphi)[t/y]]\!] = \langle 1_\Gamma, t\rangle^*(\exists_A^{\Gamma,B,A}[\![\varphi]\!]) = \\ \exists_A^{\Gamma,A}(\langle \pi_0, t \circ \pi_0, \pi_1\rangle^*[\![\varphi]\!]) = [\![\exists\, x : A\,.\,(\varphi[t/y])]\!]\ .$$

This is precisely the equation we wanted.

The Beck-Chevalley condition says that (interpretations of) the quantifiers commute with pullbacks, in just the way that the syntactic operations of applying quantifiers to formulas commute with substitutions of terms, which are interpreted as pullbacks.

Exercise 2.5.5 In Set we can identify $\mathsf{Sub}(-)$ with powersets because $\mathsf{Sub}(X) \cong \mathcal{P}X$. Then quantifiers along a function $f : A \to B$ are functions

$$\exists_f : \mathcal{P}A \to \mathcal{P}B\ , \qquad\qquad \forall_f : \mathcal{P}A \to \mathcal{P}B\ .$$

Verify that

$$\exists_f U = f_* U = \big\{b \in B \bigm| \exists\, a : A\,.\,(fa = b \wedge a \in U)\big\}\ ,$$
$$\forall_f U = \big\{b \in B \bigm| \forall\, a : A\,.\,(fa = b \Rightarrow a \in U)\big\}\ .$$

Thus $\exists_f U$ is just the usual direct image of U by f. But have you seen $\forall_f U$ before? It can also be written as $\forall_f U = \big\{b \in B \bigm| f^*\{b\} \subseteq U\big\}$. What is the meaning of $\exists_q$ and $\forall_q$ when $q : A \to A/{\sim}$ is a canonical quotient map that maps an element $x \in A$ to its equivalence class $qx = [x]$ under an equivalence relation $\sim$ on A?

2.5.2 Universal quantifiers in LCCC

Recall that a cartesian closed category is a category that has products and exponentials. We consider those categories for which every slice is cartesian closed.

Definition 2.5.6 A category $\mathcal{C}$ is *locally cartesian closed (lccc)* when it has a terminal object and every slice $\mathcal{C}/A$ is cartesian closed.

A slice category $\mathcal{C}/A$ always has a terminal object, namely the identity morphism $1_A : A \to A$.

Proposition 2.5.7 *A category $\mathcal{C}$ has pullbacks if, and only if, every slice $\mathcal{C}/A$ has binary products.*

Proof. This is obvious, since the universal property of pullbacks in $\mathcal{C}$ over an object A is exactly the universal property of products in $\mathcal{C}/A$. ∎

Thus a locally cartesian closed category has all finite limits because it has a terminal object and pullbacks. In addition, a locally cartesian closed category is cartesian closed because $\mathcal{C} \cong \mathcal{C}/1$.

We describe how exponentials in a slice $\mathcal{C}/A$ can be computed in terms of *change of base functors* and *dependent products*. Given a morphism $f : B \to A$ in $\mathcal{C}$, the "change of base along f" is a construction

$$f^* : \mathcal{C}/A \to \mathcal{C}/B\ ,$$

which maps $c : C \to A$ to $f^*c : f^*C \to B$, as in the pullback

$$\begin{tikzcd}
f^*C \arrow[r, "c^*f"] \arrow[d, "f^*c"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "c"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

and it maps a morphism

$$\begin{tikzcd}
C \arrow[rr, "h"] \arrow[dr, "c"'] & & D \arrow[dl, "d"] \\
& A &
\end{tikzcd}$$

to the unique $f^*[h] : f^*C \to f^*D$ which makes the following diagram commutative:[6]

$$\begin{tikzcd}
f^*C \arrow[rrr, "c^*f"] \arrow[dr, "f^*[h]"] \arrow[ddr, "f^*c"'] & & & C \arrow[dl, "f"'] \arrow[ddl, "c"] \\
& f^*D \arrow[r, "d^*f"] \arrow[d, "f^*d"] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "d"'] & \\
& B \arrow[r, "f"'] & A &
\end{tikzcd}$$

The mapping $f^* : \mathcal{C}/A \to \mathcal{C}/B$ is usually referred to as the "change of base functor f^*", even though in general it is only a *pseudofunctor*. If f^* were a functor, it would have to satisfy the equation

$$f^*[k \circ h] = f^*[k] \circ f^*[h]$$

but in general it only satisfies this *up to isomorphism*, since pullbacks are only assumed to be determined up to isomorphism. Thus f^* is in general a "functor up to isomorphism". For

---

[6] We use the funny notation $f^*[h]$ to distinguish the action of f^* on morphisms in $\mathcal{C}/A$ from the usual pullback of an object in $\mathcal{C}/A$ along f.

certain purposes this is a serious technical nuisance, most notably in modeling dependent type theory. However, for our aims f^* will do just fine because we will only be concerned with adjunctions and other "up to isomorphism" constructions. For example, we can safely speak of left and right adjoints to the "functor" f^* because adjunctions are natural isomorphisms between functors, not equations. For presicse definition of pseudofunctors and related notions see [**?**, Definition 7.5.1].

Exercise 2.5.8 Show that the change of base functor f^* always has a left adjoint $\Sigma_f : \mathcal{C}/B \to \mathcal{C}/A$, called a *dependent sum along* f. It maps an object $c : C \to B$ to the object $\Sigma_f c = f \circ c : C \to A$, and a morphism $h : C \to D$ with domain $c : C \to B$ and codomain $d : D \to B$ in $\mathcal{C}/B$ to the morphism $h : C \to D$ with domain $f \circ c : C \to A$ and codomain $f \circ d : D \to A$ in $\mathcal{C}/A$.

