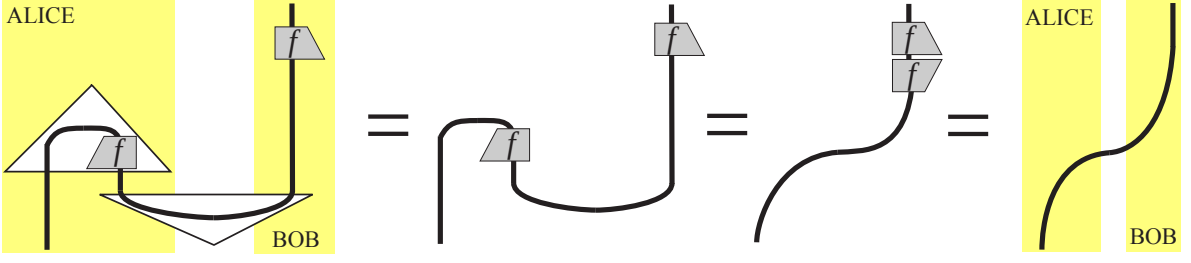
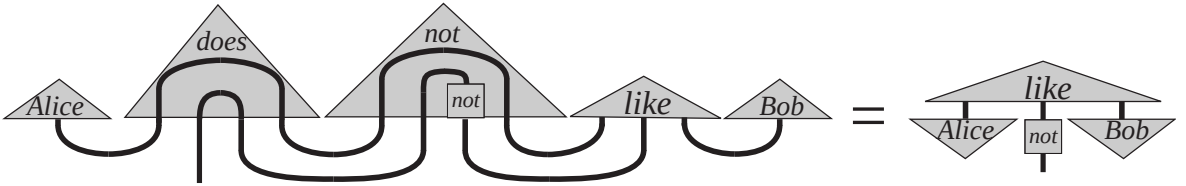
Quantum Picturalism

(or, The Logic of Quantum Mechanics – take 2)

PSA – Montreal — November 2010





Bob Coecke

Oxford University Computing Laboratory

EPSRC Engineering and Physical Sciences Research Council

ONR Science & Technology

QICS

Sixth Framework Programme

fq(x) Foundational Questions Institute

John Templeton Foundation

The Leverhulme Trust

AXA Protection Financière

The ‘Hilbert space’ quantum formalism

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Schrödinger (1935): the stuff which is the true soul of quantum theory is **'how quantum systems compose'**.

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1. Game plan: Which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**?

2. Additional question: Does such an interaction structure appear elsewhere in **"our classical reality"**?

Outcome 1a: "Sheer ratio of results to assumptions"

Hans Halvorson (2010) Editorial to: *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, Cambridge University Press.

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Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. $2\times$ ICALP'10)

EG: Ross Duncan & Simon Perdrix (2010) *Rewriting measurement-based quantum computations with generalised flow*. ICALP'10.

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Outcome 1c: **Simple intuitive (but rigorous) diagrammatic language**, meanwhile adopted by others:

"... we join the ***quantum picturalism*** revolution [1]"

Lucien Hardy (2010) *A formalism-local framework for general probabilistic theories including quantum theory*. arXiv:1005.5164

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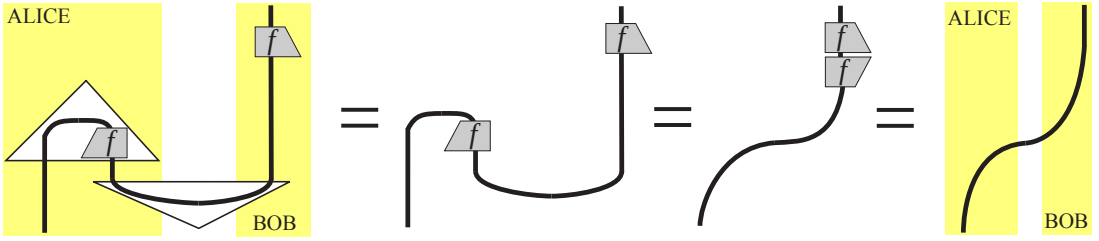
Outcome 2a:

Behaviors of matter:



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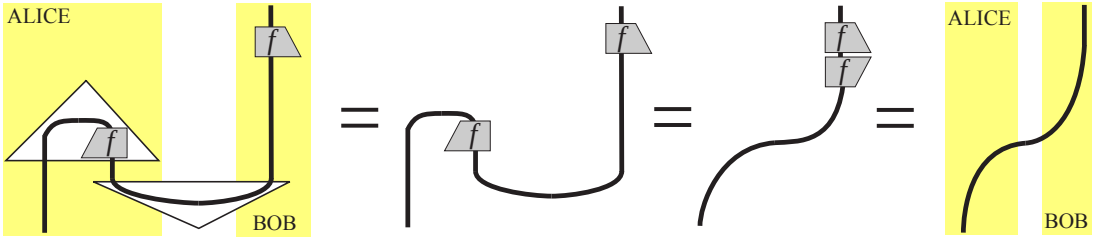
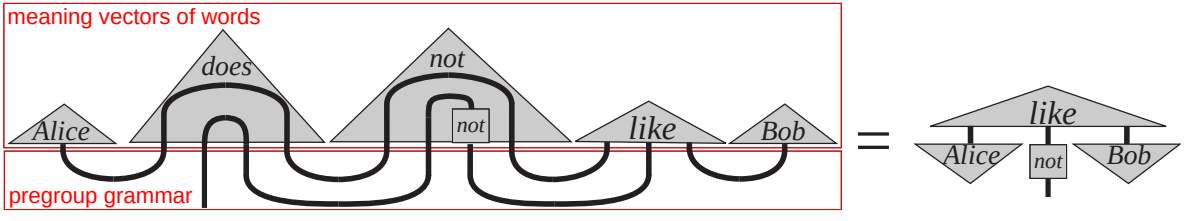


Meaning in language:



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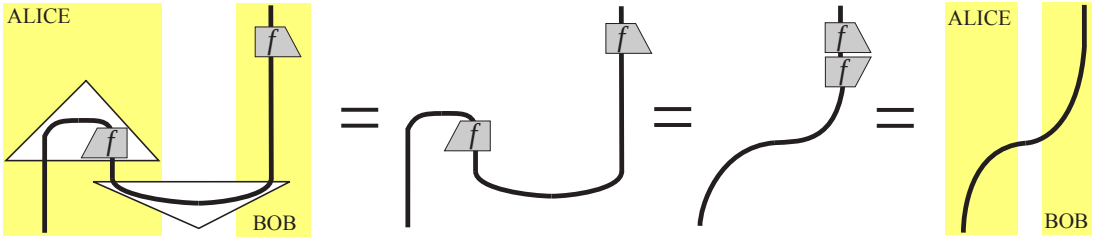

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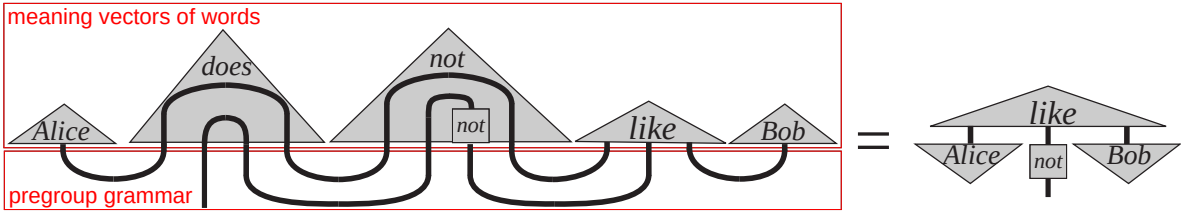
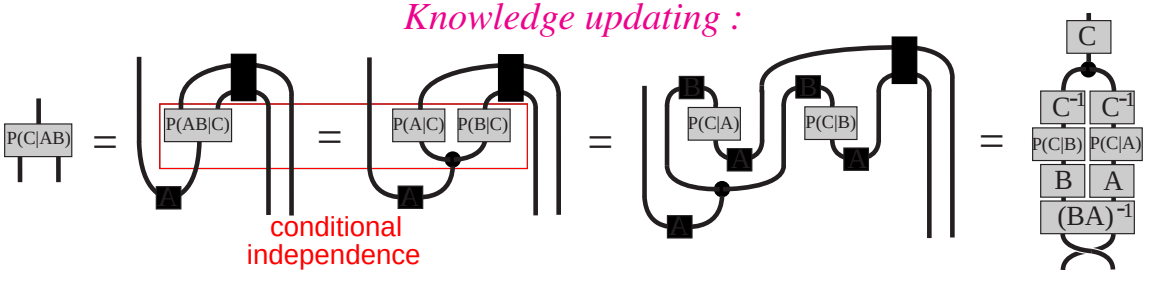


Knowledge updating :



Outcome 2b: The structure is a **true (quantum) logic**:

Quantomatic

http://dream.inf.ed.ac.uk/projects/quantomatic/

quantomatic

Salcombe – ...ncyclopedia Azimuth DPD (UK) – T...king Detail Nightmare Be...w's Parties ATP curated ...w's Parties FIFA.com – T...outh Africa The Waterma...e, Cornwall

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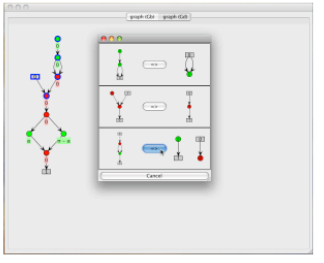
Quantomatic

Overview

Recent graph-based formalisms of quantum computation provide an abstract and symbolic way to represent and simulate computations. However, manual manipulation of such graphs is slow and error prone. This project employs a formalism, based on compact closed categories, that supports mechanised reasoning about such graphs. This gives a compositional account of graph rewriting that preserves the underlying categorical semantics.



Using this representation, we are developing a generic system with a fixed logical kernel that supports reasoning about models of compact closed category. A salient feature of the system is that it provides a formal and declarative account of derived results that can include ellipses-style notation. The main application is to develop a graph-based language for reasoning about quantum computation: Quantomatic.

The System

We have an implementation built using libraries from IsaPlanner. This builds on PolyML, Java, and Graphviz/dot. To build and run Quantomatic, you need to be able to run the commands: `ant`, `java`, `javac`, `poly`, and `dot`.

