Groupoidification in Physics

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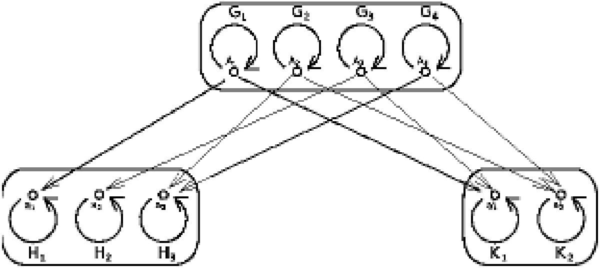
Program: "Categorify" a quantum mechanical description of states and processes.

We propose to represent:

- configuration spaces of physical systems by **groupoids** (or *stacks*), based on local symmetries
- process relating two systems through time by a **span** of groupoids, including a groupoid of "histories"



We are "doing physics in" the †-monoidal (2-)category Span(**Gpd**). This relates to more standard picture in **Hilb** by two representations:

- **Degroupoidification** (Baez-Dolan): $D : Span_1(\mathbf{Gpd}) \to \mathbf{Hilb}$, explains "Physics in **Hilb**"
- **2-Linearization** (Morton): captures more structure by $\Lambda : Span_2(\mathbf{Gpd}) \to \mathbf{2Hilb}$, suggests "Physics in **2Hilb**."

Both invariants rely on a **pull-push** process, and some form of **adjointness.**

Definition

A **groupoid G** is a category in which all morphisms are invertible.

Often, we consider groupoids IN spaces, manifolds, etc. (i.e. with manifolds of objects, morphisms).

Example

Some relevant groupoids:

- Any set S can be seen as a groupoid with only identity morphisms
- Any group G is a groupoid with one object
- Given a set S with a group-action $G \times S \to S$ yields a transformation groupoid $S /\!\!/ G$ whose objects are elements of S; if $g(s) = s'$ then there is a morphism $g_s : s \to s'$
- Any groupoid, as a category, is a union of transformation groupoids (represents "local symmetry")

- A **stack** is a groupoid taken *up to* (Morita-)equivalence
- this coincides with Morita equivalence for $C^{\star}$ algebras, in the case of groupoid algebras.
- equivalent groupoids are "physically indistinguishable". (E.g. full action groupoid; skeleton, with quotient space of objects - no need to decide which is "the" stack)

Our proposal is that *configuration spaces* for physical systems should be (topological, smooth, measured, etc.) stacks.

Note: "configurations" here are roughly "pure states" E.g. *energy levels* for harmonic oscillator.

Definition

A **span** in a category **C** is a diagram of the form:

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$

We'll use $\mathbf{C} = \mathbf{Gpd}$, so s and t are functors (i.e. also map morphisms, representing symmetries).
Spans can be composed by *weak* pullback. (a modified "fibred product") Span(**Gpd**) gets a monoidal structure from the product in **Gpd**, and has duals for morphisms and 2-morphisms.

We can look at this two ways:

- Span **C** is the *universal* 2-category containing **C**, and for which every morphism has a (two-sided) adjoint. The fact that arrows have adjoints means that Span(**C**) is a †-monoidal category (which our representations should preserve).
- Physically, X will represent an object of *histories* leading the system A to the system B. Maps s and t pick the starting and terminating *configurations* in A and B for a given history (in the sense internal to **C**).

(These reasons are closely connected: adjointness is the reversal of time orientation of histories.)

Degroupoidification works like this:
To *linearize* a (finite) groupoid, just take the free vector space on its space of isomorphism classes of objects, $\mathbb{C}^{\underline{A}}$ (or $L^2(\underline{A})$ for more physical situations).

Then there is a pair of linear maps associated to map $f : A \to B$:

- $f^* : \mathbb{C}^B \to \mathbb{C}^A$, with $f^*(g) = g \circ f$ (precomposition)
- $f_* : \mathbb{C}^A \to \mathbb{C}^B$, with $f_*(g)(b) = \sum_{f(a)=b} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(a)} g(a)$ (weighted image of functions)

(There are also integral versions; versions with $U(1)$-phased groupoids, etc. for more physical situations)

These are adjoint with respect to a naturally occurring inner product.

Definition

The functor

$$D : \mathit{Span}(\mathbf{Gpd}) \to \mathbf{Vect}$$

is defined by

$$D(G) = \mathbb{C}(\underline{G})$$

and

$$D(X, s, t) = t_* \circ s^*$$

This gives multiplication by a matrix counting (with "groupoid cardinality") the number of histories from x to y:

$$D(X)_{([a],[b])} = |(s,t)^{-1}(a,b)|_g$$

This is a "sum over histories". (For more physics, such as action principle, use *$U(1)$-groupoids*.