A right adjoint to f^*, when it exists, is called a *dependent product along* f,

$$\Pi_f : \mathcal{C}/B \to \mathcal{C}/A \;.$$

Now an exponential of $b : B \to A$ and $c : C \to A$ in $\mathcal{C}/A$ can be computed in terms of Π_b and b^*. For any $d : D \to A$, we have $b \times_A d = (b^* d) \circ b = \Sigma_b(b^* d)$, hence

$$\begin{array}{c} b \times_A d \to c \\ \hline\hline \Sigma_b(b^* d) \to c \\ \hline\hline b^* d \to b^* c \\ \hline\hline d \to \Pi_b(b^* c) \end{array}$$

Therefore, $c^b = \Pi_b(b^* c)$.

We have proved that if a cartesian category $\mathcal{C}$ has dependent product $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ along every morphism $f : A \to B$ then it is locally cartesian closed. The converse holds as well, that is every lccc has dependent products. For a proof see [**?**, Theorem I.9.4].

Exercise 2.5.9 In Set consider the dependent sum and product along a function $!_I : I \to 1$. Show that for $a : A \to I$ the set $\Pi_{!_I} A$ is the set of right inverses of a:

$$\Pi_{!_I} A = \left\{ s : I \to A \;\middle|\; a \circ s = 1_I \right\} \;.$$

If $(A_i)_{i \in I}$ is a family of sets indexed by I and we take

$$A = \textstyle\coprod_{i \in I} A_i = \left\{ \langle i, x \rangle \in I \times \bigcup_{i \in I} A_i \;\middle|\; i \in I \wedge x \in A_i \right\}$$

with $a = \pi_0 : \langle i, x \rangle \mapsto i$ then $\Pi_{!_I} A$ is precisely the cartesian product $\Pi_{i \in I} A_i$. Calculate what Π_f is in Set for a general $f : J \to I$ and conclude that Set is locally cartesian closed.

Proposition 2.5.10 *In an lccc $\mathcal{C}$, for any $f : A \to B$ the change of base functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ preserves the ccc structure.*

Proof. We need to show that f^* preserves terminal objects, binary products, and exponentials in slices. Because f^* is a right adjoint it preserves limits, hence it preserves terminal objects and binary products. To see that it preserves exponentials we first show that $f^* \circ \Pi_g \cong \Pi_{f^*g} \circ (g^*f)^*$ for $g : C \to B$. Given any $d : D \to C$, and $e : E \to A$:

$$\begin{array}{c}
e \to f^*(\Pi_g d) \\
\hline\hline
\Sigma_f e \to \Pi_g d \\
\hline\hline
g^*(\Sigma_f e) \to d \\
\hline\hline
g^*(f \circ e) \to d \\
\hline\hline
(g^*f) \circ ((f^*g)^*e) \to d \\
\hline\hline
(f^*g)^*e \to (g^*f)^*d \\
\hline\hline
e \to \Pi_{f^*g}((g^*f)^*d)
\end{array}$$

By Yoneda Lemma it follows that $f^*(\Pi_g d) \cong \Pi_{f^*g}((g^*f)^*d)$. Now we have, for any $d : D \to A$ and $c : C \to A$,

$$f^*c^d = f^*(\Pi_d(d^*c)) = \Pi_{f^*d}((d^*f)^*(d^*c)) = \Pi_{f^*d}((f^*d)^*(f^*c)) = (f^*c)^{(f^*d)} \, .$$

■

Exercise 2.5.11 State precisely which instance of Yoneda Lemma is used in the conclusion of the last proof.

Exercise 2.5.12 In the preceding proof we used the fact that $(d^*f)^*(d^*c) \cong (f^*d)^*(f^*c)$ and $g^*(f \circ e) \cong (g^*f) \circ ((f^*g)^*e)$. Prove that this is really so.

Locally cartesian closed categories are an important example of categories with universal quantifiers.

Proposition 2.5.13 *A locally cartesian closed category has universal quantifiers.*

Proof. Suppose $\mathcal{C}$ is locally cartesian closed. First observe that a morphism $m : M \to A$ is mono if, and only if, the morphism

$$\begin{tikzcd}
M \arrow[rr, "m"] \arrow[dr, "m"'] & & A \arrow[dl, "1_A"] \\
& A &
\end{tikzcd}$$

is mono in $\mathcal{C}/A$. Because right adjoints preserve monos, $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$ preserve monos for any $f : A \to B$, that is, if $m : M \rightarrowtail A$ is mono then $\Pi_f m : \Pi_f M \to B$ is mono in $\mathcal{C}$. Therefore, we may define $\forall_f$ as the restriction of Π_f to $\mathsf{Sub}(A)$. To be more precise, a subobject $[m : M \rightarrowtail A]$ is mapped by $\forall_f$ to the subobject $[\Pi_f m : \Pi_f M \rightarrowtail B]$. This works because for any monos $m : M \rightarrowtail A$ and $n : N \rightarrowtail B$ we have

$$\begin{array}{cr}
f^*[m : M \rightarrowtail A] \leq [n : N \rightarrowtail B] & \text{in } \mathsf{Sub}(B) \\
\hline\hline
f^* m \to n & \text{in } \mathcal{C}/B \\
\hline\hline
m \to \Pi_f n & \text{in } \mathcal{C}/A \\
\hline\hline
[m] \leq \forall_f [n] & \text{in } \mathsf{Sub}(A)
\end{array}$$

The Beck-Chevalley condition for $\forall_f$ follows from Proposition 2.5.10. Indeed, if $g : C \to B$ and $m : M \rightarrowtail C$ then

$$f^*(\Pi_g m) \cong \Pi_{f^*g}((g^* f)^* m) ,$$

therefore

$$f^*(\forall_g [m : M \rightarrowtail C]) = \forall_{f^*g}((g^* f)^* [m : M \rightarrowtail C]) ,$$

as required. ∎

2.5.3 Implication

Recall that the rules of inference for implication state that $\Rightarrow$ is right adjoint to $\wedge$:

$$\frac{\Gamma \mid \vartheta \ \texttt{pred} \qquad \Gamma \mid \varphi \ \texttt{pred}}{\Gamma \mid (\vartheta \Rightarrow \varphi) \ \texttt{pred}} \qquad\qquad \begin{array}{c} \Gamma \mid \psi, \vartheta \vdash \varphi \\ \hline\hline \Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi \end{array}$$