For Mac, you can install Graphviz/dot from the Graphviz download site. Once installed, test it by making sure that the `dot` command can be executed in a shell (that it is in the path). On ubuntu/debian linux you quickly and easily install graphiz by typing:

```
sudo apt-get install graphviz
```

Assuming you have java, a recent java compiler, and graphviz installed, you can install everything else with the following shell commands:

```
### Make the Quanto directory, for all things quanto, and go there
mkdir Quanto; cd Quanto

### download and build PolyML
svn co https://polyml.svn.sourceforge.net/svnroot/polyml/fixes-5.3 polyml
cd polyml/polyml; ./configure --prefix=$(cd ..; pwd); make; make install; cd ../..

### download and build Quantomatic
svn co https://isaplanner.svn.sourceforge.net/svnroot/isaplanner/trunk/isaplib isaplib
svn co https://isaplanner.svn.sourceforge.net/svnroot/isaplanner/trunk/quantomatic quantomatic
cd quantomatic/gui; ant; cd ..
```

– I can give you a demo backstage –

A MINIMAL LANGUAGE FOR QUANTUM REASONING

Abramsky & Coecke (2004) *A categorical semantics for quantum protocols*.
arXiv:quant-ph/0402130

Coecke (2005) *Kindergarten quantum mechanics*.
arXiv:quant-ph/0510032

— *(physical) data in the language* —

Systems:

$$A \qquad B \qquad C$$

Processes:

$$A \xrightarrow{f} A \qquad A \xrightarrow{g} B \qquad B \xrightarrow{h} C$$

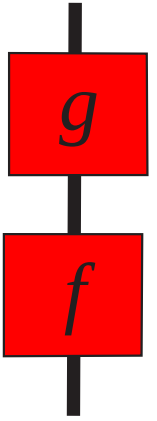
Compound systems:

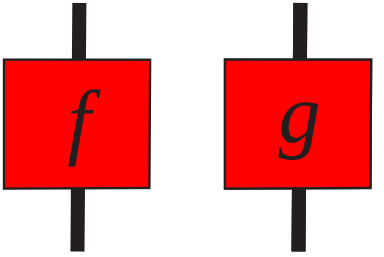
$$A \otimes B \qquad \mathrm{I} \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Temporal composition:

$$A \xrightarrow{h \circ g} C \;:=\; A \xrightarrow{g} B \xrightarrow{h} C \qquad A \xrightarrow{1_A} A$$

— *graphical notation* —

$g \circ f \equiv$ 

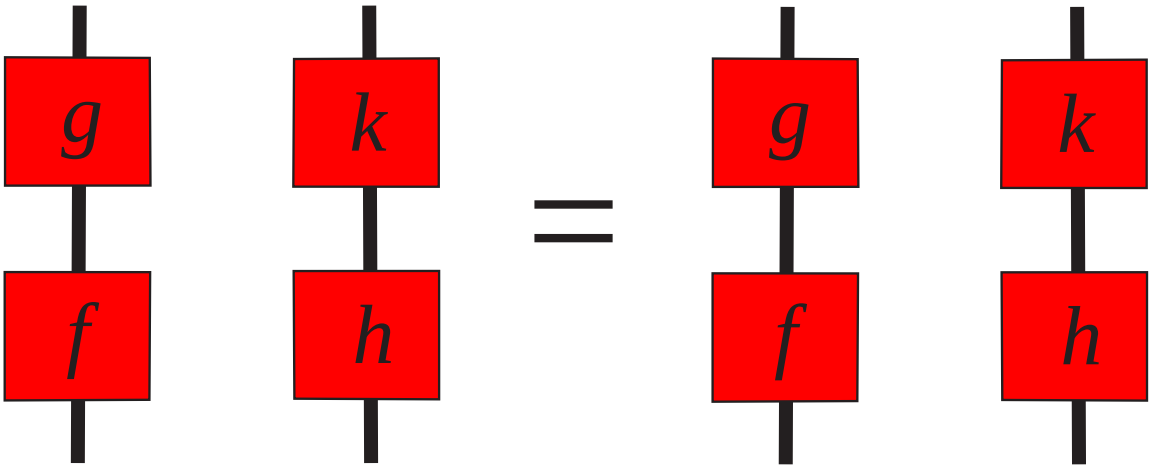
$f \otimes g \equiv$ 

Roger Penrose (1971) *Applications of negative dimensional tensors.*
In: Combinatorial Mathematics and its Applications. Academic Press.

André Joyal & Ross Street (1991) *The geometry of tensor calculus* I.
Advances in Mathematics **88**, 55–112.

— *merely a new notation?* —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



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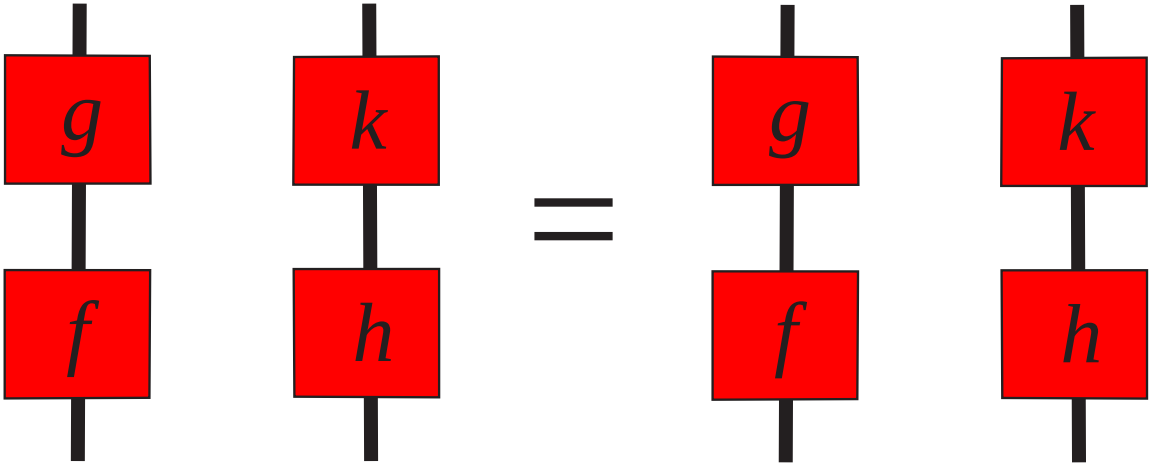

peel potato and then fry it,
while,
clean carrot and then boil it

=

peel potato while clean carrot,
and then,
fry potato while boil carrot

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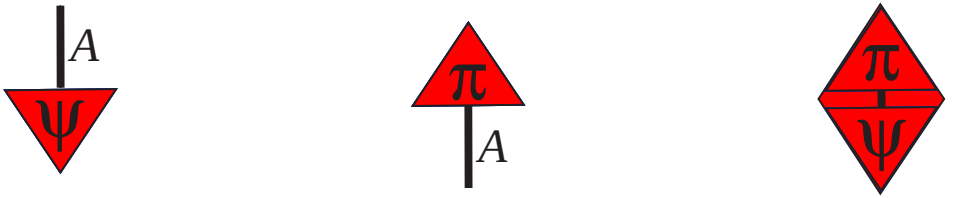

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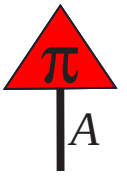

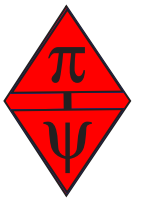

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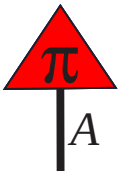

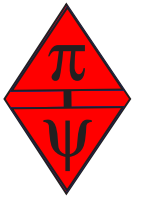

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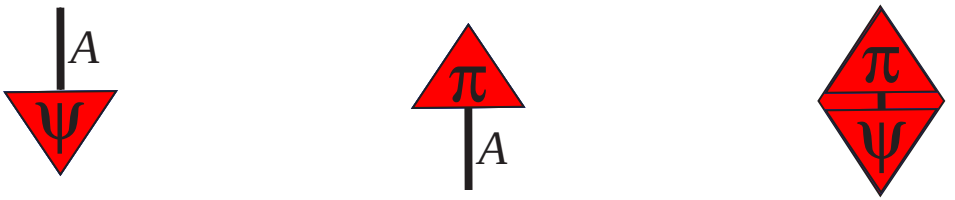

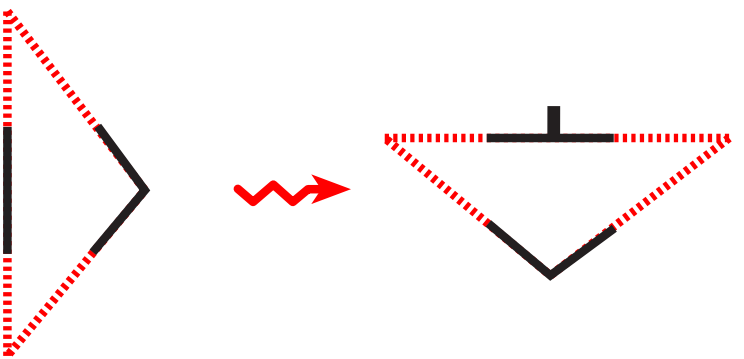

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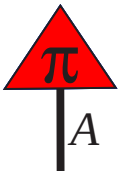

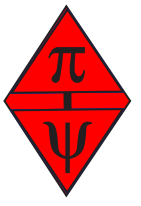

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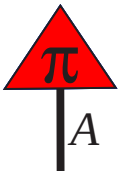

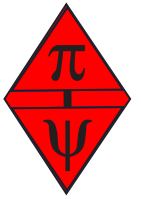

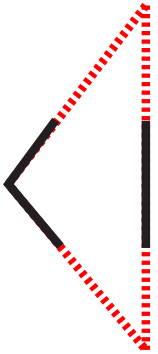

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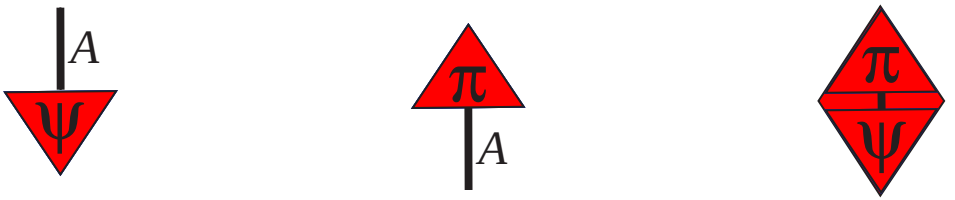

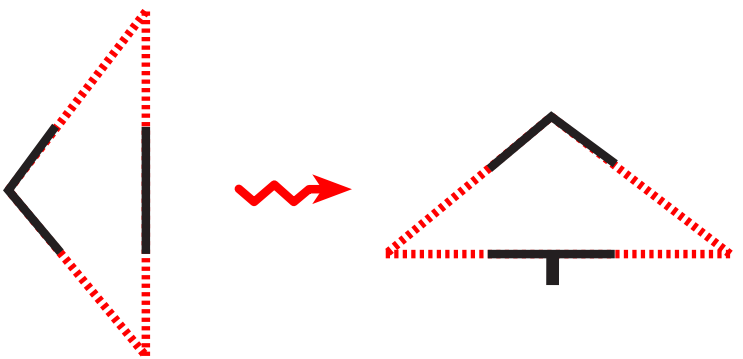

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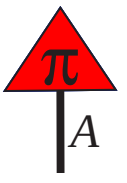

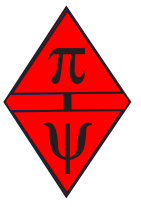

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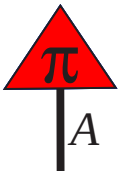

