

Introduction to Categorical Logic

[DRAFT: SEPTEMBER 14, 2009]

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September 14, 2009

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Chapter 1

Algebraic Theories

Algebraic theories are descriptions of structures determined by operations and equations. There are familiar examples from elementary algebra, such as groups, but also many concepts that are not evidently algebraic, such as adjoint functors, can be given algebraic formulations. Thus the scope of algebraic theories is actually much greater than first appears. On the other hand, all such algebraic notions have in common some quite deep and general properties, from the existence of free algebras to Lawvere’s duality theory. The most important of these are presented in this chapter. The development also serves as a first example and template for the scheme of “functorial semantics,” to be applied to other logical notions in later chapters.

1.1 Algebraic Theories

We begin with a general approach to algebraic structures such as groups, rings, modules, and lattices. These are characterized by axiomatizations which involve only variables, constants, operations, and equations. It is important that the operations are defined everywhere, which excludes two important examples: fields because the inverse of 0 is undefined, and categories because composition is defined only for some pairs of morphisms.

Let us start with the quintessential algebraic theory—the theory of groups. A group can be described as a set G with a binary operation $\cdot : G \times G \rightarrow G$, satisfying the two axioms:

$$\begin{aligned} &\forall x, y, z \in G. (x \cdot y) \cdot z = x \cdot (y \cdot z) \\ &\exists e \in G. \forall x \in G. \exists y \in G. (e \cdot x = x \cdot e = x \wedge x \cdot y = y \cdot x = e) \end{aligned}$$

Taking a closer look at the logical form of these axioms, we see that the second one, which expresses the existence of a unit and inverse elements, is somewhat unsatisfactory because it involves nested quantifiers. Not only does this complicate the interpretation, but it is not really necessary, since the unit element and inverse operation in a group are uniquely determined. Thus we can add them to the structure and reformulate as follows. We require

the unit to be a distinguished *constant* $e \in G$ and the inverse to be an *operation* $^{-1} : G \rightarrow G$. We then obtain an equivalent formulation in which all axioms are now *equations*:

$$\begin{aligned} x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\ x \cdot e &= x & e \cdot x &= x \\ x \cdot x^{-1} &= e & x^{-1} \cdot x &= e \end{aligned}$$

Notice that the universal quantifier $\forall x \in G$ is no longer needed in stating the axioms, since we interpret all variables as ranging over all elements of G . Nor do we really need to explicitly mention the particular set G in the specification. Finally, since the constant e can be regarded as a nullary operation, i.e., a function $e : 1 \rightarrow G$, the specification of the group concept consists solely of operations and equations. This leads us to the general definition of an algebraic theory.

Definition 1.1.1 A *signature* Σ for an algebraic theory consists of a family of sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. The elements of Σ_k are called the *k-ary operations*. In particular, the elements of Σ_0 are the *nullary operations* or *constants*.

The *terms* of a signature Σ are expressions constructed inductively by the following rules:

1. variables x, y, z, \dots , are terms,
2. if t_1, \dots, t_k are terms and $f \in \Sigma_k$ is a k -ary operation then $f(t_1, \dots, t_k)$ is a term.

Definition 1.1.2 (cf. Definition ??) An *algebraic theory* $\mathbb{T} = (\Sigma_{\mathbb{T}}, A_{\mathbb{T}})$ is given by a signature $\Sigma_{\mathbb{T}}$ and a set $A_{\mathbb{T}}$ of *axioms*, which are equations between terms (formally, pairs of terms).

Algebraic theories are also called *equational theories*.

Example 1.1.3 The theory of a commutative ring with unit is an algebraic theory. There are two nullary operations (constants) 0 and 1, a unary operation $-$, and two binary operations $+$ and \cdot . The equations are:

$$\begin{aligned} (x + y) + z &= x + (y + z) & (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ x + 0 &= x & x \cdot 1 &= x \\ 0 + x &= x & 1 \cdot x &= x \\ x + (-x) &= 0 & (x + y) \cdot z &= x \cdot z + y \cdot z \\ (-x) + x &= 0 & z \cdot (x + y) &= z \cdot x + z \cdot y \\ x + y &= y + x & x \cdot y &= y \cdot x \end{aligned}$$

Example 1.1.4 The “empty” theory with no operations and no equations is the theory of a set.

Example 1.1.5 The theory with one constant and no equations is the theory of a *pointed set*, cf. Example ??.

Example 1.1.6 Let R be a ring. There is an algebraic theory of left R -modules. It has one constant 0 , a unary operation $-$, a binary operation $+$, and for each $a \in R$ a unary operation \bar{a} , called *scalar multiplication by a* . The following equations hold:

$$\begin{aligned} (x + y) + z &= x + (y + z) , & x + y &= y + x , \\ x + 0 &= x , & 0 + x &= x , \\ x + (-x) &= 0 , & (-x) + x &= 0 . \end{aligned}$$

For every $a, b \in R$ we also have the equations

$$\bar{a}(x + y) = \bar{a}x + \bar{a}y , \quad \bar{a}(\bar{b}x) = \overline{(ab)}x , \quad \overline{(a + b)}x = \bar{a}x + \bar{b}x .$$

Scalar multiplication by a is usually written as $a \cdot x$ instead of $\bar{a}x$. If we replace the ring R by a field \mathbb{F} we obtain the algebraic theory of a vector space over \mathbb{F} (even though the theory of fields is not algebraic!).