Exercise 2.5.14 Show that the above two-way rule can be replaced by the following introduction and elimination rules:

$$\frac{\Gamma \mid \psi \vartheta \vdash \varphi}{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi} \qquad\qquad \frac{\Gamma \mid \psi \vdash \vartheta \Rightarrow \varphi \qquad \Gamma \mid \psi \vdash \vartheta}{\Gamma \mid \psi \vdash \varphi}$$

We expect that in order to interpret implication in a cartesian category $\mathcal{C}$ we require $\mathsf{Sub}(A)$ to be a Heyting algebra for every $A \in \mathcal{C}$. However, we must not forget that implication interacts with substitution by the rule

$$(\vartheta \Rightarrow \varphi)[t/x] \ = \ \vartheta[t/x] \Rightarrow \varphi[t/x] .$$

Semantically this means that implication is *stable* under pullbacks.

Definition 2.5.15 A cartesian category $\mathcal{C}$ has *implications* when, for every $A \in \mathcal{C}$, the poset $\mathsf{Sub}(A)$ is a Heyting algebra with stable implication $\Rightarrow$. This means that for $U, V \in \mathsf{Sub}(A)$ and $f : B \to A$,

$$f^*(U \Rightarrow V) = (f^* U \Rightarrow f^* V) .$$

Proposition 2.5.16 *If a cartesian category has universal quantifiers then it has implications.*

Proof. Let $[u : U \rightarrowtail A]$ and $[v : V \rightarrowtail A]$ be subobjects of A. Define

$$([u] \Rightarrow [v]) = \forall_u(u^*[v]) \; .$$

Then for any subobject $[w : W \rightarrowtail A]$

$$\begin{array}{lr}
[w] \leq [u] \Rightarrow [v] & \text{in } \mathsf{Sub}(A) \\
\hline\hline
[w] \leq \forall_u(u^*[v]) & \text{in } \mathsf{Sub}(A) \\
\hline\hline
u^*[w] \leq u^*[v] & \text{in } \mathsf{Sub}(U) \\
\hline\hline
u^*w \rightarrow u^*v & \text{in } \mathcal{C}/U \\
\hline\hline
\Sigma_u(u^*w) \rightarrow v & \text{in } \mathcal{C}/A \\
\hline\hline
u \circ (u^*w) \rightarrow v & \text{in } \mathcal{C}/A \\
\hline\hline
[u] \wedge [w] \leq [v] & \text{in } \mathsf{Sub}(A)
\end{array}$$

Stability of $\Rightarrow$ follows from Beck-Chevalley condition for $\forall$. ■

Exercise 2.5.17 Prove the last claim of the proof.

Corollary 2.5.18 *Any LCCC has universal quantifiers and implications.*

Thus in particular, all presheaf categories $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ have these logical operations. We shall compute these explicitely in section **??** below.

Exercise 2.5.19 Fill in the details of the following alternate approach to quantification in LCCCs.

1. Show that an LCCC is a category $\mathcal{C}$ with a terminal object, in which the composition functor
$$f_! : \mathcal{C}/A \rightarrow \mathcal{C}/B,$$
for every $f : A \rightarrow B$ in $\mathcal{C}$, has a right adjoint
$$f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A,$$
which in turn has a right adjoint
$$f_* : \mathcal{C}/A \rightarrow \mathcal{C}/B.$$

2. Show that the left adjoints $f_!$ satisfy the Beck-Chevalley condition (this is simple).

3. Show that the right adjoints f_* satisfy the Beck-Chevalley condition, using the fact that the left adjoints do (swap all the arrows in the square required to commute by their respective left adjoints).

4. Define the universal quantifier as the restriction of the right adjoint $\pi_* : \mathcal{C}/A \times B \to \mathcal{C}/B$, taken along a projection, to the subcategory of monomorphisms (equivalently, subobjects):
$$i : \mathsf{Sub}(B) \hookrightarrow \mathcal{C}/B$$
5. Show that if the inclusion functor i has a left adjoint, then the existential quantifier can also be defined, in terms of the left adjoint $\pi_!$.

2.6 Regular logic

In this section we consider the question when a cartesian category has existential quantifiers. It turns out that this is closely related to the notion of a *regular category*, a concept which first arose in the context of abelian categories and axiomatic homology theory, quite independently of categorical logic.

2.6.1 Regular categories

Throughout this section we work in a cartesian category $\mathcal{C}$. The *kernel pair* of a morphism $f : A \to B$ is the pair of morphisms $k_1, k_2 : K \to A$ obtained as in the following pullback

$$\begin{tikzcd}
K \arrow[r, "k_2"] \arrow[d, "k_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Note that a kernel pair determines an equivalence relation $\langle k_1, k_2\rangle : K \rightarrowtail A \times A$, in the sense that it satisfies the reflexivity, symmetry and transitivity conditions. Indeed, an equivalence relation in a general cartesian category is precisely a model of the cartesian theory of an equivalence relation, in the sense of example 2.3.3.

Exercise 2.6.1 Prove this.

In Set the mono $\langle k_1, k_2\rangle : K \rightarrowtail A \times A$ is the equivalence relation $\sim$ on A defined by

$$x \sim y \iff fx = fy\ .$$

The quotient by the equivalence relation determined by the kernel pair k_1, k_2 is their coequalizer $q : A \to Q$, if it exists,

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, two heads, "q"] & Q
\end{tikzcd}$$

Such a coequalizer is called a *kernel quotient*.