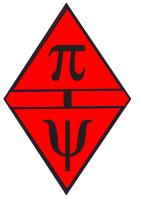

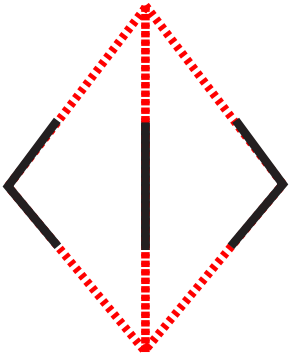

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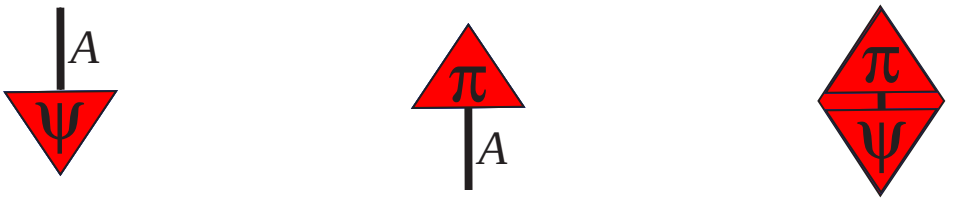
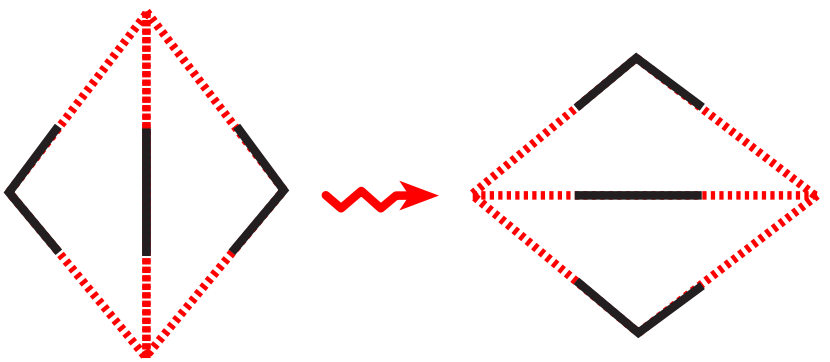
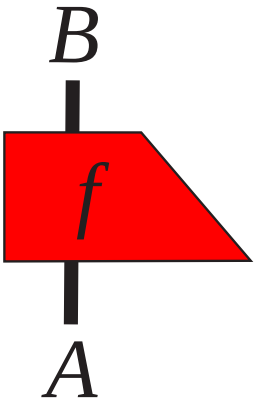




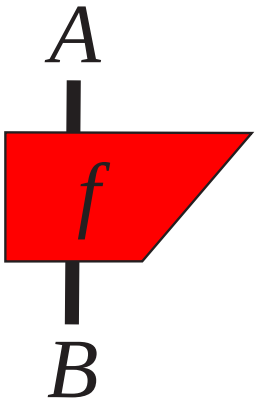
— *adjoint* —

$$f \ : A \to B$$



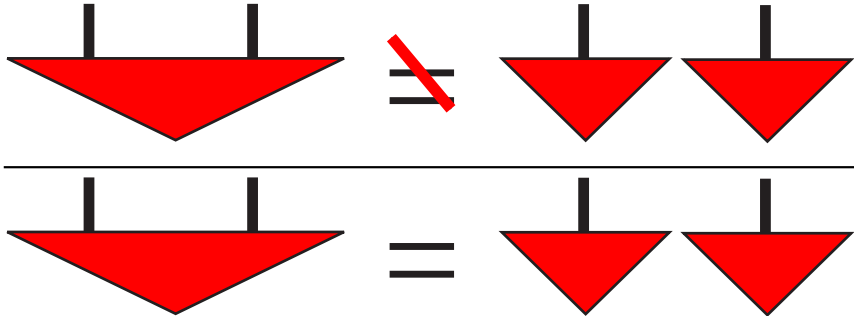
— *adjoint* —

$$f^\dagger : B \to A$$

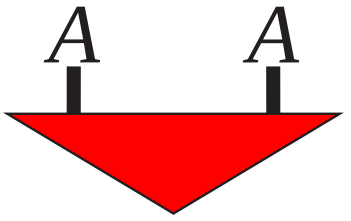


— *asserting (pure) entanglement* —

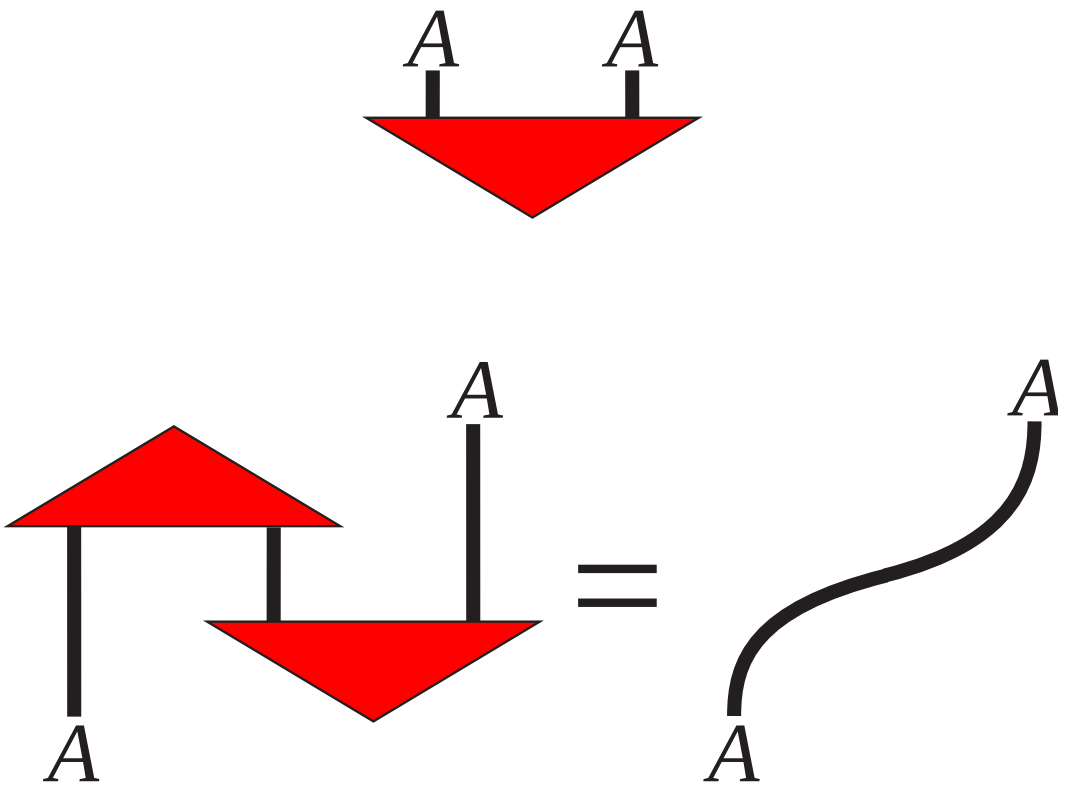
$$\frac{quantum}{classical} = \frac{\not=}{=}$$