Example 1.1.7 In computer science, inductive datatypes are examples of algebraic theories. For example, the datatype of binary trees with leaves labeled by integers might be defined as follows in a programming language:

```
type tree = Leaf of int | Node of tree * tree
```

This corresponds to the algebraic theory with a constant `Leaf` n for each integer n and a binary operation `Node`. There are no equations. Actually, when computer scientists define a datatype like this, they have in mind a particular model of the theory, namely the *free* one.

Example 1.1.8 An obvious non-example is the theory of posets, formulated with a binary relation symbol $x \leq y$ and the usual axioms of reflexivity, transitivity and anti-symmetry, namely:

$$\begin{aligned} x &\leq x \\ x \leq y \wedge y \leq z &\Rightarrow x \leq z \\ x \leq y \wedge y \leq x &\Rightarrow x = y \end{aligned}$$

On the other hand, using an operation of greatest lower bound or “meet” $x \wedge y$, one can make the equational theory of “ \wedge -semilattices”:

$$\begin{aligned} x \wedge x &= x \\ x \wedge y &= y \wedge x \\ x \wedge (y \wedge z) &= (x \wedge y) \wedge z \end{aligned}$$

Then, defining a partial ordering $x \leq y \iff x \wedge y = x$ we arrive at the notion of a “poset with meets”, which *is* equational (of course, the same can be done with joins $x \vee y$ as well). We’ll have a proof later (in section ??) that there is no reformulation of the general theory of posets into an equivalent equational one however.

ADD (Andrej):

Another example of “fixing” a non-algebraic theory: Riesz spaces (partially ordered vector spaces) are not algebraic, but assuming they have min and max makes them algebraic (because the poset becomes a (distributive) lattice).

Exercise 1.1.9 Let G be a group. Formulate the notion of a (left) G -set (i.e. a functor $G \rightarrow \mathbf{Set}$) as an algebraic theory.

1.1.1 Models of Algebraic Theories

Let us now consider what a *model* of an algebraic theory is. In classical algebra, a group is given by a set G , an element $e \in G$, a function $m : G \times G \rightarrow G$ and a function $i : G \rightarrow G$, satisfying the group axioms:

$$\begin{aligned} m(x, m(y, z)) &= m(m(x, y), z) \\ m(x, ix) &= m(ix, x) = e \\ m(x, e) &= m(e, x) = x \end{aligned}$$

This notion can easily be generalized so that we can speak of models of group theory in categories other than \mathbf{Set} . This is accomplished simply by translating the equations between certain elements into equations between the operations themselves: thus a group is given by an object $G \in \mathbf{Set}$ and three morphisms

$$e : 1 \rightarrow G, \quad m : G \times G \rightarrow G, \quad i : G \rightarrow G.$$

Associativity of m is expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \pi_2} & G \times G \\ \pi_0 \times m \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array} \quad (1.1)$$

Similarly, the axioms for the unit and the inverse are expressed by commutativity of the following diagrams:

$$\begin{array}{ccc}
 G \times 1 & \xrightarrow{1_G \times e} & G \times G \xleftarrow{e \times 1_G} 1 \times G \\
 & \searrow \pi_0 & \downarrow m \swarrow \pi_1 \\
 & & G
 \end{array}
 \qquad
 \begin{array}{ccccc}
 G & \xrightarrow{\langle 1_G, i \rangle} & G \times G & \xleftarrow{\langle i, 1_G \rangle} & G \\
 \downarrow !_G & & \downarrow m & & \downarrow !_G \\
 1 & \xrightarrow{e} & G & \xleftarrow{e} & 1
 \end{array}
 \quad (1.2)$$

Moreover, this formulation makes sense in any category \mathcal{C} with finite products. So we can define a *group in \mathcal{C}* to consist of an object G equipped with arrows:

$$\begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \xleftarrow{i} G \\
 & & \uparrow e \\
 & & 1
 \end{array}$$

such that the above diagrams (1.1) and (1.2) expressing the group equations commute.

There is also an obvious corresponding generalization of a group homomorphism in \mathbf{Set} to homomorphisms of groups in \mathcal{C} . Namely, an arrow in \mathcal{C} between groups $h : M \rightarrow N$ is a homomorphism if it commutes with the interpretations of the basic operations m , i , and e ,

$$h \circ m^M = m^N \circ h^2 \qquad h \circ i^M = i^N \circ h \qquad h \circ e^M = e^N$$

as indicated in:

$$\begin{array}{ccc}
 M^2 & \xrightarrow{h^2} & N^2 \\
 m^M \downarrow & & \downarrow m^N \\
 M & \xrightarrow{h} & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 M & \xrightarrow{h} & N \\
 i^M \downarrow & & \downarrow i^N \\
 M & \xrightarrow{h} & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 1 & \xrightarrow{=} & 1 \\
 e^M \downarrow & & \downarrow e^N \\
 M & \xrightarrow{h} & N
 \end{array}$$

Together with the evident composition and identity arrows inherited from \mathcal{C} , this gives a category of groups in \mathcal{C} which we denote:

Group(\mathcal{C})

In general, we define an *interpretation* I of a theory \mathbb{T} in a category \mathcal{C} with finite products to consist of an object $I \in \mathcal{C}$ and, for each basic operation f of arity k , a morphism $f^I : I^k \rightarrow I$. (More formally, I is the tuple consisting of an underlying set $|I|$ and the interpretations f^I , but we shall write simply I for $|I|$.) In particular, basic constants are interpreted as morphisms $1 \rightarrow I$. The interpretation can be extended to all terms as follows: a general term t is always interpreted together with a *context* of variables x_1, \dots, x_n , where the variables appearing in t are among the variables appearing in the context. We write

$$x_1, \dots, x_n \mid t \quad (1.3)$$

to indicate that the term t is to be understood in context x_1, \dots, x_n . The interpretation of a term in context (1.3) is a morphism $t^I : I^n \rightarrow I$, determined by the following specification:

1. The interpretation of a variable x_i is the i -th projection $\pi_i : I^n \rightarrow I$.
2. A term of the form $f(t_1, \dots, t_k)$ is interpreted as the composite:

$$I^n \xrightarrow{(t_1^I, \dots, t_k^I)} I^k \xrightarrow{f^I} I$$

where $t_i^I : I^n \rightarrow I$ is the interpretation of the subterm t_i , for $i = 1, \dots, k$, and f^I is the interpretation of the basic operation f .