Because $f \circ k_1 = f \circ k_2$, f factors through q by a unique morphism $m : Q \to A$,

$$\begin{tikzcd}
 & & B \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[ur, "f"] \arrow[r, two heads, "q"'] & Q \arrow[u, "m"']
\end{tikzcd} \tag{2.5}$$

It is of some interest to know when m is guaranteed to be a mono. For example, in Set the function $m : Q \to B$ is defined by $m[x] = fx$, where $Q = A/{\sim}$ as above. In this case m is injective because $m[x] = m[y]$ implies $fx = fy$, hence $x \sim y$ and $[x] = [y]$.

Definition 2.6.2 A category with finite limits is *regular* when it has kernel quotients and stable regular epis, meaning that:

1. every kernel pair has a coequalizer, and
2. a pullback of a regular epi is a regular epi.

Recall that an epi is *regular* if it is a coequalizer. In diagrams we denote regular epis by arrows with triangular heads, for example

$$\begin{tikzcd}
A \arrow[r, -{Triangle[open]}, "e"] & B
\end{tikzcd}$$

Exercise 2.6.3 Suppose $e : A \to B$ is a regular epi. Prove that it is the coequalizer of its kernel pair.

Let us return to (2.5) and show that in a regular category m is mono. Consider the following diagram, in which h_1, h_2 are constructed as the kernel pair of m, and the other three squares are constructed as pullbacks:

$$\begin{tikzcd}
K \arrow[r, -{Triangle[open]}, "p_2"] \arrow[d, -{Triangle[open]}, "p_1"'] \arrow[dr, -{Triangle[open]}, "r"] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[r] \arrow[d, -{Triangle[open]}, "s_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "q"] \\
\cdot \arrow[r, -{Triangle[open]}, "s_1"'] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & H \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[d, "m"] \\
A \arrow[r, -{Triangle[open]}, "q"'] & Q \arrow[r, "m"'] & B
\end{tikzcd}$$

Because all the smaller squares are pullbacks the large square is a pullback as well, therefore the morphism across the top is $k_2 : K \to A$, and the left-hand vertical morphism is $k_1 : K \to A$. Morphisms s_1, s_2, p_1, and p_2 are all regular epis because they are pullbacks of regular epis. The morphism $r = s_2 \circ p_2 = s_1 \circ p_1$ is epi because it is a composition of regular epis. Observe that

$$h_1 \circ r = q \circ k_1 = q \circ k_2 = h_2 \circ r \ ,$$

and because r is epi, $h_1 = h_2$. But this means that m is monic, since given $u, v : U \to Q$ with $m \circ u = m \circ v$ there exists $w : U \to H$ such that $u = w \circ h_1 = w \circ h_2 = v$.

Proposition 2.6.4 *In a regular category every morphism $f : A \to B$ factors as a composition of a regular epi q and a mono m,*

$$\begin{tikzcd}
A \arrow[r, "q"', -{Triangle[open]}] \arrow[rr, bend left, "f"] & Q \arrow[r, tail, "m"'] & B
\end{tikzcd}$$

The factorization in unique up to isomorphism.

Proof. By uniqueness of factorization we mean that if

$$\begin{tikzcd}
A \arrow[r, "q'"', -{Triangle[open]}] \arrow[rr, bend left, "f"] & Q' \arrow[r, tail, "m'"'] & B
\end{tikzcd}$$

is another such factorization, then there exists an isomorphism $i : Q \to Q'$ such that $q' = i \circ q$ and $m = m' \circ m$.

As the factorization of f we take the one constructed in (2.5). Then q is regular epi by construction, and we have just shown that m is mono. So it only remains to show that the factorization is unique. Suppose f also factors as $f = m' \circ q'$ where q' is regular epi and m is mono. Consider the following diagram, in which k_1, k_2 is the kernel pair of f, q is the coequalizer of k_1 and k_2, and h_1, h_2 is the kernel pair of q' so that q' is the coequalizer of h_1 and h_2:

$$\begin{tikzcd}
 & H \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] & \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[r, "q", -{Triangle[open]}] \arrow[d, "q'"', -{Triangle[open]}] & Q \arrow[d, tail, "m"] \arrow[dl, dotted, shift right, "i"'] \\
 & Q' \arrow[r, tail, "m'"'] \arrow[ur, dotted, shift right, "j"'] & B
\end{tikzcd}$$

Because $m' \circ q' \circ k_1 = m \circ q \circ k_1 = m \circ q \circ k_2 = m' \circ q' \circ k_2$ and m' is mono, $q' \circ k_1 = q' \circ k_2$. So there exists a unique $i : Q \to Q'$ such that $q' = i \circ q$. But then $m' \circ i \circ q = m' \circ q' = f = m \circ q$ and because q is epi, $m' \circ i = m$.

We prove that i is iso by constructing its inverse j. Because $m \circ q \circ h_1 = m' \circ q \circ h_1 = m' \circ q \circ h_2 = m \circ q \circ h_2$ and m is mono, $q \circ h_1 = q \circ h_2$. So there exists a unique $j : Q' \to Q$ such that $q = j \circ q'$. Now we have $i \circ j \circ q' = i \circ q = 1_{Q'} \circ q'$, from which we conclude that $i \circ j = 1_{Q'}$ because q' is epi. Similarly, $j \circ i \circ q = j \circ q' = 1_Q \circ q$, therefore $j \circ i = 1_Q$. ■

A factorization $f = m \circ q$ as in the previous Proposition determines a subobject

$$\mathsf{im}(f) = [m : Q \rightarrowtail B] \in \mathsf{Sub}(B) \ ,$$

called the *image of* f. It is characterized as the least subobject $[u : U \rightarrowtail B]$ of B through which f factors.