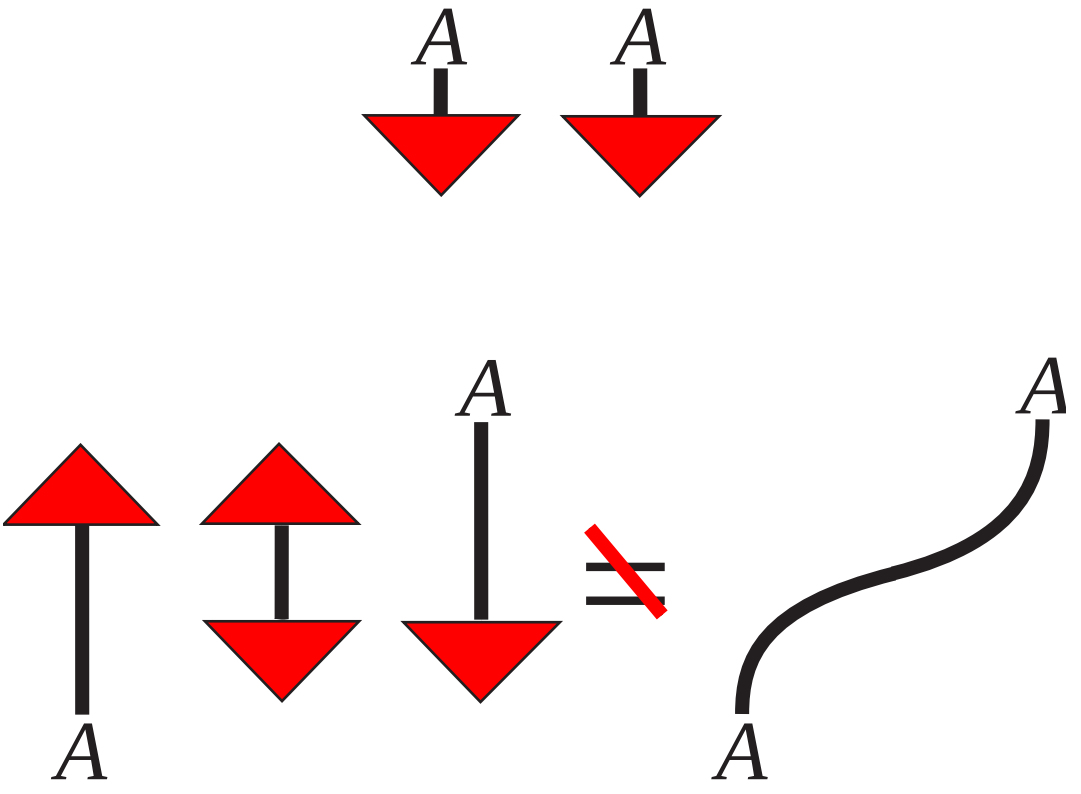
— *quantum-like* —



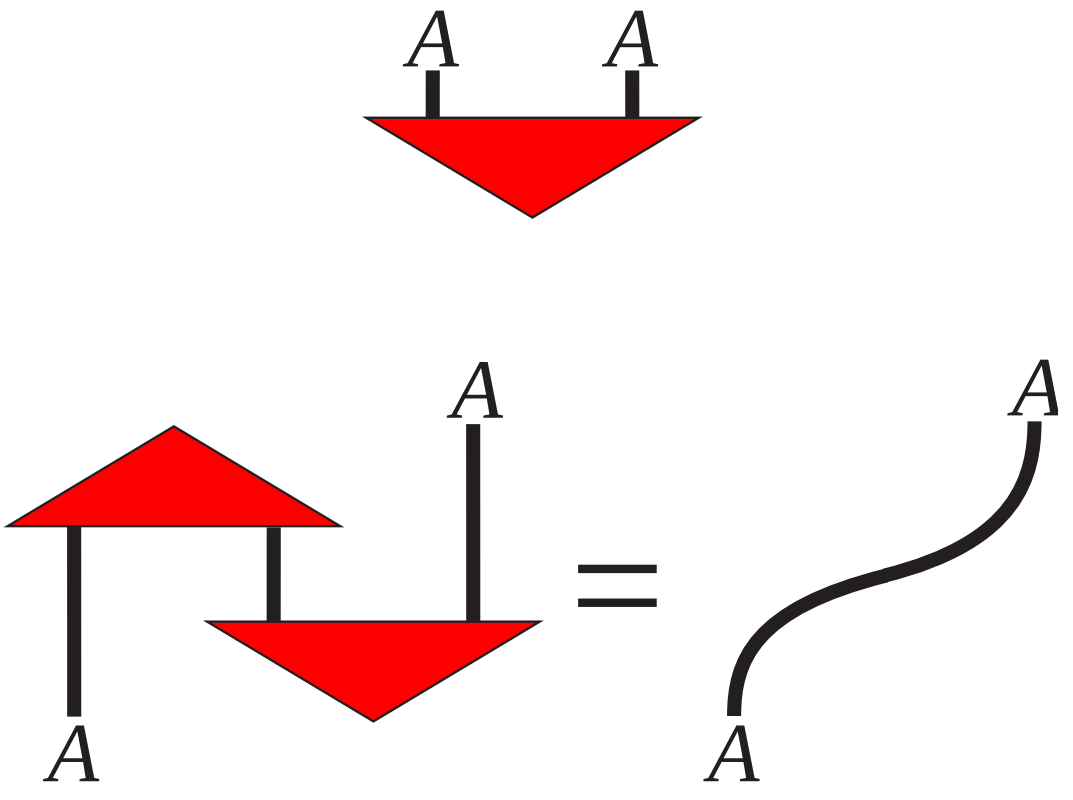
— *quantum-like* —



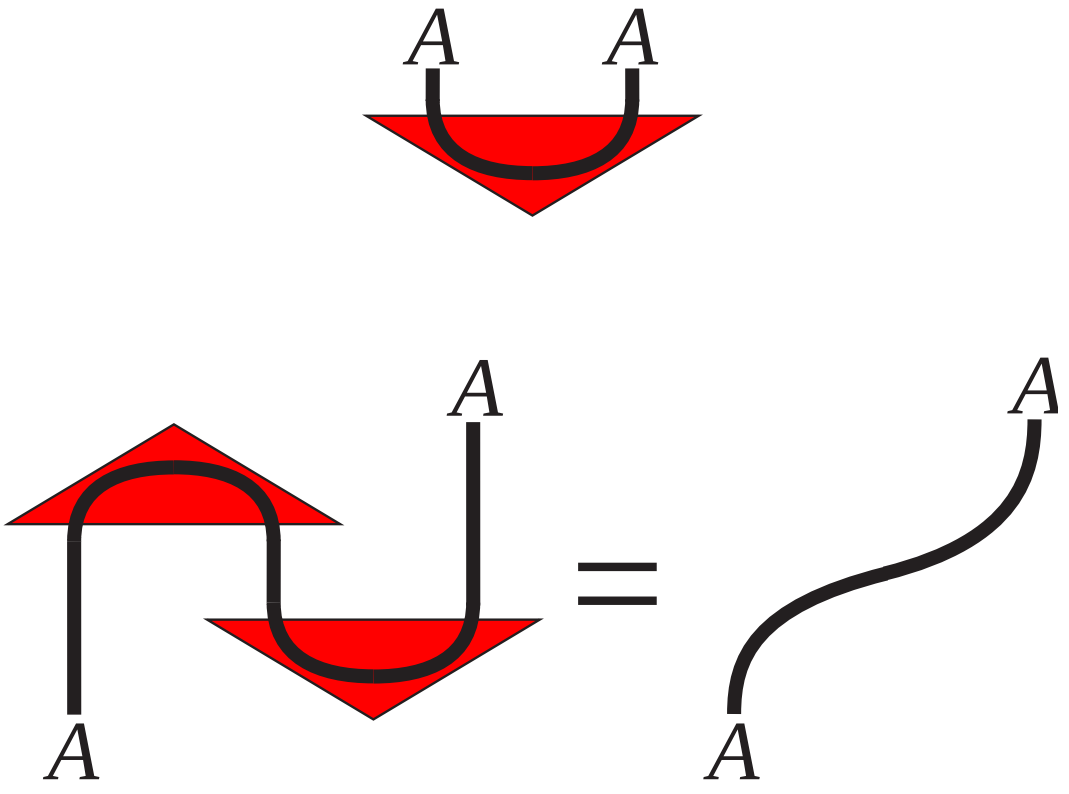
— *quantum-like* —



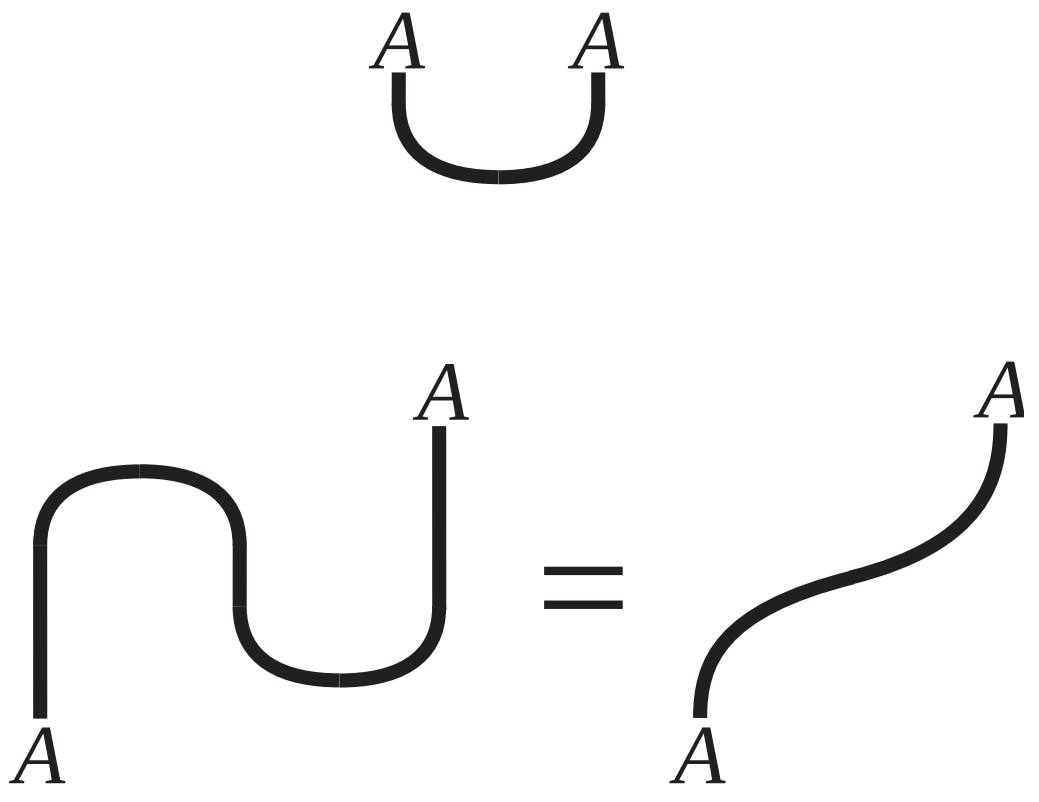
— *quantum-like* —



— *quantum-like* —

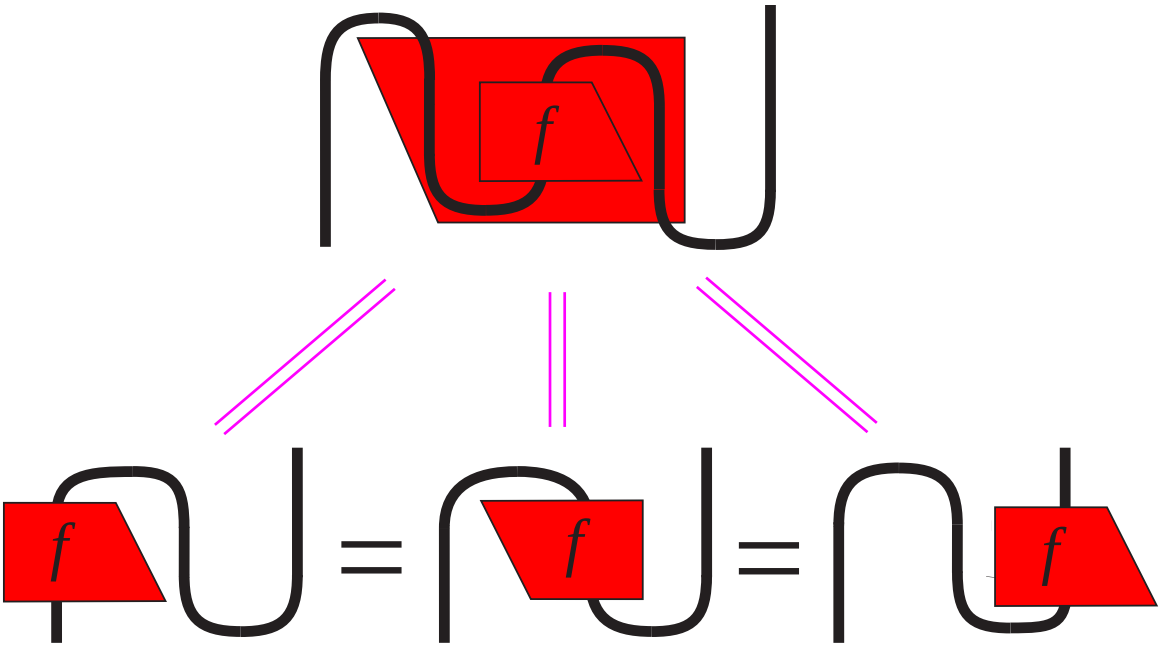


— *quantum-like* —

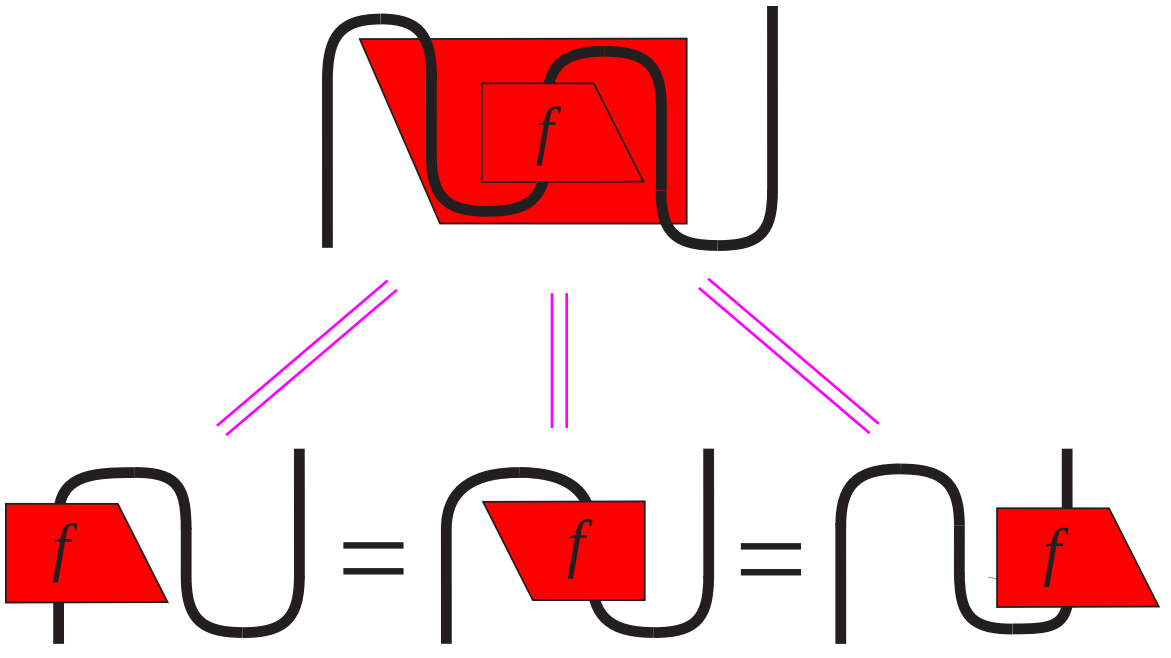


— *quantum-like* —

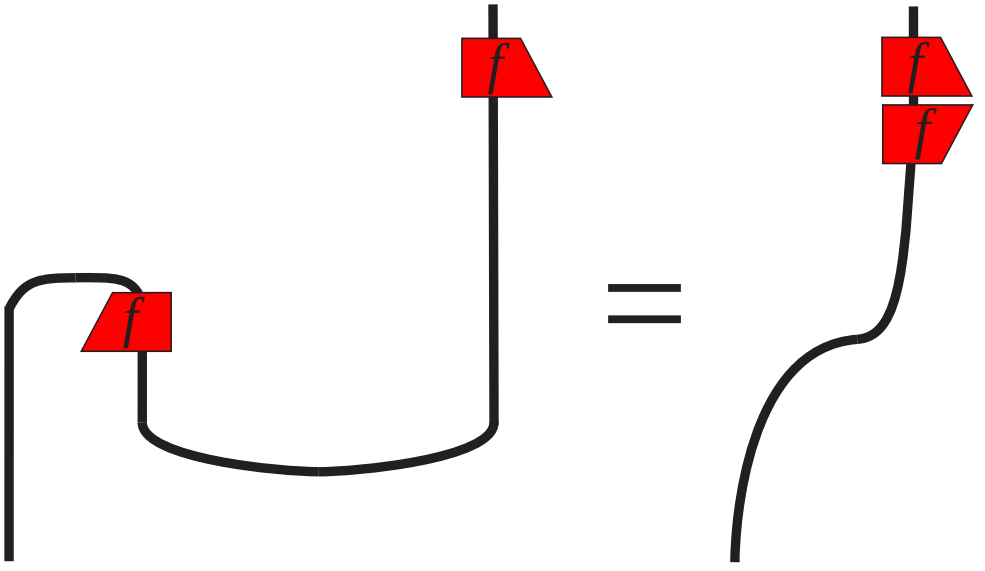
— *sliding* —



— *sliding* —



In QM: cups = Bell-states, caps =Bell-effects, π-rotations = transpose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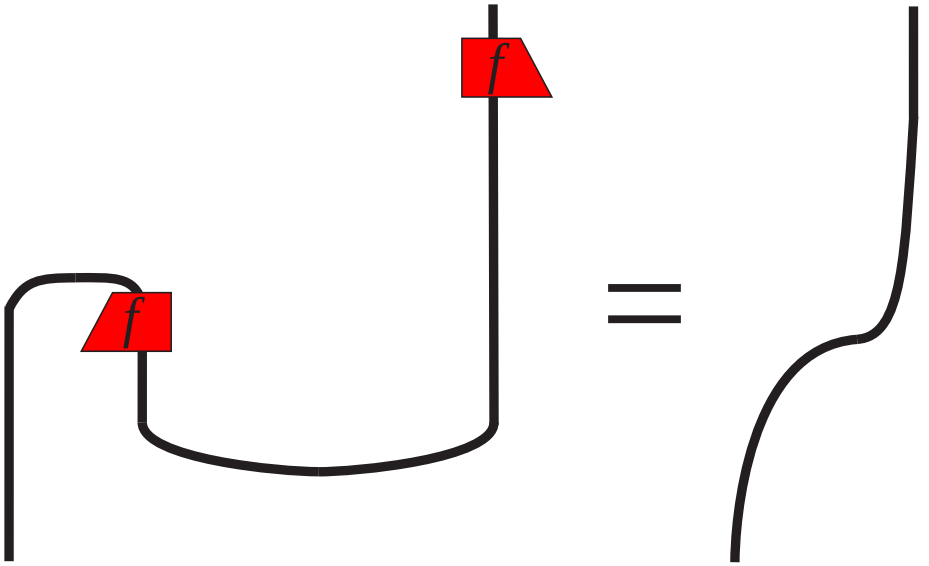