Degroupoidification ignores the fact that **Gpd** is a 2-category (with *groupoids*, *functors*, and *natural transformations*).
The **2-morphisms** of $Span_2(\mathbf{Gpd})$ are (iso. classes of) spans of *span maps*:

$$\begin{tikzcd}
 & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\
A & Y \arrow[l] \arrow[r] \arrow[u, "\sigma"'] \arrow[d, "\tau"] & B \\
 & X' \arrow[ul, "s'"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

These have duals, just like the 1-morphisms.
We want a representation of $Span_2(\mathbf{Gpd})$ that captures more than D, and preserves the adjointness property for both kinds of morphism.

First, this representation lives in **2Hilb**:

Definition

A finite dimensional **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear abelian category generated by finitely many simple objects. A 2-Hilbert space (Baez) is an abelian $H^\star$-category.

That is, 2-vector spaces have a "direct sum" $\oplus$, and $hom(x, y)$ is a vector space for objects x and y. A 2-Hilbert space, in addition, has $hom(x, y)$ a Hilbert space, and a star structure:

$$\mathrm{hom}(x, y) \cong (\mathrm{hom}(y, x))^*$$

which we think of as finding the "adjoint of a morphism".
A **2-linear map** is a functor preserving all this structure.

Lemma

If $\mathbf{B}$ *is an essentially finite groupoid, the representation category* $Rep(\mathbf{B})$ *is a 2-Hilbert space.*

The "basis elements" (generators) of $[\mathbf{B}, \mathbf{Vect}]$ are labeled by $([b], V)$, where $[b] \in \underline{\mathbf{B}}$ and V an irreducible rep of $Aut(b)$.
Baez, Freidel et. al. conjecture the following for the infinite-dimensional case (incompletely understood):

Conjecture

Any 2-Hilbert space is of the following form: $\mathbf{Rep}(\mathcal{A})$*, the category of representations of a von Neumann algebra* $\mathcal{A}$ *on Hilbert spaces. The star structure takes the adjoint of a map.*

This includes the example above, by way of the groupoid algebra $C_c(X)$.

In this context:

- For our physical interpretation $\mathcal{A}$ is the algebras of **symmetries** of a system. The algebra of **observables** will be its commutant - which depends on the choice of representation!
- Basis elements are irreducible representations of the vN algebra - physically, these can be interpreted as **superselection sectors**. Any representation is a direct sum/integral of these.
- Then 2-linear maps are functors, but can also be represented as **Hilbert bimodules** between algebras. The simple components of these bimodules are like matrix entries.

Definition

A **state** for an object *A* in a monoidal category is a morphism from the monoidal unit, $\psi : I \to A$.

- $A \in$ **Hilb**: state determines a vector by $\psi : \mathbb{C} \to H$
- $A \in$ **2Hilb**: a state determines an object (e.g. a representation of groupoid/algebra - an irreducible one is a **superselection sector**)
- $A \in$ Span(**Gpd**), the unit is **1**, the terminal groupoid, so

$$1 \xleftarrow{!} S \xrightarrow{\Psi} A$$

is a "groupoid over *A*", actually Ψ

A state in Span(**Gpd**) determines either of the others, using *D* or Λ.

Theorem

If $\mathbf{X}$ and $\mathbf{B}$ are essentially finite groupoids, a functor $f : \mathbf{X} \to \mathbf{B}$ gives two 2-linear maps:

$$f^* : \Lambda(\mathbf{B}) \to \Lambda(\mathbf{X})$$

namely composition with f, with $f^*F = F \circ f$ and

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{B})$$

called "pushforward along f". Furthermore, f_* is the two-sided adjoint to f^* (i.e. both left-adjoint and right-adjoint).

In fact, there are left and right adjoints, f_* and $f_!$, but the *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \to f_*(F)(b)$$

is given by the *exterior trace map* (which uses a modified group average).

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\boldsymbol{a}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, we get that $\Lambda(X, s, t)$ is represented by the matrix with coefficients:

$$\Lambda(X, s, t)_{([a],V),([b],W)} \simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

This is a intertwiner space is the categorified analog of the counting done by D: this constructs a Hilbert space as a *direct sum over histories* (generally, direct integral).

In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

Theorem

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1},\mathbf{1})$*:*

$$\begin{tikzcd} & A \arrow[dl, "!"'] \arrow[dr, "!"] & \\ \mathbf{1} & X \arrow[u, "s"] \arrow[d, "t"'] & \mathbf{1} \\ & B \arrow[ul, "!"] \arrow[ur, "!"'] & \end{tikzcd}$$

where $\mathbf{1}$ *is the (terminal) groupoid with one object and one morphism,* Λ *on 2-morphisms is just the degroupoidification functor* D*.*

The groupoid cardinality comes from the modified group average in N.