Proposition 2.6.5 *In a regular category* $\mathcal{C}$*, the image* $\mathsf{im}(f)$ *of a morphism* $f : A \to B$ *is the least subobject* $[u : U \rightarrowtail B]$ *of* B *such that* f *factors through* u*.*

Proof. Suppose f factors through $v : V \rightarrowtail B$ as

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[rr, bend left, "f"] & V \arrow[r, tail, "v"'] & B \end{tikzcd}$$

and consider the factorization of f, as in (2.5). Since $v \circ g \circ k_1 = f \circ k_1 = f \circ k_2 = v \circ g \circ k_2$ and v is mono, $g \circ k_1 = g \circ k_2$, therefore there exists a unique $\overline{g} : Q \to V$ such that $g = \overline{g} \circ q$. Now $v \circ \overline{g} \circ q = v \circ g = f = m \circ q$ and because q is epi, $v \circ \overline{g} = m$ as required. ■

A morphism $f : A \to B$ is sometimes called a *generalized element* or *generalized point of* B. If $U \leq B$ is a subobject of B, we write $f \in U$ when f factors through U. With this notation we have

$$f \in U \iff \mathsf{im}(f) \leq U \ .$$

At first sight it may seem unreasonable to use the phrases "element" and "point" for something that clearly is a morphism. But this is in perfect accordance with mathematical practice. For example, in mechanics we often say things like "the velocity $\dot{p}$ of a point mass p moving in space" which just means that p is a vector in $\mathbb{R}^3$ parametrized by time—in other words a generalized point $p : \mathbb{R} \to \mathbb{R}^3$.

In set theory the axiom of extensionality say that "a set is determined by its elements", that is, for any sets A and B,

$$A = B \iff \forall x \, . \, (x \in A \iff x \in B) \ .$$

The corresponding statement in an arbitrary category is that objects A and B are isomorphic if, and only if, they have naturally isomorphic sets of *generalized* elements:

$$A \cong B \iff \mathsf{Hom}(-, A) \cong \mathsf{Hom}(-, B) \ .$$

This is Corollary **??**, which says that the Yoneda embedding reflects isomorphisms.

Let us consider some examples of regular categories. The category Set is regular. It is complete and cocomplete, so it has finite limits and coequalizers. The pullback of a regular epi is again regular epi because in Set every epi is regular, and it is always the case that the pullback of an epi is epi in Set (pullback is a left adjoint).

In general, any presheaf category $\widehat{\mathcal{C}}$ is also regular, because it is complete and cocomplete, and (co)limits are computed pointwise. Thus, again, every epi is regular, and epis are stable under pullbacks.

The next example deserves to be a proposition.

Proposition 2.6.6 *The category* $\mathsf{Mod}_{\mathsf{Set}}(\mathbb{A})$ *of set-theoretic models of an algebraic theory* $\mathbb{A}$ *is regular.*

Proof. We sketch a proof, for details see [**?**, Theorem 3.5.4]. Recall that the objects of $\mathsf{Mod}(\mathbb{A}) = \mathsf{Mod}_{\mathsf{Set}}(\mathbb{A})$ are $\mathbb{A}$-algebras, which are structures $X = (|X|, f_1, f_2, \ldots)$ where $|X|$ is the carrier set and $f_1, f_2, \ldots$ are the basic operations on $|X|$. Every such $\mathbb{A}$-algebra is also required to satisfy the equational axioms of $\mathbb{A}$. A morphism $f : X \to Y$ is a function $f : |X| \to |Y|$ that preserves the basic operations.

The category of $\mathbb{A}$-algebras $\mathsf{Mod}(\mathbb{A})$ has small limits, which are computed as in Set. Thus the product of $\mathbb{A}$-algebras X and Y has as its carrier set $|X \times Y| = |X| \times |Y|$, and the basic operations of $X \times Y$ are computed separately on each component. An equalizer of morphisms $f, g : X \to Y$ has as its carrier set the equalizer of $f, g : |X| \to |Y|$, and the basic operations inherited from X.

To see that coequalizers of kernel pairs exist, consider a morphism $h : X \to Y$. We can form the quotient $\mathbb{A}$-algebra Q whose carrier set is $|Q| = |X|/{\sim}$, where $\sim$ is the relation defined by

$$x \sim y \iff hx = hy \ .$$

A basic operation $f_Q : Q^k \to Q$ is induced by the basic operation $f_X : X^k \to X$ by

$$f_Q\langle [x_1], \ldots, [x_k] \rangle = [f_X\langle x_1, \ldots, x_k \rangle] \ .$$

It is easily verified that Q is an $\mathbb{A}$-algebra and that the canonical quotient map $q : |X| \to |Q|$ is the coequalizer of the kernel pair of h.

Lastly regular epis in $\mathsf{Mod}(\mathbb{A})$ are stable because pullbacks and kernel pairs are computed as in Set, and a morphism $f : X \to Y$ is regular epi in $\mathsf{Mod}(\mathbb{A})$ if, and only if, the underlying function $f : |X| \to |Y|$ is regular epi in Set. ■

We now know that categories of groups, rings, modules, $\mathcal{C}^\infty$-rings and other algebraic categories are regular. The preceding proposition is useful also for showing that certain structures can*not* be axiomatized by algebraic theories. The category of posets is an example of a category that is not regular; therefore the theory of partial orders cannot be axiomatized solely by equations.

Exercise 2.6.7 Show that Poset is not regular. (Hint: find a regular epi that is not stable under pullback.)

Exercise* 2.6.8 Is Top regular? Hint: is there is a topological quotient map $q : X \twoheadrightarrow X'$ and a space Y such that $q \times 1_Z : X \times Y \twoheadrightarrow X' \times Y$ is not a quotient map?

Definition 2.6.9 A functor $F : \mathcal{C} \to \mathcal{D}$ is *regular* if it preserves finite limits and regular epis. It follows that F preserves image factorizations. The category of regular functors $\mathcal{C} \to \mathcal{D}$ and natural transformations is denoted by $\mathsf{Reg}(\mathcal{C}, \mathcal{D})$.

Example 2.6.10 ("Fuzzy logic") Let H be a complete Heyting algebra; thus H is a cartesian closed poset with all small joins $\bigvee_i p_i$. The category of *H-presets* has as objects all pairs $(X, e_X : X \to H)$ where X is a set and e_X is a function, called the *existence predicate of X*. For $x \in X$, $e_X(x)$ can be thought of as "the amount by which x exists". A *morphism of presets* is a function $f : X \to Y$ satisfying, for all $x \in X$,

$$e_X(x) \leq e_Y(fx).$$

This is a regular category, with the following structure.