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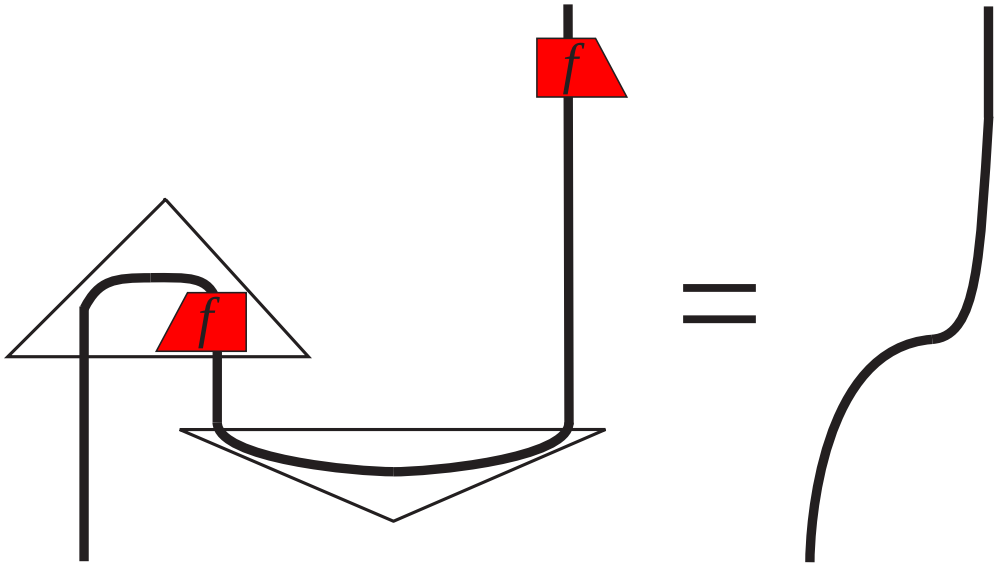
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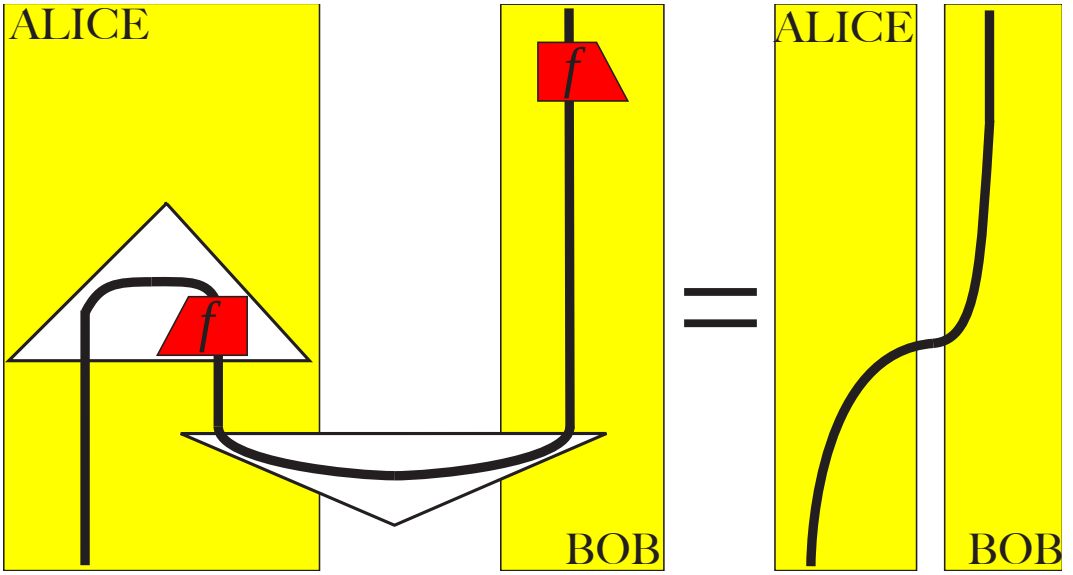
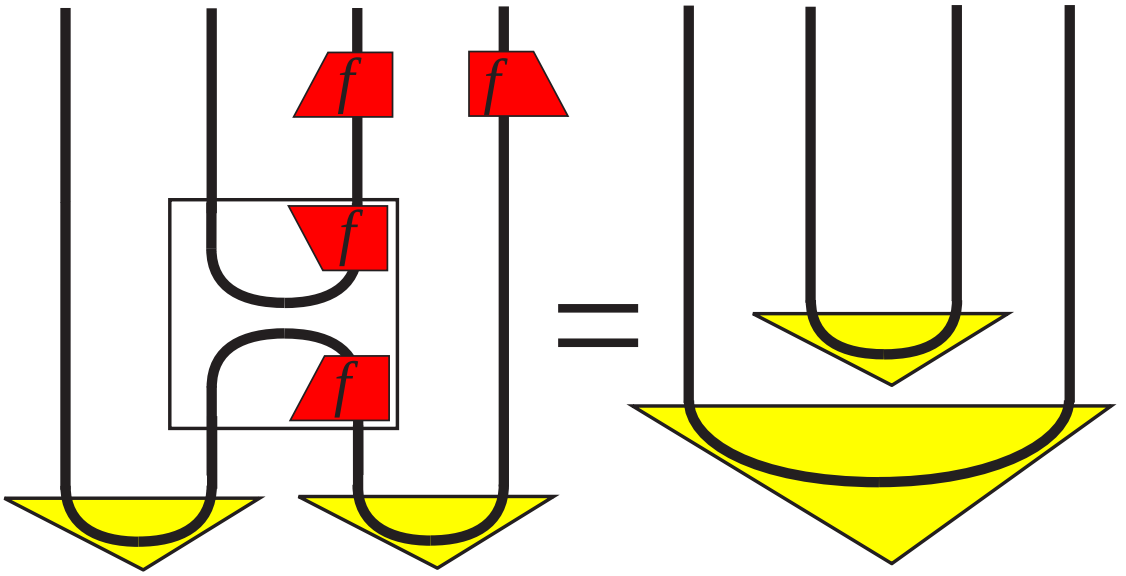
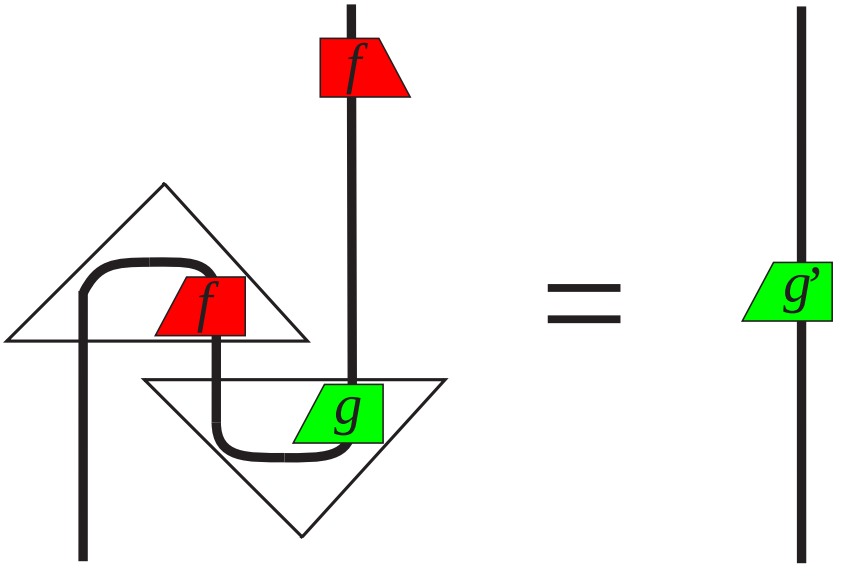


$\Rightarrow$ **quantum teleportation**



$\Rightarrow$ **Entanglement swapping**



$\Rightarrow$ **gate teleportation computation**

— *dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

— *dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

Thm. [Selinger '08] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the category of finite dimensional Hilbert spaces, linear maps, tensor product, and adjoints*.