It is clear that the interpretation of a term really depends on the context, and when necessary we shall write $t^I = [x_1, \dots, x_n \mid t]^I$. For example, the term $f x_1$ is interpreted as a morphism $f^I : I \rightarrow I$ in context x_1 , and as the morphism $f^I \circ \pi_1 : I^2 \rightarrow I$ in the context x_1, x_2 .

Suppose u and v are terms in context x_1, \dots, x_n . Then we say that the equation $u = v$ is *satisfied* by the interpretation I if u^I and v^I are the same morphism in \mathcal{C} . In particular, if $u = v$ is an axiom of the theory, and x_1, \dots, x_n are all the variables appearing in u and v , we say that I *satisfies the axiom* $u = v$ if $[x_1, \dots, x_n \mid u]^I$ and $[x_1, \dots, x_n \mid v]^I$ are the same morphism,

$$I^n \xrightarrow{\begin{array}{c} [x_1, \dots, x_n \mid u]^I \\ [x_1, \dots, x_n \mid v]^I \end{array}} I \quad (1.4)$$

which we also write as:

$$I \models u = v \quad \iff \quad u^I = v^I.$$

Of course, we can now define as usual:

Definition 1.1.10 (cf. Definition ??) A *model* M of an algebraic theory \mathbb{T} in a category \mathcal{C} with finite products is an interpretation I of the theory that satisfies the axioms of \mathbb{T} ,

$$I \models u = v,$$

for all $(u = v) \in A_{\mathbb{T}}$.

A *homomorphism* of models $h : M \rightarrow N$ is an arrow in \mathcal{C} that commutes with the interpretations of the basic operations,

$$h \circ f^M = f^N \circ h^k$$

for all $f \in \Sigma_{\mathbb{T}}$, as indicated in:

$$\begin{array}{ccc} M^k & \xrightarrow{h^k} & N^k \\ f^M \downarrow & & \downarrow f^N \\ M & \xrightarrow{h} & N \end{array}$$

The category of \mathbb{T} -models in \mathcal{C} is written,

$$\text{Mod}(\mathbb{T}, \mathcal{C}).$$

A model of the empty theory \mathbb{T}_0 in a category \mathcal{C} with finite products is just an object $A \in \mathcal{C}$, and similarly for homomorphisms, so

$$\text{Mod}(\mathbb{T}_0, \mathcal{C}) = \mathcal{C}.$$

A model of the theory $\mathbb{T}_{\text{Group}}$ of groups in \mathcal{C} is a group in \mathcal{C} , in the above sense, and similarly for homomorphisms, so:

$$\text{Mod}(\mathbb{T}_{\text{Group}}, \mathcal{C}) = \text{Group}(\mathcal{C}).$$

In particular, a model in Set is just a group in the usual sense:

$$\text{Mod}(\mathbb{T}_{\text{Group}}, \text{Set}) = \text{Group}(\text{Set}) = \text{Group}.$$

An example of a new kind is provided the following.

Example 1.1.11 A model of the theory of groups in a functor category $\text{Set}^{\mathbb{C}}$ is a functor from \mathbb{C} into groups,

$$\text{Group}(\text{Set}^{\mathbb{C}}) \cong \text{Hom}(\mathbb{C}, \text{Group}).$$

Indeed, for each object $C \in \mathbb{C}$ there is an evaluation functor,

$$\text{eval}_C : \text{Set}^{\mathbb{C}} \rightarrow \text{Set}$$

with $\text{eval}_C(F) = F(C)$, and evaluation preserves products since these are computed point-wise in the functor category. Moreover, every arrow $h : C \rightarrow D$ in \mathbb{C} gives rise to an obvious natural transformation $h : \text{eval}_C \rightarrow \text{eval}_D$. Thus for any group G in $\text{Set}^{\mathbb{C}}$, we have groups $\text{eval}_C(G)$ for each $C \in \mathbb{C}$ and group homomorphisms $h_G : C(G) \rightarrow D(G)$, comprising a functor $G : \mathbb{C} \rightarrow \text{Group}$. Conversely, it is clear that any such functor $H : \mathbb{C} \rightarrow \text{Group}$ arises in this way from a group H in $\text{Set}^{\mathbb{C}}$, at least up to isomorphism.

Exercise 1.1.12 Verify the details of the isomorphism of categories

$$\text{Mod}(\mathbb{T}, \text{Set}^{\mathbb{C}}) \cong \text{Hom}(\mathbb{C}, \text{Mod}(\mathbb{T}, \text{Set}))$$

discussed in example 1.1.11.

Exercise 1.1.13 Determine what a group is in the following categories: the category of finite sets FinSet , the category of topological spaces Top , the category of graphs Graph , and the category of groups Group .

Hint: Only the last case is tricky. Before thinking about it, prove the following lemma [?, Lemma 3.11.6]. Let G be a set provided with two binary operations \cdot and \star and a common unit e , so that $x \cdot e = e \cdot x = x \star e = e \star x = x$. Suppose the two operations commute, i.e., $(x \star y) \cdot (z \star w) = (x \cdot z) \star (y \cdot w)$. Then they coincide, are *commutative* and associative.