- the terminal object is $\top : 1 \to H$,
- the product of $e_A : A \to H$ and $e_B : B \to H$ is
$$e_A \wedge e_B : A \times B \to H,$$
where $(e_a \wedge e_B)(a, b) = e_A(a) \wedge e_B(b)$,
- the equalizer of two maps $f, g : A \to B$ is their equalizer as functions, $A' = \{a \mid f(a) = g(a)\} \hookrightarrow A$, with the restriction of $e_A : A \to H$ to $A' \subseteq A$.
- a map $f : A \to B$ is a regular epi if and only if it is a surjective function and for all $b \in B$:
$$e_B(b) = \bigvee_{f(a)=b} e_A(a)$$

Exercise 2.6.11 Verify that H-presets form a regular category, and compute the regular epi-mono factorization of a map.

Remark 2.6.12 (Exactness) A regular category $\mathcal{C}$ is said to be *exact* [**?**] if *every* equivalence relation has a quotient (not just the kernel pairs). It can be shown fairly easily that categories of models of algebraic theories are exact: an equivalence relation in such a category is a congruence relation with respect to the algebraic operations, and its (set) quotient is then necessarily also a homomorphism, and thus a coequalizer of algebras.

Exact completions, H-sets.

2.6.2 Images and existential quantifiers

Recall that $\mathsf{Sub}(A)$ is equivalent to the category $\mathsf{Mono}(A)$ of monos into A. If we compose an equivalence functor $\mathsf{Sub}(A) \to \mathsf{Mono}(A)$ with the inclusion $\mathsf{Mono}(A) \to \mathcal{C}/A$ we obtain an inclusion functor $I : \mathsf{Sub}(A) \to \mathcal{C}/A$. In the other direction we have the "image functor" $\mathsf{im} : \mathcal{C}/A \to \mathsf{Sub}(A)$ which maps an object $b : B \to A$ in $\mathcal{C}/A$ to the subobject $\mathsf{im}(f) = [\mathsf{im}(f) \rightarrowtail A]$.

Exercise 2.6.13 In order to show that im is in fact a functor, prove that $f = g \circ h$ implies $\mathsf{im}(f) \leq \mathsf{im}(g)$.

Proposition 2.6.5 says that the image functor is left adjoint to the inclusion functor $I : \mathsf{Sub}(A) \to \mathcal{C}/A$,

$$\mathsf{im} \dashv I \ .$$

Furthermore, images are stable in the sense that the following diagram commutes for all $f : A \to B$:

$$\begin{tikzcd}
\mathcal{C}/A \arrow[d, "\mathsf{im}_A"'] & \mathcal{C}/B \arrow[l, "f^*"'] \arrow[d, "\mathsf{im}_B"] \\
\mathsf{Sub}(A) & \mathsf{Sub}(B) \arrow[l, "f^*"]
\end{tikzcd} \tag{2.6}$$

The functor f^* on the top is the change of base functor, and the functor f^* on the bottom is the pullback functor. To see this, consider $g : C \to B$ and the following diagram:

$$\begin{tikzcd}
f^*C \arrow[r] \arrow[d, two heads] \arrow[dd, bend right=60, "f^*g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, two heads] \arrow[dd, bend left=60, "g"] \\
\mathsf{im}(f) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{im}(g) \arrow[d, tail] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

On the right-hand side we have the factorization of g, which is then pulled back along f. Because monos and regular epis are stable, this gives us a factorization of f^*g, hence

$$\mathsf{im}(f^*g) = f^*(\mathsf{im}(g)) \ .$$

Proposition 2.6.14 *A regular category has existential quantifiers. The existential quantifier along $f : A \to B$ is*

$$\exists_f[m : M \rightarrowtail A] = \mathsf{im}(f \circ m) \ .$$

Proof. First we verify that $\exists_f \dashv f^*$. For subobjects $[u : U \rightarrowtail A]$ and $[v : V \rightarrowtail B]$:

$$\begin{array}{lr}
\exists_f[u] \leq [v] & \text{in } \mathsf{Sub}(B) \\
\hline\hline
\mathsf{im}(f \circ u) \leq [v] & \text{in } \mathsf{Sub}(B) \\
\hline\hline
f \circ u \to I[v] & \text{in } \mathcal{C}/B \\
\hline\hline
f \circ u \to v & \text{in } \mathcal{C}/B \\
\hline\hline
\Sigma_f u \to v & \text{in } \mathcal{C}/B \\
\hline\hline
u \to f^* v & \text{in } \mathcal{C}/A \\
\hline\hline
[u] \leq f^*[v] & \text{in } \mathsf{Sub}(A)
\end{array}$$

In the second step in the above derivation we used the adjunction between $\mathsf{im} : \mathcal{C}/B \to \mathsf{Sub}(B)$ and the inclusion $\mathsf{Sub}(B) \to \mathcal{C}/B$.

The Beck-Chevalley condition follows from stability of image factorizations. Given a pullback

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and a mono $u : U \rightarrowtail C$, (2.6) gives

$$f^*(\exists_g[u]) = f^*(\mathsf{im}(g \circ u)) = \mathsf{im}(f^*(g \circ u)) = \mathsf{im}(k \circ (h^* u)) = \exists_k(h^* u) = \exists_k(h^*[u]),$$

as required. ■

2.6.3 Regular theories

A regular category has finite limits and image factorizations, therefore it allows us to interpret a type theory with the terminal type and binary products, and a logic with equality, conjunction, and existential quantifiers. This system is called *regular logic.*

Definition 2.6.15 A (many-sorted) *regular theory* $\mathbb{T}$ is a (many-sorted) type theory together with axioms expressed in the fragment of logic built from $\top$, $=$, $\wedge$, and $\exists$.