— *dagger compact categories* —

In words: *Any equation involving:*

- *states, operations, effects*
- *unitarity, adjoints (e.g. self-adjoint), projections*
- *Bell-states/effects, transpose, conjugation*
- *inner-product, trace, Hilbert-Schmidt norm*
- *positivity, completely positive maps, ...*

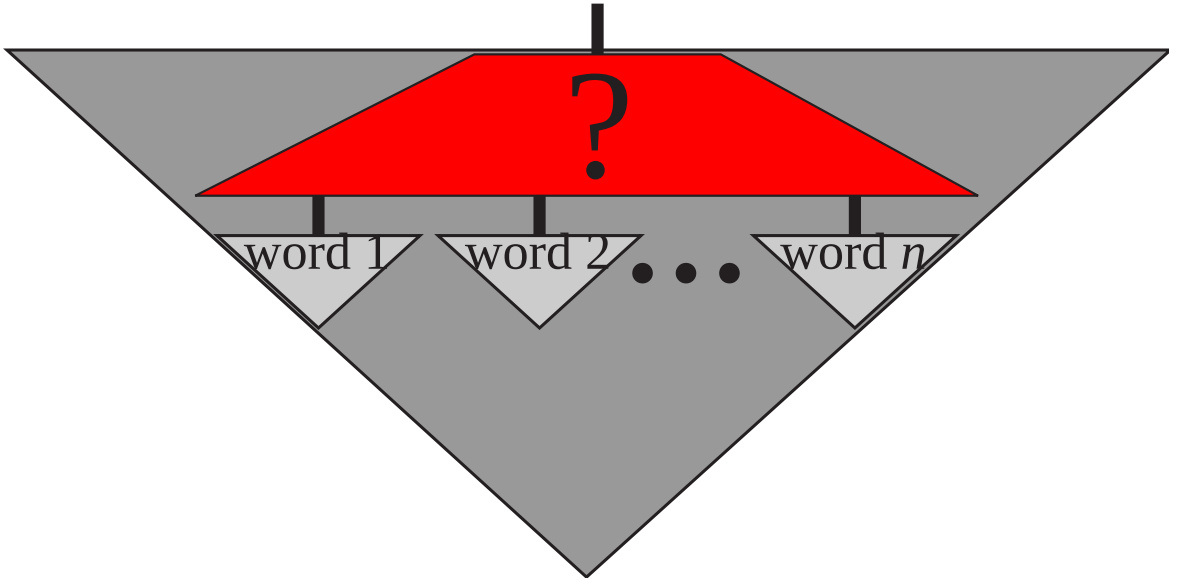
holds in quantum theory if and only if it can be derived in the graphical language via homotopy.

A SLIGHTLY DIFFERENT LANGUAGE FOR NATURAL LANGUAGE MEANING

Coecke, Sadrzadeh & Clark (2010) *Mathematical Foundations for a Compositional Distributional Model of Meaning*.
arXiv:1003.4394

— *the from-words-to-a-sentence process* —

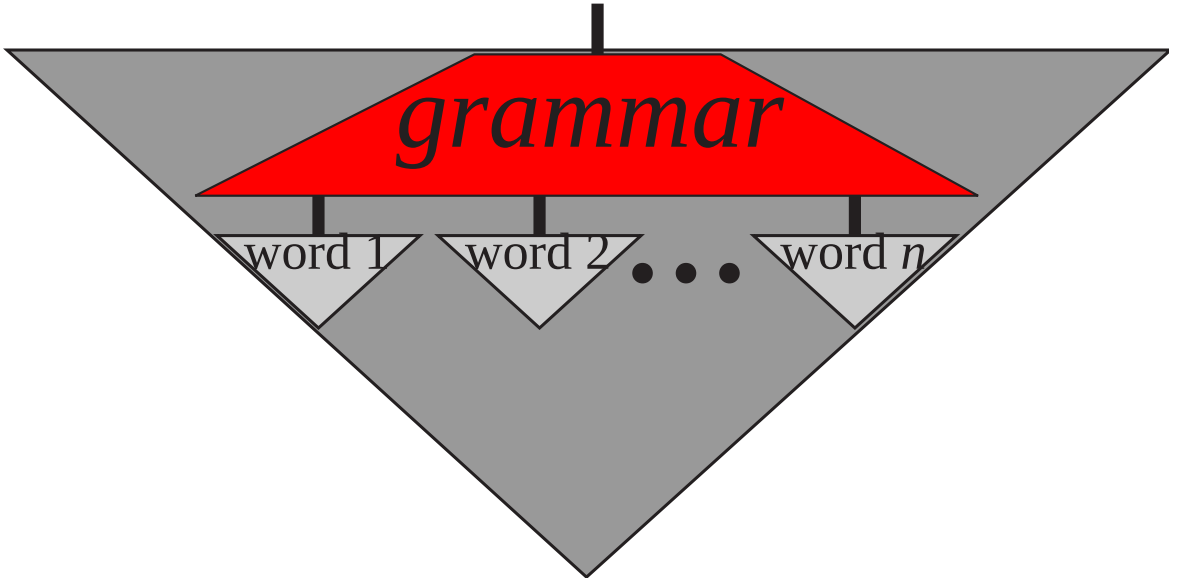
Consider meanings of words (e.g. vectors as in Google):



How do we/machines compute meaning of sentences?

— *the from-words-to-a-sentence process* —

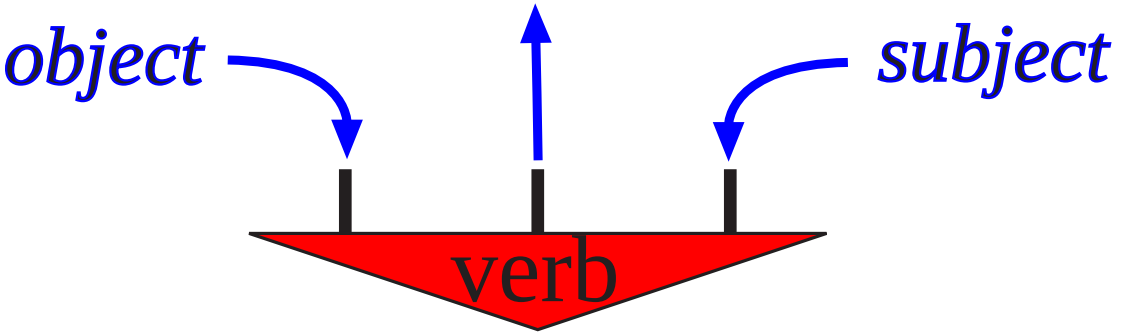
Consider meanings of words (e.g. vectors as in Google):



How do we/machines compute meaning of sentences?

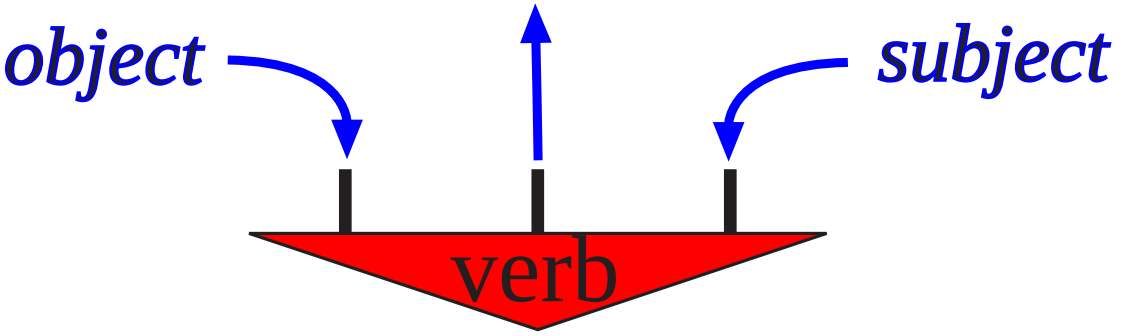
— *the from-words-to-a-sentence process* —

Information flow within a verb:

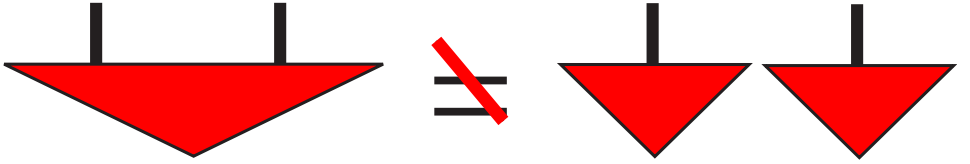


— *the from-words-to-a-sentence process* —

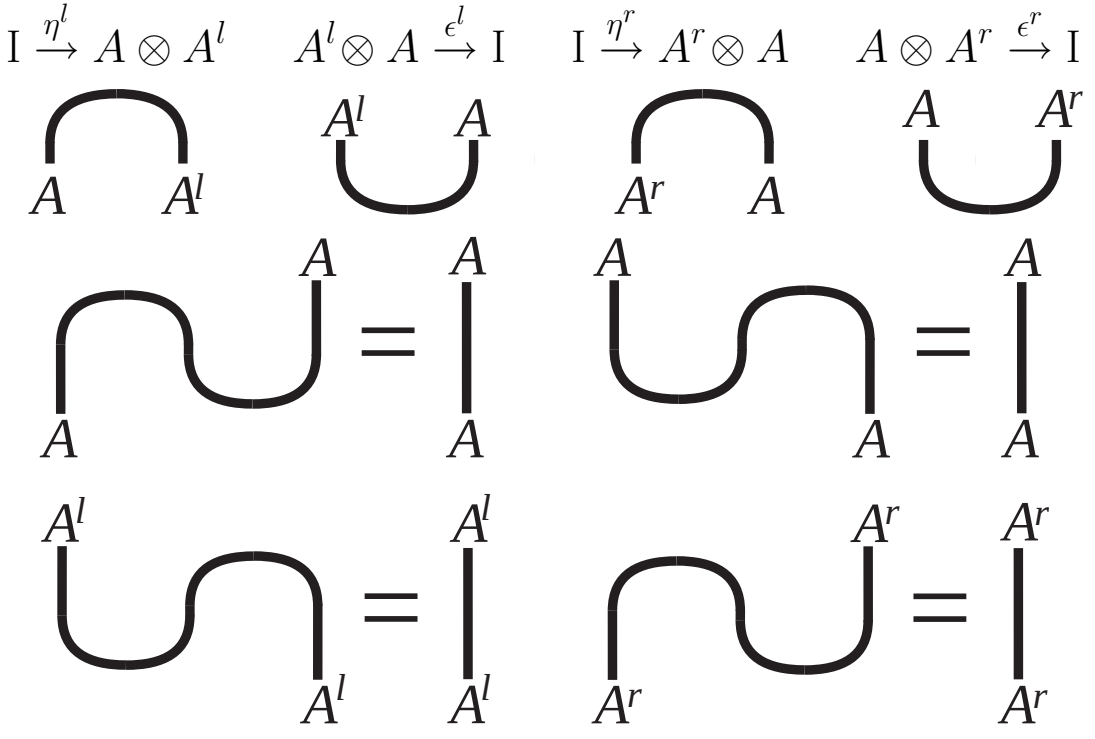
Information flow within a verb:



Again we have:

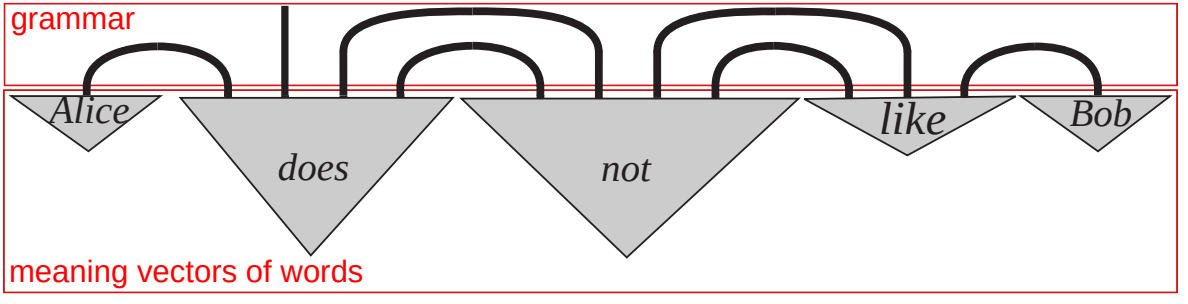


— *going non-symmetric* —

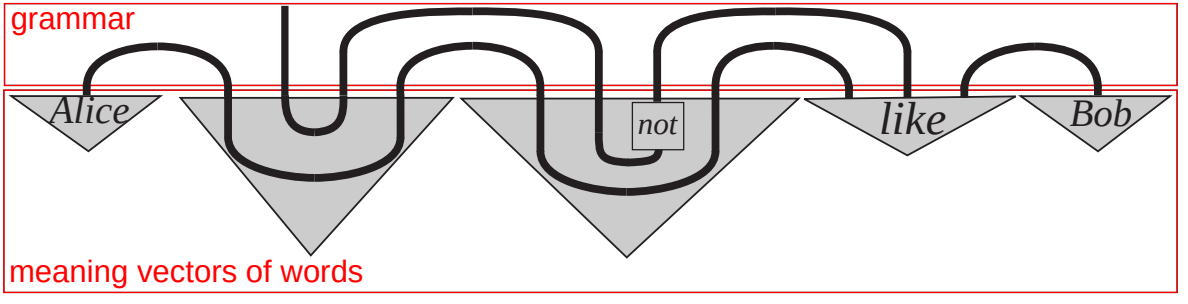


$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$

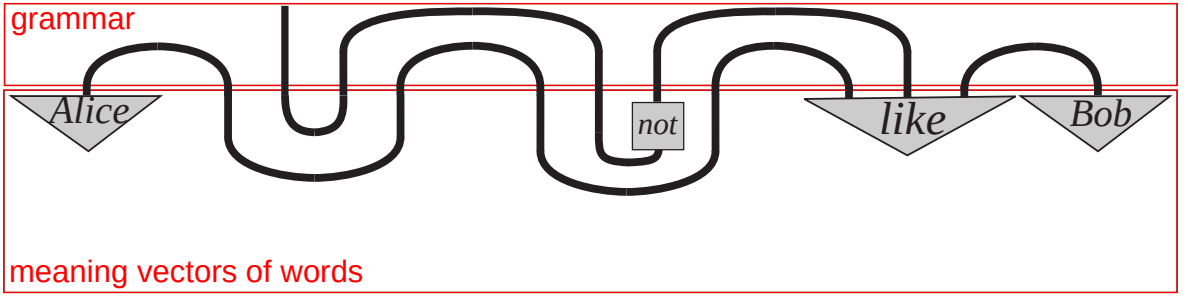
$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$



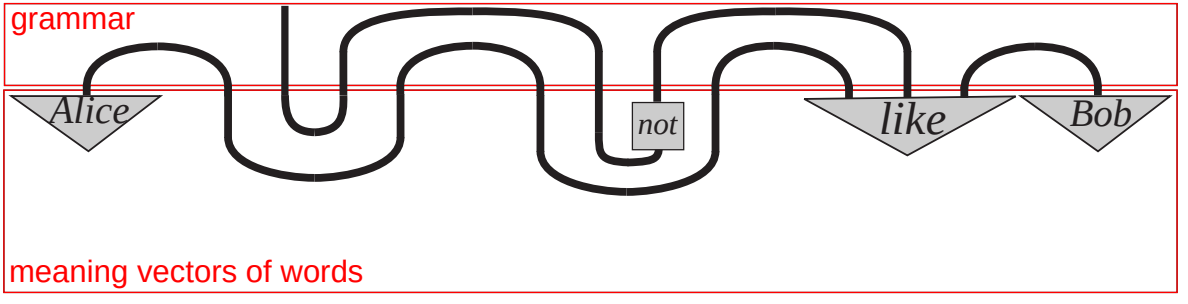
$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$

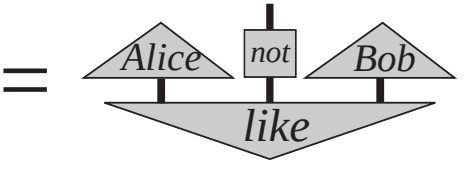


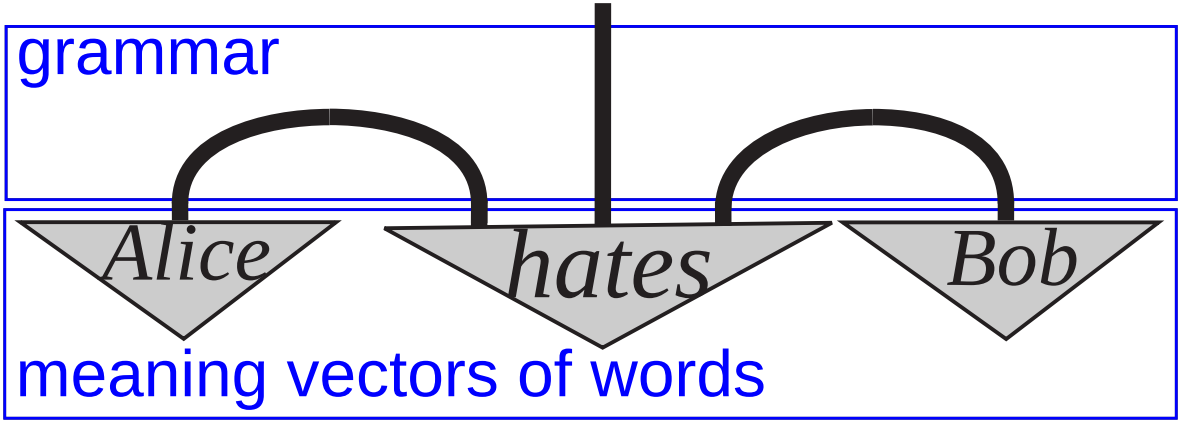
$$— \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} —$$

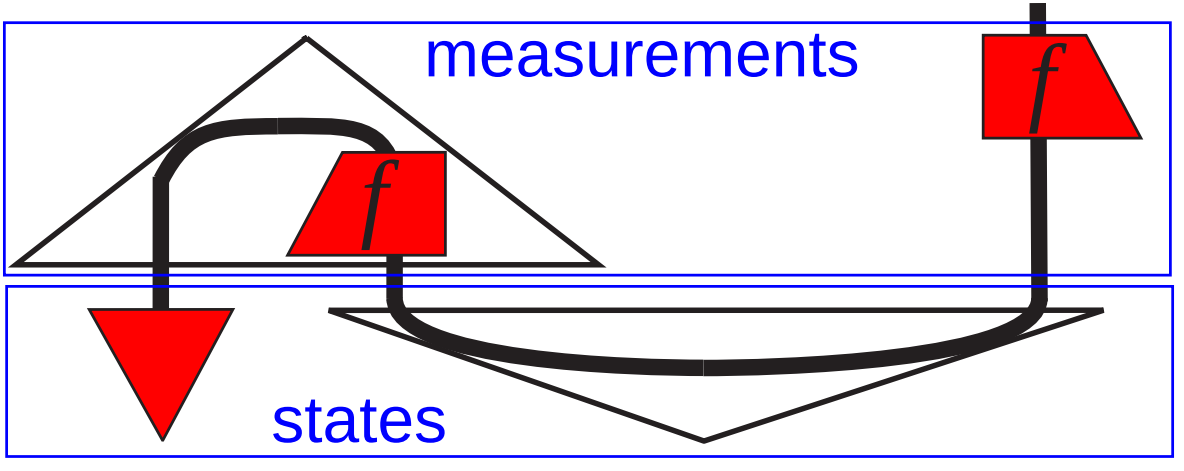


$$- \overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob} -$$



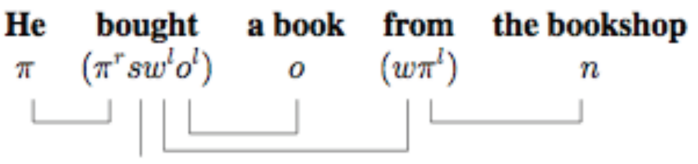



grammar
Alice
hates
Bob
meaning vectors of words

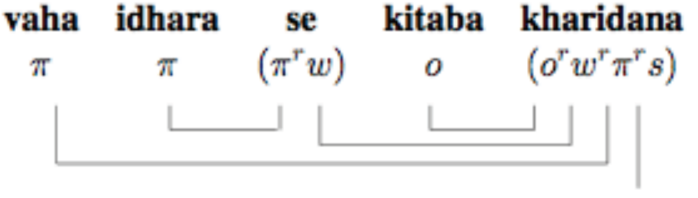

measurements
f
f
states

— *analogy: "non-local" info-flows* —

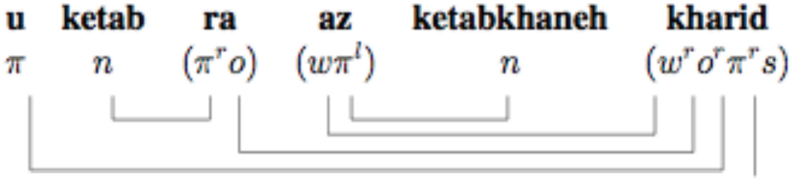
English (& French):



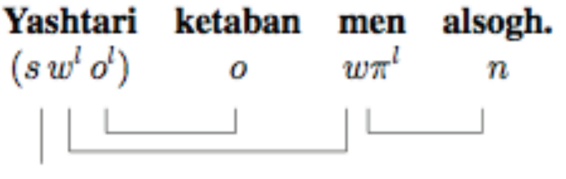
Hindi:



Persian:



Arabic (and Hebrew):

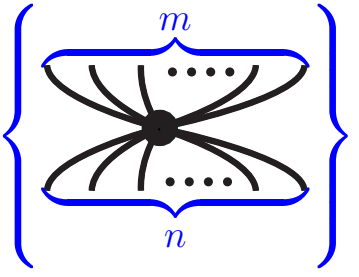


Mehrnoosh Sadrzadeh (2008) *Pregroup analysis of Persian sentences*.