1.1.2 Theories as categories

The syntactically presented notion of an algebraic theory, say of groups, is a notational convenience, but as a specification of, say, the mathematical concept of a group it has some defects. We want to find a *presentation-free* notion that captures the group concept without tying it to a specific syntactic presentation. The notion we seek can be given by a category with a certain universal mapping property which determines it uniquely (up to equivalence). This also results in a reformulation of the usual conception of syntax and semantics — so distinctive of conventional logic — bringing it more in line with other fields of modern mathematics.

Let us consider group theory again. The algebraic axiomatization in terms of unit, multiplication and inverse is not the only possible one. For example, an alternative formulation uses the unit e and a binary operation \odot , called *double division*, along with a single axiom [?]:

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z .$$

The usual group operations are related to double division as follows:

$$x \odot y = x^{-1} \cdot y^{-1} , \quad x^{-1} = x \odot e , \quad x \cdot y = (x \odot e) \odot (y \odot e) .$$

There may be various reasons why we prefer to work with one formulation of group theory rather than another, but this should not be reflected in the general idea of what a group is. We want to avoid particular choices of basic constants, operations, and axioms. This is akin to the situation where an algebra is presented by generators and relations: the algebra itself is regarded as independent of any particular choice of presentation. Similarly, one usually prefers a basis-free theory of vector spaces: it is better to formulate the idea of a vector space without speaking explicitly of vector bases, even though every vector space has one. Without a doubt, vector bases are important, but they really are an auxiliary concept.

As a first step, we could simply take *all* operations built from unit, multiplication, and inverse as basic, and *all* valid equations of group theory as axioms. But we can go a step further and collect all the operations into a category, thus forgetting about which ones were “basic” and which ones “derived”, and which equalities were “axioms”. We first describe this construction of a category $\mathcal{C}_{\mathbb{T}}$ for a general algebraic theory \mathbb{T} , and then determine another characterization of it.

As objects of $\mathcal{C}_{\mathbb{T}}$ we take *contexts*, i.e. sequences of variables,

$$[x_1, \dots, x_n] . \quad (n \geq 0)$$

A morphism from $[x_1, \dots, x_m]$ to $[x_1, \dots, x_n]$ is an n -tuple (t_1, \dots, t_n) , where each t_k is a term in the context, $x_1, \dots, x_m \mid t_k$. Two such morphisms (t_1, \dots, t_n) and (s_1, \dots, s_n) are equal if, and only if, the axioms of the theory imply that $t_k = s_k$ for every $k = 1, \dots, n$,

$$\mathbb{T} \vdash t_k = s_k$$

Strictly speaking, morphisms are thus *equivalence classes* of terms in context

$$[x_1, \dots, x_m \mid t_1, \dots, t_n] : [x_1, \dots, x_m] \longrightarrow [x_1, \dots, x_n],$$

where two terms are equivalent when the theory proves them to be equal. Since it is rather cumbersome to work with equivalence classes, we shall work with the terms directly, but keeping in mind that equality between them is equivalence. The composition of morphisms

$$\begin{aligned} (t_1, \dots, t_m) &: [x_1, \dots, x_k] \rightarrow [x_1, \dots, x_m] \\ (s_1, \dots, s_n) &: [x_1, \dots, x_m] \rightarrow [x_1, \dots, x_n] \end{aligned}$$

is the morphism (r_1, \dots, r_n) whose i -th component is obtained by simultaneously substituting in s_i the terms t_1, \dots, t_m for the variables x_1, \dots, x_m :

$$r_i = s_i[t_1, \dots, t_m/x_1, \dots, x_m] \quad (1 \leq i \leq n)$$

The identity morphism on $[x_1, \dots, x_n]$ is (x_1, \dots, x_n) . It is easy to verify that these specifications are well-defined on equivalence classes, and make $\mathcal{C}_{\mathbb{T}}$ a category.

Definition 1.1.14 The category $\mathcal{C}_{\mathbb{T}}$ just defined is called the *syntactic category* of the theory \mathbb{T} .

The syntactic category $\mathcal{C}_{\mathbb{T}}$ contains the same “algebraic” information as the theory \mathbb{T} from which it was built, but in a syntax-invariant way. Any two different presentations of \mathbb{T} — like the ones for groups mentioned above — will give rise to essentially the same category $\mathcal{C}_{\mathbb{T}}$. But there is also a much more important sense in which $\mathcal{C}_{\mathbb{T}}$ represents \mathbb{T} , as we next show.

Exercise 1.1.15 Show that the syntactic category $\mathcal{C}_{\mathbb{T}}$ has all finite products.

1.1.3 Models as Functors

Let \mathbb{T} be an arbitrary algebraic theory and $\mathcal{C}_{\mathbb{T}}$ the syntactic category constructed from \mathbb{T} as in the foregoing section. It is easy to show that the product in $\mathcal{C}_{\mathbb{T}}$ of two objects $[x_1, \dots, x_n]$ and $[x_1, \dots, x_m]$ is the object $[x_1, \dots, x_{n+m}]$, and that $\mathcal{C}_{\mathbb{T}}$ has all finite products (including $\mathbf{1} = [-]$, the empty context). Moreover, there is a \mathbb{T} -model U in $\mathcal{C}_{\mathbb{T}}$ consisting of the language itself: The underlying object is the context $U = [x_1]$ of length one, and each operation symbol f of, say, arity k is interpreted as itself,

$$f^U = [x_1, \dots, x_k \mid f(x_1, \dots, x_k)] : U^k = [x_1, \dots, x_k] \longrightarrow [x_1] = U.$$

The axioms are of course all satisfied, since for any terms s, t :

$$U \models s = t \iff s^U = t^U \iff \mathbb{T} \vdash s = t. \quad (1.5)$$

This *syntactic model* U in $\mathcal{C}_{\mathbb{T}}$ is “universal” in the following sense: any model M in any category \mathcal{C} with finite products is the image of U under an essentially unique, finite product

preserving functor $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$. In a certain sense, then, $\mathcal{C}_{\mathbb{T}}$ is the free finite product category with a model of \mathbb{T} . We now proceed to make this more precise.