Thus an *interpretation* of a regular theory $\mathbb{T}$ in a regular category $\mathcal{C}$ is therefore given by the following data:

1. Each basic sort A is interpreted as an object $[\![A]\!]$.
2. Each basic constant f with signature $(A_1, \dots, A_n; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!]$.

3. Each basic relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathsf{Sub}([\![A_1]\!] \times \cdots \times [\![A_1]\!])$.

An interpretation is then extended to all terms and formulas of the theory $\mathbb{T}$. The cartesian part of logic is interpreted as was explained in Section **??**. Existential quantification is interpreted by the existential quantifiers in the category,

$$[\![\Gamma \mid \exists\, x : A \,.\, \varphi]\!] = \exists_A [\![\Gamma, x : A \bigm| \varphi]\!] \;,$$

where

$$\exists_A = \exists_\pi : \mathsf{Sub}([\![\Gamma]\!] \times [\![A]\!]) \to \mathsf{Sub}([\![\Gamma]\!])$$

is the existential quantifier along the projection $\pi : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$.

If all the axioms of $\mathbb{T}$ are valid in a given interpretation, then we say that the interpretation is a *model* of $\mathbb{T}$. Once again we shall show that semantics of regular categories is *functorial*, i.e., that the models of a theory $\mathbb{T}$ can be viewed as structure-preserving functors,

$$\mathsf{Reg}(\mathcal{S}(\mathbb{T}), \mathcal{C}) \simeq \mathsf{Mod}_{\mathbb{T}}(\mathcal{C}) \;.$$

where on the left-hand side $\mathcal{S}(\mathbb{T})$ is a suitable regular category, called the *classifying category for* $\mathbb{T}$, and $\mathsf{Reg}(-,-)$ indicates that we take regular functors. Again, the morphisms between models will correspond to natural transformations between their classifying functors.

2.6.4 Classifying category of a regular theory

We now sketch the construction of the classifying categoriy $\mathcal{S}(\mathbb{T})$ of an arbitrary regular theory $\mathbb{T}$. An object is represented by a formula in context,

$$[\Gamma \mid \varphi],$$

where $\Gamma \mid \varphi$ `pred`. Two such objects $[\Gamma \mid \varphi]$ and $[\Gamma \mid \psi]$ are equal if $\mathbb{T}$ proves both

$$\Gamma \mid \varphi \vdash \psi \;, \qquad\qquad \Gamma \mid \psi \vdash \varphi \;.$$

Objects which differ only in the names of free variables are also considered equal. A morphism

$$[x : A \mid \varphi] \xrightarrow{\;\rho\;} [y : B \mid \psi]$$

is represented by a formula $x : A, y : B \mid \rho$ such that $\mathbb{T}$ proves that ρ is a *functional relation* from φ to ψ:

$$x : A \mid \varphi \vdash \exists\, y : B \,.\, \rho \qquad \text{(total)}$$

$$x : A, y : B, z : B \mid \rho \wedge \rho[z/y] \vdash y = z \qquad \text{(single-valued)}$$

$$x : A, y : B \mid \rho \vdash \varphi \wedge \psi \qquad \text{(well-defined)}$$

Two functional relations ρ and σ represent the same morphism if $\mathbb{T}$ proves both

$$x : A, y : B \mid \rho \vdash \sigma \,, \qquad\qquad x : A, y : B \mid \sigma \vdash \rho \,,$$

Relations which only differ in the names of free variables are considered equal.

The identity morphism on $[x : A \mid \varphi]$ is represented by the relation

$$x : A, y : A \mid (x = y) \wedge \varphi \,.$$

Composition of morphisms

$$\begin{tikzcd}
{[x : A \mid \varphi]} \arrow[r, "\rho"] & {[y : B \mid \psi]} \arrow[r, "\tau"] & {[z : C \mid \theta]}
\end{tikzcd}$$

is given by the relational product

$$x : A, z : C \mid \exists\, y : B \,.\, (\rho \wedge \tau) \ \ .$$

We leave a detailed proof that $\mathcal{S}(\mathbb{T})$ is a category as an exercise. Let us show that composition of morphisms is associative. Given morphisms

$$\begin{tikzcd}
{[x : A \mid \varphi]} \arrow[r, "\rho"] & {[y : B \mid \psi]} \arrow[r, "\tau"] & {[z : C \mid \theta]} \arrow[r, "\sigma"] & {[u : D \mid \zeta]}
\end{tikzcd}$$

we need to derive in context $x : A, u : D$

$$\xi \ = \ \exists\, y : B \,.\, (\varphi \wedge (\exists\, z : C \,.\, (\tau \wedge \sigma)))$$

from

$$\exists\, z : C \,.\, ((\exists\, y : B \,.\, (\rho \wedge \tau)) \wedge \sigma)$$

and vice versa. In one direction we have:

$$\begin{array}{c}
x : A, u : D \mid \exists\, y : B \,.\, (\rho \wedge \exists\, z : C \,.\, (\tau \wedge \sigma)) \vdash \xi \\
\hline\hline
x : A, u : D, y : B \mid \rho \wedge \exists\, z : C \,.\, (\tau \wedge \sigma) \vdash \xi \\
\hline\hline
x : A, u : D, y : B \mid \rho, \exists\, z : C \,.\, (\tau \wedge \sigma) \vdash \xi \\
\hline\hline
x : A, u : D, z : C, y : B \mid \rho, \tau \wedge \sigma \vdash \xi \\
\hline\hline
x : A, u : D, z : C, y : B \mid \rho, \tau, \sigma \vdash \xi \\
\hline\hline
x : A, u : D, z : C, y : B \mid \rho \wedge \tau, \sigma \vdash \xi \\
\hline\hline
x : A, u : D, z : C \mid (\exists\, y : B \,.\, (\rho \wedge \tau)), \sigma \vdash \xi \\
\hline\hline
x : A, u : D, z : C \mid (\exists\, y : B \,.\, (\rho \wedge \tau)) \wedge \sigma \vdash \xi \\
\hline\hline
x : A, u : D \mid \exists\, z : C \,.\, ((\exists\, y : B \,.\, (\rho \wedge \tau)) \wedge \sigma) \vdash \xi
\end{array}$$

The other direction is equally routine.

