Homework 3

1. A *natural isomorphism* is a natural transformation all of the components of which is an isomorphism. Show that such a natural transformation is an isomorphism in the functor category.

2. Show that in any category with products, there is a natural (in all three arguments A, B, C) isomorphism $A \times (B \times C) \cong (A \times B) \times C$.

3. Show that in any CCC $\mathbf{C}$, for any fixed object A there is a functor $(-)^A : \mathbf{C} \to \mathbf{C}$. Show moreover that there is a contravariant functor $A^{(-)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.

4. Use the Yoneda Lemma to show that the following hold in any CCC with coproducts:

   (a) $C^{A+B} \cong C^A \times C^B$

   (b) $(A \times B)^C \cong A^C \times B^C$

   (c) $(A^B)^C \cong A^{(B \times C)}$

5. Consider the category $\mathbf{Sets}^{\omega}$ of "sets through time".

   (a) Show that the product of two such (variable) sets A and B is given by:

   $$(A \times B)(n) = A(n) \times B(n)$$

   with the expected arrows $(A \times B)(m) \to (A \times B)(n)$ for all $m \leq n$.

   (b) Show that same is true for coproducts $A + B$, so this category also has all coproducts.

   (c) (Harder:) Show that the exponentials are given by:

   $$B^A(n) = \{f : A|^n \to B|^n\}$$

   where $A|^n$ is the restriction of A to the upper segment of ω above n, i.e. $\{n \leq n+1 \leq n+2 \leq \ldots\}$, with the evident restrictions $B^A(m) \to B^A(n)$ for $m \leq n$.