First, observe that any FP-functor $F : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$ takes U to a model FU in \mathcal{C} , just because FP-functors preserve models of algebraic theories, as is easily seen. Moreover, any natural transformation $\vartheta : F \rightarrow G$ between FP-functors determines a homomorphism of models $h = \vartheta_U : FU \rightarrow GU$, since clearly, for any basic operation f ,

$$h \circ f^{FU} = f^{GU} \circ h^k$$

by naturality. In more detail, suppose $f : U \times U \rightarrow U$ is a basic operation, then there is a commutative diagram,

$$\begin{array}{ccc}
 FU \times FU & \xrightarrow{h \times h} & GU \times GU \\
 \downarrow \cong & & \downarrow \cong \\
 F(U \times U) & \xrightarrow{\vartheta_{U \times U}} & G(U \times U) \\
 \downarrow Ff & & \downarrow Gf \\
 FU & \xrightarrow{h = \vartheta_U} & GU
 \end{array}$$

f^{FU} (left curved arrow) and f^{GU} (right curved arrow) connect the top and bottom nodes of the squares.

where the upper square commutes by preservation of products, and the lower one by naturality. Thus the operation “evaluation at U ” determines a functor,

$$\text{eval}_U : \text{Hom}_{\text{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \text{Mod}(\mathbb{T}, \mathcal{C}) \quad (1.6)$$

from the category of finite product preserving functors $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$, with natural transformations as arrows, into the category of \mathbb{T} -models in \mathcal{C} .

Proposition 1.1.16 *The functor (1.6) is an equivalence of categories, natural in \mathcal{C} .*

Proof. Let M be any model in an FP-category \mathcal{C} . Then the assignment $f \mapsto f^M$ given by the interpretation determines a functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$, defined on objects by

$$M^\sharp[x_1, \dots, x_k] = M^k$$

and on morphisms by

$$M^\sharp(t_1, \dots, t_n) = \langle t_1^M, \dots, t_n^M \rangle.$$

In detail, M^\sharp is defined on morphisms

$$[x_1, \dots, x_k \mid t] : [x_1, \dots, x_k] \rightarrow [x_1, \dots, x_n]$$

in $\mathcal{C}_{\mathbb{T}}$ by the following rules:

1. The morphism

$$(x_i) : [x_1, \dots, x_k] \rightarrow [x_1]$$

is mapped to the i -th projection

$$\pi_i : M^k \rightarrow M.$$

2. The morphism

$$(f(t_1, \dots, t_m)) : [x_1, \dots, x_k] \rightarrow [x_1]$$

is mapped to the composite

$$M^k \xrightarrow{(M^\sharp t_1, \dots, M^\sharp t_m)} M^m \xrightarrow{M^\sharp f} M$$

where $M^\sharp t_i : M^k \rightarrow M$ is the value of M^\sharp on the morphisms $(t_i) : [x_1, \dots, x_k] \rightarrow [x_1]$, for $i = 1, \dots, m$, and $M^\sharp f = f^M$ is the interpretation of the basic operation f .

3. The morphism

$$(t_1, \dots, t_n) : [x_1, \dots, x_k] \rightarrow [x_1, \dots, x_n]$$

is mapped to the morphism $\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle$ where $M^\sharp t_i$ is the value of M^\sharp on the morphism $(t_i) : [x_1, \dots, x_k] \rightarrow [x_1]$, and

$$\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle : M^k \longrightarrow M^n$$

is the evident n -tuple in the FP-category \mathcal{C} .

That $M : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$ really is a functor now follows from the assumption that the interpretation M is a model, which means that all the equations of the theory are satisfied by it, so that the above specification is well-defined on equivalence classes. Observe that the functor M is defined in such a way that it obviously preserves finite products, and that there is an isomorphism of models,

$$M^\sharp(U) \cong M.$$

Thus we have shown that the functor “evaluation at U ”,

$$\text{eval}_U : \text{Hom}_{\text{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \text{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.7}$$

is essentially surjective on objects, since $\text{eval}_U(M^\sharp) = M^\sharp(U)$.

We leave the verification that it is full and faithful as an easy exercise.

Exercise 1.1.17 Verify this.