Example

In the case where $\mathbf{A} = \mathbf{B} = \mathbf{FinSet_0}$ (equivalently, the symmetric groupoid $\coprod_{n\geq 0} \Sigma_n$ - note no longer finite), we find

$$D(\mathbf{FinSet_0}) = \mathbb{C}[[t]]$$

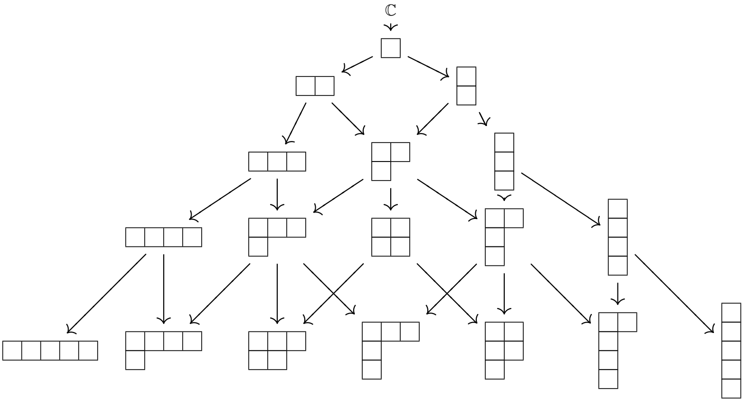
where t^n marks the basis element for object $[n]$. This gets a canonical inner product and can be treated as the Hilbert space for the *quantum harmonic oscillator* ("Fock Space").

The operators $\mathbf{a} = \partial_t$ and $\mathbf{a}^\dagger = M_t$, generate the *Weyl algebra* of operators for the QHO. These are given under D by the span A:

$$\begin{tikzcd} & \mathbf{FinSet_0} \arrow[dl, "\cup\star"'] \arrow[dr, "id"] & \\ \mathbf{FinSet_0} & & \mathbf{FinSet_0} \end{tikzcd}$$

and its dual $A^\dagger$. Composites of these give a categorification of operators explicitly in terms of *Feynman diagrams*.

The image of this picture under Λ involves representation theory of the symmetric groups as $\Lambda(\mathbf{FinSet_0}) \cong \prod_n Rep(\Sigma_n)$, and gives rise to "paraparticle statistics":



Example

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \rightarrow \mathbf{2Vect}$$

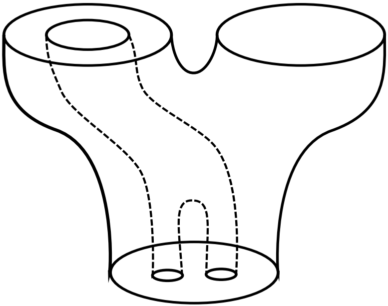
where $\mathbf{nCob_2}$ is a 2-category of cobordisms.

One construction uses *gauge theory*, for gauge group G (here a finite group). Given M, the groupoid $\mathcal{A}_0(M, G) = hom(\pi_1(M), G) /\!/ G$ has:

- **Objects**: Flat connections on M
- **Morphisms** Gauge transformations

Then $\mathcal{A}_0(-, G) : \mathbf{nCob_2} \rightarrow Span_2(\mathbf{Gpd})$, and there is an ETQFT $Z_G = \Lambda \circ \mathcal{A}_0(-, G)$.

This relies on the fact that cobordisms in $\mathbf{nCob_2}$ can be transformed into products of cospans:

<table>
<tr><th>$\mathbf{nCob_2}$</th><th>$Span^2(Top)$</th></tr>
<tr><td></td><td>

$$\begin{tikzcd}
S^1 \arrow[r, "i_A"] \arrow[d, "i_1"'] & (A \coprod D) \arrow[d, "\iota_1"'] & S^1 \coprod S^1 \arrow[l, "i'_A \otimes i_D"'] \arrow[d, "i_2"] \\
Y \arrow[r, "\iota_3"] & M & Y \arrow[l, "\iota_4"'] \\
S^1 \coprod S^1 \arrow[u, "i_2"] \arrow[r, "i_2"'] & Y \arrow[u, "\iota_2"'] & S^1 \arrow[l, "i_1"] \arrow[u, "i_1"']
\end{tikzcd}$$

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Then $\mathcal{A}_0(-, G)$ maps these into $Span^2(\mathbf{Gpd})$.

- View S^1 as the boundary around a system (e.g. particle).
- Irreducible objects of $Z_G(S^1) \simeq [G /\!\!/ G, \mathbf{Vect}]$ are labelled by $([g], W)$, for $[g]$ a conjugacy class in G and W an irrep of its stabilizer subgroup
- For $G = SU(2)$, this is an angle $m \in [0, 2\pi]$, a particle; and an irrep of $U(1)$ (or $SU(2)$ for $m = 0$) is labelled by an integer j
- This theory then looks like 3D quantum gravity coupled to particles with mass and spin. with *mass m* and *spin j*
- Under the topology change of the pair of pants, a pair of such reps is taken to one with nontrivial representations (superselection sectors) for all $[mm']$ for any representatives of $[m]$, $[m']$ (each possible total mass and spin for the combined system).

Dynamics (maps between Hilbert spaces) space arises from the 2-morphisms - componentwise in each 2-linear map.