Exercise 2.6.16 Extend the definition of $\mathcal{S}(\mathbb{T})$ to morphisms between objects with arbitrary contexts,

$$[\Gamma \mid \varphi] \xrightarrow{\rho} [\Delta \mid \psi]$$

(use relations $\Gamma, \Delta \mid \rho$), and provide a proof that $\mathcal{S}(\mathbb{T})$ is a category.

The category $\mathcal{S}(\mathbb{T})$ is regular. We sketch the constructions required for regularity.

- The terminal object is $[\cdot \mid \top]$.
- The product of $[x : A \mid \varphi]$ and $[y : B \mid \psi]$, where x and y are distinct variables, is the object
  $$[x' : A, y' : B \mid \varphi[x'/x] \land \psi[y'/y]] \ .$$
  The first projection from the product is
  $$x' : A, y' : B, x : A \mid x' = x \land \varphi[x'/x] \land \psi[y'/y]$$
  and the second projection is
  $$x : A, y : B, y' : B \mid y' = y' \land \varphi[x'/x] \land \psi[y'/y] \ .$$
- An equalizer of morphisms
  $$\begin{tikzcd}[x : A \mid \varphi] \arrow[r, shift left, "\rho"] \arrow[r, shift right, "\tau"'] & {[y : B \mid \psi]}\end{tikzcd}$$
  is
  $$[x : A \mid \varphi \land \exists\, y : B \,.\, (\rho \land \tau)] \xrightarrow{\varepsilon} [x' : A \mid \varphi[x'/x]]$$
  where ε is the morphism
  $$x : A, x' : A \mid (x = x') \land \varphi \land \exists\, y : B \,.\, (\rho \land \tau) \ .$$
- Finally, let us consider coequalizers of kernel pairs. The kernel pair of $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is
  $$\begin{tikzcd}K \arrow[r, shift left, "\kappa_1"] \arrow[r, shift right, "\kappa_2"'] & {[x : A \mid \varphi]}\end{tikzcd}$$
  where K is the object
  $$[u : A, v : A \mid \varphi[u/x] \land \varphi[v/x] \land \exists\, y : B \,.\, (\rho[u/x] \land \rho[v/x])] \ ,$$
  the morphism κ_1 is
  $$u : A, v : A, x : A \mid (u = x) \land \varphi$$
  and κ_2 is
  $$u : A, v : A, x : A \mid (v = x) \land \varphi \ .$$
  Now the coequalizer of κ_1 and κ_2 is the morphism
  $$[x : A \mid \varphi] \xrightarrow{\rho} [y : B \mid \exists\, x : A \,.\, \rho] \ .$$

Exercise 2.6.17 Show that in $\mathcal{S}(\mathbb{T})$ the regular-epi mono factorization of a morphism $\rho : [x : A \mid \varphi] \to [y : B \mid \psi]$ is given by

$$\begin{tikzcd}[x : A \mid \varphi] \arrow[r, "\rho"] & {[y : B \mid \exists\, x : A \,.\, \rho]} \arrow[r, "\iota"] & {[z : B \mid \psi[z/y]]}\end{tikzcd}$$

where ι is the morphism

$$y : B, z : B \mid (y = z) \wedge (\exists\, x : A \,.\, \rho) \wedge \psi[z/y] \;.$$

Theorem 2.6.18 (Functorial semantics for regular logic) *For any regular theory* $\mathbb{T}$*, the syntactic category* $\mathcal{S}(\mathbb{T})$ *classigies* $\mathbb{T}$*-models in regular categories,*

$$\mathsf{Reg}(\mathcal{S}(\mathbb{T}), \mathcal{C}) \simeq \mathsf{Mod}_{\mathbb{T}}(\mathcal{C}).$$

In particular, there is a universal model U *in* $\mathcal{S}(\mathbb{T})$.

As was the case for algebraic theories, we then also have the following,

Corollary 2.6.19 *Semantics of regular logic in regular categories is sound and complete: a regular theory* $\mathbb{T}$ *proves*

$$\Gamma \mid \varphi \vdash \psi$$

if, and only if, every model of $\mathbb{T}$ *satisfies it.*

Proof. Soundness is proved by induction on the proof of $\Gamma \mid \varphi \vdash \psi$. All that needs to be checked is that rules of inference preserve validity. This was done for cartesian logic in Section **??**. The inference rule for existential quantifiers preserves validity as well, because it translates to the universal property of the adjunction $\exists \dashv \pi_0$.

Completeness follows from the fact that $\mathbb{T}$ has a universal model U in $\mathcal{S}(\mathbb{T})$. In this model a sort A is interpreted by the object $[x : A \mid \top]$ and a basic constant f with signature $(A_1, \ldots, A_n; B)$ is interpreted by the relation

$$x_1 : A_1, \ldots, x_n : A_n, y : B \mid f(x_1, \ldots, x_n) = y \;.$$

A relation symbol R with signature $(A_1, \ldots, A_n)$ is interpreted by the subobject represented by the morphism

$$\rho : [x_1 : A_1, \ldots, x_n : A_n \mid R(x_1, \ldots, x_n)] \longrightarrow [y_1 : A_1, \ldots, y_n : A_n \mid \top]$$

where ρ is the formula

$$R(x_1, \ldots, x_n) \wedge x_1 = y_1 \wedge \cdots \wedge x_n = y_n \;.$$

The model U has the property

$$U \models \Gamma \mid \varphi \vdash \psi \qquad \Longleftrightarrow \qquad \mathbb{T} \text{ proves } \Gamma \mid \varphi \vdash \psi \;.$$

■