Naturality in \mathcal{C} means the following. Suppose M is a model of \mathbb{T} in any category \mathcal{C} with finite products (“FP-category”). Any finite product-preserving functor (“FP-functor”) $F : \mathcal{C} \rightarrow \mathcal{D}$ to another FP-category \mathcal{D} then takes M to a model $F(M)$ in \mathcal{D} . The interpretation is given by setting $f^{F(M)} = F(f^M)$ for the basic operations f (and composing with the canonical isos coming from preservation of products, $F(M) \times F(M) \cong F(M \times M)$, etc.). Since equations are described by commuting diagrams, F takes a model to a model, and the same is true for homomorphisms. Thus $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a functor on \mathbb{T} -models,

$$\mathbf{Mod}(\mathbb{T}, F) : \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathbf{Mod}(\mathbb{T}, \mathcal{D}).$$

By naturality of (1.6) we mean that the following square commutes, up to natural isomorphism:

$$\begin{array}{ccc} \mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) & \xrightarrow{\text{eval}_U} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \\ \mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, F) \downarrow & & \downarrow \mathbf{Mod}(\mathbb{T}, F) \\ \mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) & \xrightarrow{\text{eval}_U} & \mathbf{Mod}(\mathbb{T}, \mathcal{D}) \end{array} \quad (1.8)$$

But this is clear, since for any FP-functor $M : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$ we have:

$$\begin{aligned} \text{eval}_U \circ \mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M) &= (\mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M))(U) \\ &= (F \circ M)(U) \\ &= F(M(U)) \\ &= F(\text{eval}_U(M)) \\ &\cong \mathbf{Mod}(\mathbb{T}, F)(\text{eval}_U(M)) \\ &= \mathbf{Mod}(\mathbb{T}, F) \circ \text{eval}_U(M). \end{aligned}$$

■

The equivalence of categories

$$\mathbf{Hom}_{\mathbf{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \quad (1.9)$$

actually determines $\mathcal{C}_{\mathbb{T}}$ and the universal model U uniquely, up to equivalence of categories and isomorphism of models. Indeed, to recover U , just put $\mathcal{C}_{\mathbb{T}}$ for \mathcal{C} and the identity functor $1_{\mathcal{C}_{\mathbb{T}}}$ on the left, to get U in $\mathbf{Mod}(\mathbb{T}, \mathcal{C}_{\mathbb{T}})$ on the right! To see that $\mathcal{C}_{\mathbb{T}}$ itself is also determined, observe that (1.9) essentially says that the functor $\mathbf{Mod}(\mathbb{T}, \mathcal{C})$ is representable, with representing object $\mathcal{C}_{\mathbb{T}}$. As usual, this fact can also be formulated in elementary terms as a universal mapping property of $\mathcal{C}_{\mathbb{T}}$, as follows:

Definition 1.1.18 The *classifying category* of an algebraic theory \mathbb{T} is an FP-category $\mathcal{C}_{\mathbb{T}}$ with a distinguished model U , called the *universal model*, such that:

(i) for any model M in any FP-category \mathcal{C} , there is an FP-functor

$$M^\sharp : \mathcal{C}_\mathbb{T} \rightarrow \mathcal{C}$$

and an isomorphism of models $M \cong M^\sharp(U)$.

(ii) for any FP-functors $F, G : \mathcal{C}_\mathbb{T} \rightarrow \mathcal{C}$ and model homomorphism $h : F(U) \rightarrow G(U)$, there is a unique natural transformation $\vartheta : F \rightarrow G$ with

$$\vartheta_U = h.$$

Observe that (i) says that the evaluation functor (1.6) is essentially surjective, and (ii) that it is full and faithful. The category $\mathcal{C}_\mathbb{T}$ is clearly determined up to equivalence by this universal mapping property in the usual way. Specifically, if (\mathcal{C}, U) and (\mathcal{D}, V) are both classifying categories for the same theory, then there are classifying functors,

$$\begin{array}{ccc} & V^\sharp & \\ \mathcal{C} & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathcal{D} \\ & U^\sharp & \end{array}$$

the composites of which are necessarily isomorphic to the respective identity functors, since e.g. $U^\sharp(V^\sharp(U)) \cong U^\sharp(V) \cong U$.

We have now shown not only that every algebraic theory has a classifying category, but also that the syntactic category is essentially determined by that distinguishing property. We record this as the following.

Theorem 1.1.19 *Every algebraic theory \mathbb{T} has the syntactic category $\mathcal{C}_\mathbb{T}$ as a classifying category.*

Example 1.1.20 Let us see what the foregoing definitions give us in the case of group theory $\mathbb{G} = \mathbb{T}_{\text{Group}}$. Recall that the category \mathbb{G} consists of contexts $[x_1, \dots, x_n]$ and terms built from variables and the basic group operations. A finite product preserving functor $M : \mathbb{G} \rightarrow \mathbf{Set}$ is then determined up to natural isomorphism by its action on the context $[x_1]$ and the terms representing the basic operations. If we set

$$\begin{array}{ll} G = M[x_1], & e = M(\cdot \mid e), \\ i = M(x_1 \mid x_1^{-1}), & m = M(x_1, x_2 \mid x_1 \cdot x_2), \end{array}$$

then (G, e, i, m) is just a group with unit e , inverse i and multiplication m . That G satisfies the axioms for groups follows from functoriality of M . Conversely, any group (G, e, i, m) determines a finite product preserving functor $M_G : \mathbb{G} \rightarrow \mathbf{Set}$ defined by

$$\begin{array}{ll} M_G[x_1, \dots, x_n] = G^n, & M_G(\cdot \mid e), \\ M_G(x_1 \mid x_1^{-1}) = i, & M_G(x_1, x_2 \mid x_1 \cdot x_2) = m. \end{array}$$

This shows that $\mathbf{Mod}_{\mathbf{Set}}(\mathbb{G})$ is indeed equivalent to \mathbf{Group} , provided both categories have the same notion of morphisms.

Suppose then that (G, e_G, i_G, m_G) and (H, e_H, i_H, m_H) are groups, and let $\phi : M_G \Longrightarrow M_H$ be a natural transformation between the corresponding functors. Then ϕ is already determined by its component at $[x_1]$ because by naturality the following diagram commutes, for $1 \leq k \leq n$:

$$\begin{array}{ccc} G^n & \xrightarrow{\phi_{[x_1, \dots, x_n]}} & H^n \\ G\pi_k = \pi_k \downarrow & & \downarrow H\pi_k = \pi_k \\ G & \xrightarrow{\phi_{[x_1]}} & H \end{array}$$

If we write $\phi' = \phi_{[x_1]}$ then it follows that $\phi_{[x_1, \dots, x_n]} = \phi' \times \dots \times \phi'$. Again, by naturality of ϕ we see that the following diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{\phi' \times \phi'} & H \times H \\ m_G \downarrow & & \downarrow m_h \\ G & \xrightarrow{\phi'} & H \end{array}$$

Similar commutative squares show that ϕ' preserves the unit and commutes with the inverse operation, therefore $\phi' : G \rightarrow H$ is indeed a group homomorphism. Conversely, a group homomorphism $\psi' : G \rightarrow H$ determines a natural transformation $\psi : G \Longrightarrow H$ whose component at $[x_1, \dots, x_n]$ is the n -fold product $\psi' \times \dots \times \psi' : G^n \rightarrow H^n$. This demonstrates that

$$\mathbf{Mod}_{\mathbf{Set}}(\mathbb{G}) \simeq \mathbf{Group} .$$

Example 1.1.21 Recall from ?? that a group G in the functor category $\mathbf{Set}^{\mathbb{C}}$ is essentially the same thing as a functor $G : \mathbb{C} \rightarrow \mathbf{Group}$. From the point of view of algebras as functors, this amounts to the observation that product-preserving functors $\mathbb{G} \rightarrow \mathbf{Hom}(\mathbb{C}, \mathbf{Set})$ correspond (by exponential transposition) to functors $\mathbb{C} \rightarrow \mathbf{Hom}_{FP}(\mathbb{G}, \mathbf{Set})$, where the latter \mathbf{Hom} -set consists just of product-preserving functors. Indeed, the correspondence extends to natural transformations to give an equivalence of categories,

$$\mathbf{Group}(\mathbf{Set}^{\mathbb{C}}) \simeq (\mathbf{Group}(\mathbf{Set}))^{\mathbb{C}} \simeq \mathbf{Group}^{\mathbb{C}} .$$

1.1.4 Completeness

Consider an algebraic theory \mathbb{T} and an equation $s = t$ between terms of the theory. If the equation can be proved from the axioms of the theory, then every model of the theory

satisfies the equation; this is just the *soundness* of the equational calculus with respect to models in categories. The converse statement is:

“Every model of \mathbb{T} satisfies $s = t$.” \Rightarrow “ \mathbb{T} proves equation $s = t$.” .

This property is called *completeness*, and (together with soundness) it says that the calculus of equations suffices for proving all (and only) the ones that hold in the semantics. This holds in an especially strong sense for categorical semantics, as shown by the following.

Theorem 1.1.22 (Completeness for algebraic theories) *Suppose \mathbb{T} is an algebraic theory. Then there exists an FP- category \mathcal{C} and a model $U \in \mathbf{Mod}_{\mathcal{C}}(\mathbb{T})$ with the property that, for every equation $s = t$ between terms of the theory \mathbb{T} ,*

$$U \models s = t \iff \mathbb{T} \vdash s = t.$$

That is, satisfaction by U is equivalent to provability in \mathbb{T} . Thus the equational calculus of algebraic theories is complete with respect to general categorical semantics.

Proof. This follows from the classifying category theorem 1.1.19 as follows: Let $\mathcal{C} = \mathcal{C}_{\mathbb{T}}$ the classifying category and U the universal model. If $\mathbb{T} \vdash s = t$, then by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ we have $s^U = t^U$. Any model M in an FP-category \mathcal{C} has a classifying functor $M^{\sharp} : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{C}$, which preserves the interpretations of s and t in the sense that (up to canonical isomorphism):

$$M^{\sharp}(s^U) = s^{M^{\sharp}(U)} = s^M$$

and similarly for t . Thus from $s^U = t^U$ we can infer $s^M = t^M$, i.e. $M \models s = t$.

Conversely, if $U \models s = t$, then $s^U = t^U$. But by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$, it then must be the case that $\mathbb{T} \vdash s = t$.

Finally, note that $M \models s = t$ for every model M if and only if this holds in the universal model U . ■

Definition 1.1.23 A single model with the property mentioned in the theorem, of satisfying all and only those equations that are provable from the theory, shall be said to be *logically generic*.

Thus, by the foregoing, the universal model is logically generic. Classically, it is seldom the case that there exists a single, logically generic model; instead, for the sake of completeness, we consider the range of all models in **Set**. Completeness with respect to a restricted range of models is of course a stronger statement than general completeness. Toward this classical result, we first consider completeness with respect to “variable models” in **Set**, i.e. models in presheaf categories $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

Proposition 1.1.24 *Let \mathbb{T} be an algebraic theory. The Yoneda embedding*

$$y : \mathcal{C}_{\mathbb{T}} \rightarrow \widehat{\mathcal{C}_{\mathbb{T}}}$$

is a generic model for \mathbb{T} .

Proof. The Yoneda embedding $y : \mathcal{C}_{\mathbb{T}} \rightarrow \widehat{\mathcal{C}_{\mathbb{T}}}$ preserves limits, and in particular finite products, hence it corresponds to a model $U' = y(U)$ of \mathbb{T} in $\widehat{\mathcal{C}_{\mathbb{T}}}$. Simply because y is a functor, U' satisfies all equations that hold in U , but because it is faithful, U' does not validate any equations that do not already hold in U . But since U is logically generic, so is U' . ■

Example 1.1.25 We consider group theory one last time. The universal group is a group that satisfies every equation that is satisfied by all groups, and no others. Let us describe it as a generalized set. Recall that the theory of a group is a category \mathbb{G} whose objects are contexts $[x_1, \dots, x_n]$, $n \in \mathbb{N}$. The carrier U of the universal group is the Yoneda embedding of the context with one variable,

$$U = y[x_1] = \mathbb{G}(-, [x_1]) .$$

This is a set parametrized by the objects of \mathbb{G} . For every $n \in \mathbb{N}$, we get a set $U_n = \mathbb{G}([x_1, \dots, x_n], [x_1])$ that consists of all terms built from n variables, modulo equations of group theory—which is precisely the free group on n generators! Unit, inverse, and multiplication on U are defined at each stage U_n as the corresponding operations on the free group on n generators.

To summarize, the universal group is the free group on n -generators, where $n \in \mathbb{N}$ is a parameter.

Finally, we consider completeness with respect to **Set**-valued models $M : \mathcal{C}_{\mathbb{T}} \rightarrow \mathbf{Set}$, which of course correspond to classical models. We need the following:

Lemma 1.1.26 *For any small category \mathcal{C} , there is a jointly faithful set of FP-functors $E_i : \mathbf{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \mathbf{Set}$, $i \in I$. That is, for any maps $f, g : A \rightarrow B$ in $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$, if $E_i(f) = E_i(g)$ for all $i \in I$, then $f = g$.*

Proof. Consider the evaluation functors $ev_C : \mathbf{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \mathbf{Set}$ for all $C \in \mathcal{C}$. These are clearly jointly faithful, and they preserve all limits and colimits, since these are constructed pointwise in presheaves. ■

Proposition 1.1.27 *Suppose \mathbb{T} is an algebraic theory. For every equation $s = t$ between terms of the theory \mathbb{T} ,*

$$M \models s = t \quad \text{for all models } M \text{ in } \mathbf{Set} \iff \mathbb{T} \vdash s = t .$$

*Thus the equational calculus of algebraic theories is complete with respect to **Set**-valued semantics.*

Exercise 1.1.28 The universal group U is a functor $\mathbb{G}^{\text{op}} \rightarrow \mathbf{Set}$. In the last example we described the object part of U . What is the action of U on morphisms?

Exercise 1.1.29 Let s be a term of group theory with variables x_1, \dots, x_n . On one hand we can think of s as an element of the free group U_n , and on the other we can consider the interpretation of s in the universal group U , namely a natural transformation $Us : U^n \Rightarrow U$. Suppose t is another term of group theory with variables x_1, \dots, x_n . Show that $Us = Ut$ if, and only if, $s = t$ in the free group U_n .

1.1.5 Functorial Semantics

Let us now summarize our treatment of algebraic theories so far. We have reformulated the traditional *logical* notions in terms of “algebraic” or *categorical* ones. The traditional approach to logic may be described as involving four different parts:

Type theory

There is an underlying type theory, which is a calculus of types and terms. For algebraic theories the calculus of types is trivial, since there is only one type which is not even explicitly mentioned. The terms are built from variables and basic operations.

Logic A variety of different kinds of logic can be considered. Algebraic theories have a very simple kind of logic that only involves equations between terms and equational reasoning.

Theory

A theory is given by basic types, basic terms, and axioms. The types and the terms are expressed in the type theory of the system, and the axioms are expressed in the logic of the system.

Interpretations and Models

The type theory and logic of a logical system can be interpreted in any category of the appropriate kind. For algebraic theories we considered categories with finite products. The interpretation is *denotational*, in the sense that the types and terms of the theory are assigned to objects and morphisms (which they “denote”) by induction on the structure of types, terms, and logical formulas. An interpretation of a theory is a *model* if it satisfies all the axioms of the theory, where in the present case the notion of satisfaction just means that the arrows interpreting the terms occurring in the equations are actually equal.

The alternative approach developed here — called *functorial semantics* — may be summarized as follows:

Theories are categories

From a theory we can construct a category which expresses essentially the same information as the theory but is syntax-invariant, in the sense that it does not depend on a particular presentation by (basic) operations and axioms. The structure of the category reflects the underlying type theory and logic. For example, single sorted algebraic theories give rise to categories with finite products.

Models are functors

A model is a (structure-preserving) functor from a (category representing a) theory to a category with appropriate structure to interpret the logic. The requirement that all axioms of the theory must be satisfied by a model translates to the requirement that the model is a functor and that it preserves the structure of the theory. For models of algebraic theories we only required that they preserve finite products, whereas functoriality ensures that all valid equations of the theory are preserved, thus satisfying the axioms.

Homomorphisms are natural transformations

We obtain a notion of homomorphisms between models for free: since models are functors, homomorphisms are natural transformations between them. Homomorphisms between models of algebraic theories turned out to be the usual notion of morphisms that preserved the algebraic structure.

Universal model

By admitting models in categories other than **Set**, functorial semantics allows the possibility of *universal models*: a model U in the classifying category \mathcal{C}_T , such that every model anywhere is a functorial image of U by an essentially unique, logic-preserving functor. Such a universal model is then “logically generic”, in the sense that it has all and only those logical properties had by all models, since such properties are preserved by the functors in question.

Logical completeness

The construction of the classifying category from the logical syntax of the theory shows the *soundness and completeness* of the theory with respect to general categorical semantics. Completeness with respect to a special class of models (e.g. **Set**-valued ones) results from an embedding theorem for the classifying category.