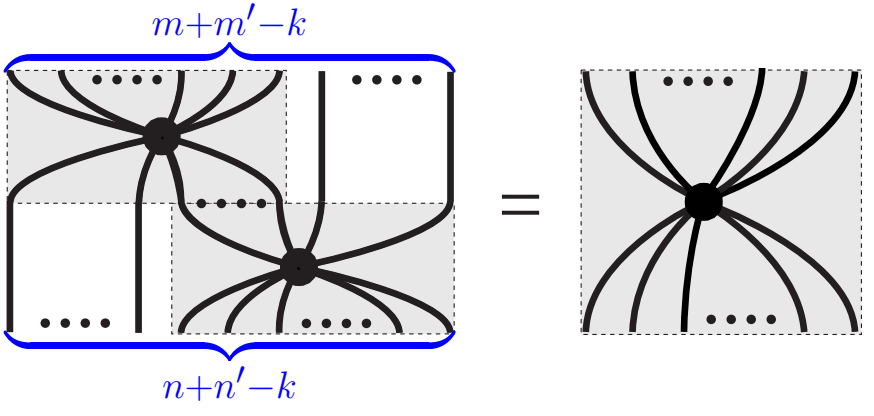
THE EXTENDED LANGUAGE: COMPLEMENTARITY & CLASSICALITY

— *observables* —

— *observables* —

'spiders' $=$ 

such that, for $k > 0$:

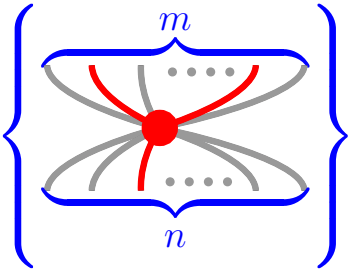


— *observables* —

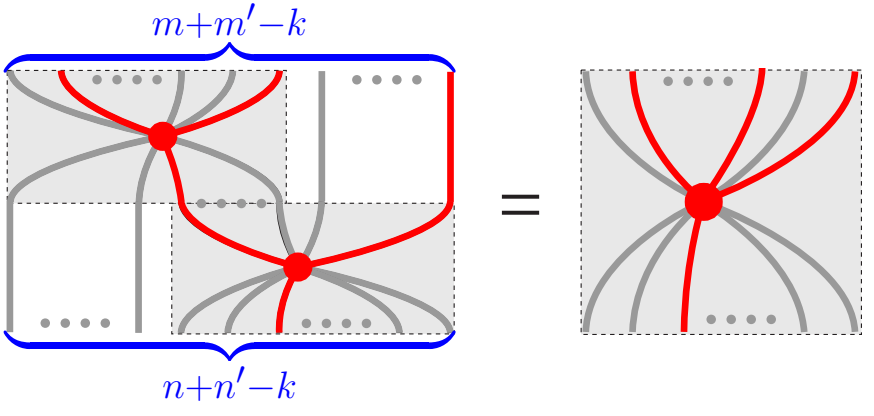
Theorem 1. In any dagger symmetric monoidal category families of spiders and dagger special commutative Frobenius algebra are in bijective correspondence.

— *observables* —

'spiders' $=$



such that, for $k > 0$:



— *observables* —

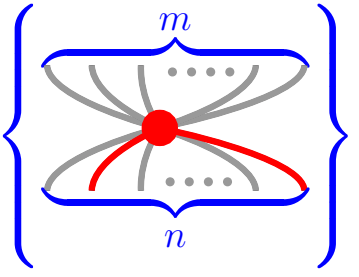
Theorem 1. In any dagger symmetric monoidal category families of spiders and dagger special commutative Frobenius algebra are in bijective correspondence.

Theorem 2. (Coecke-Pavlovic-Vicary) In **FdHilb** dagger special commutative Frobenius algebra are exactly orthonormal bases, namely those of copyable elts.

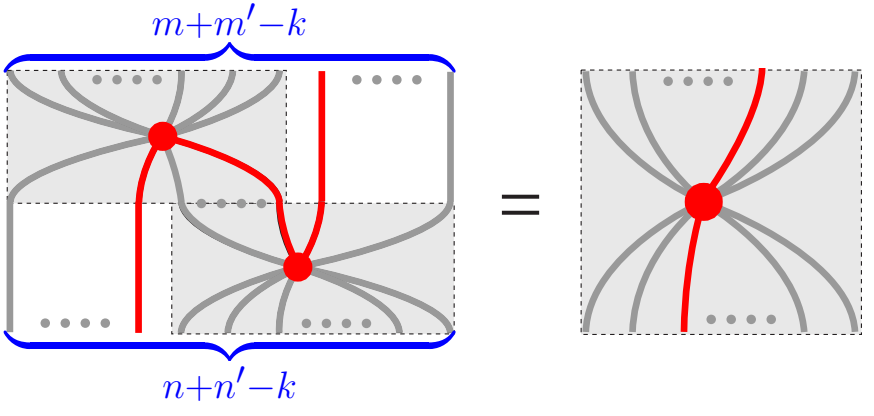
Coecke & Pavlovic (2007) *Quantum measurement without sums*. In: Mathematics of Quantum Computing and Technology. `quant-ph/0608035`

Coecke, Pavlovic & Vicary (2008) *A new description of orthogonal bases*. Mathematical Structures in Computer Science. `0810.0812`

— *observables* —

'spiders' $=$ 

such that, for $k > 0$:



— *complementarity* —

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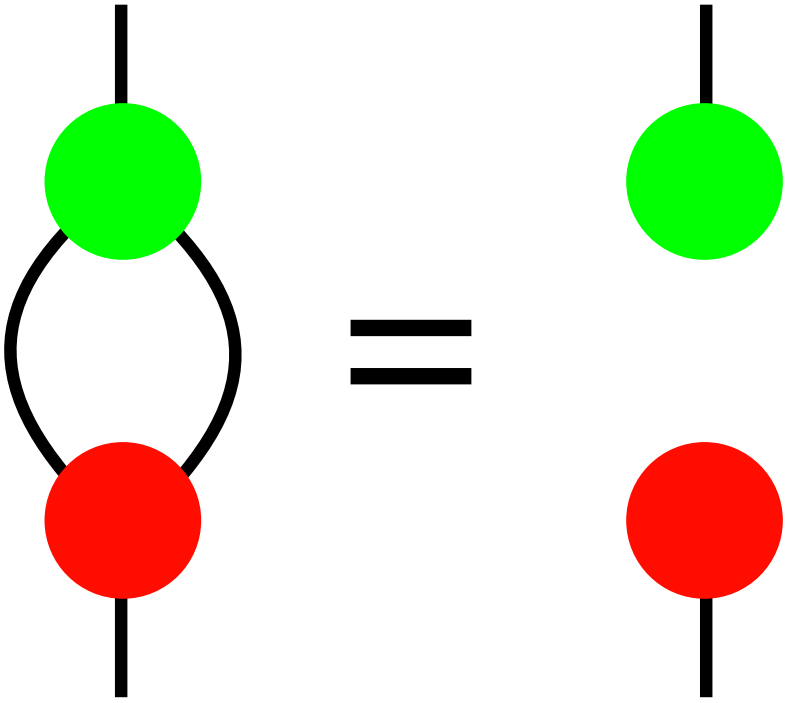
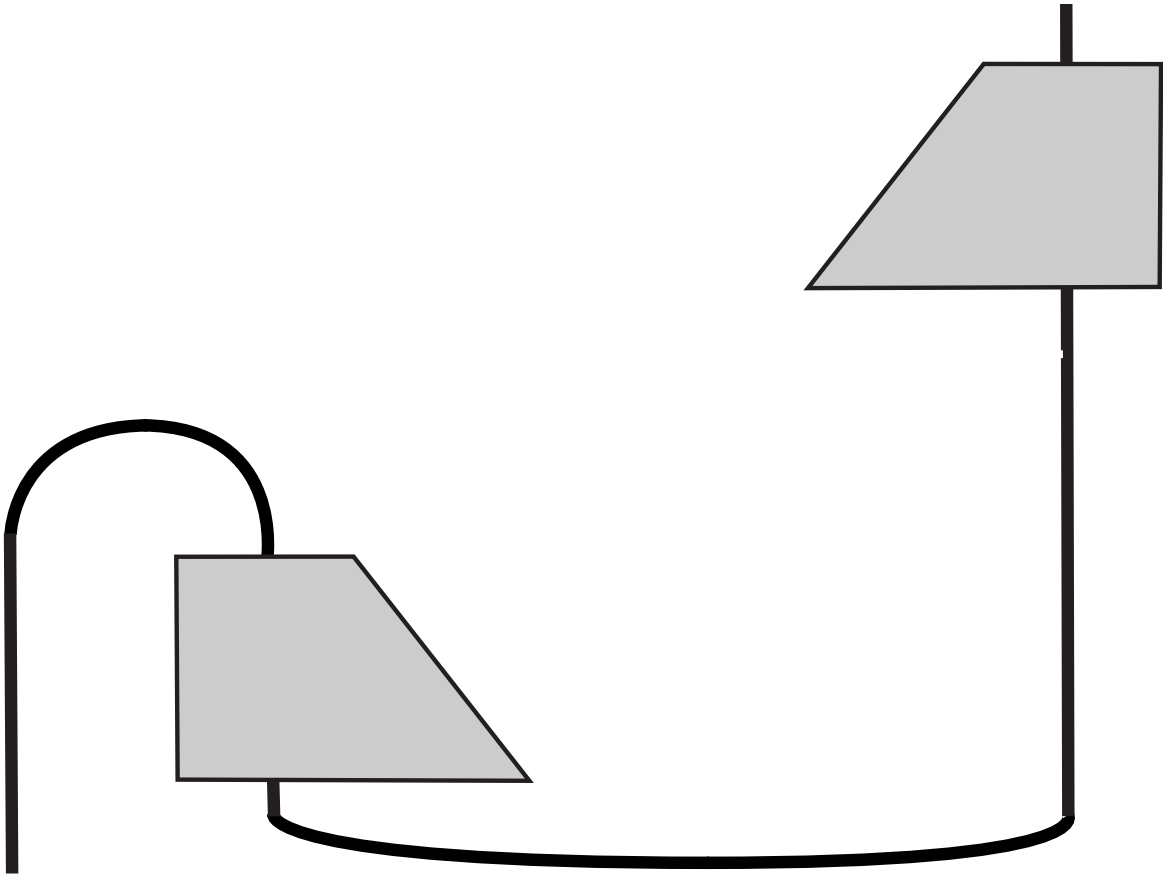
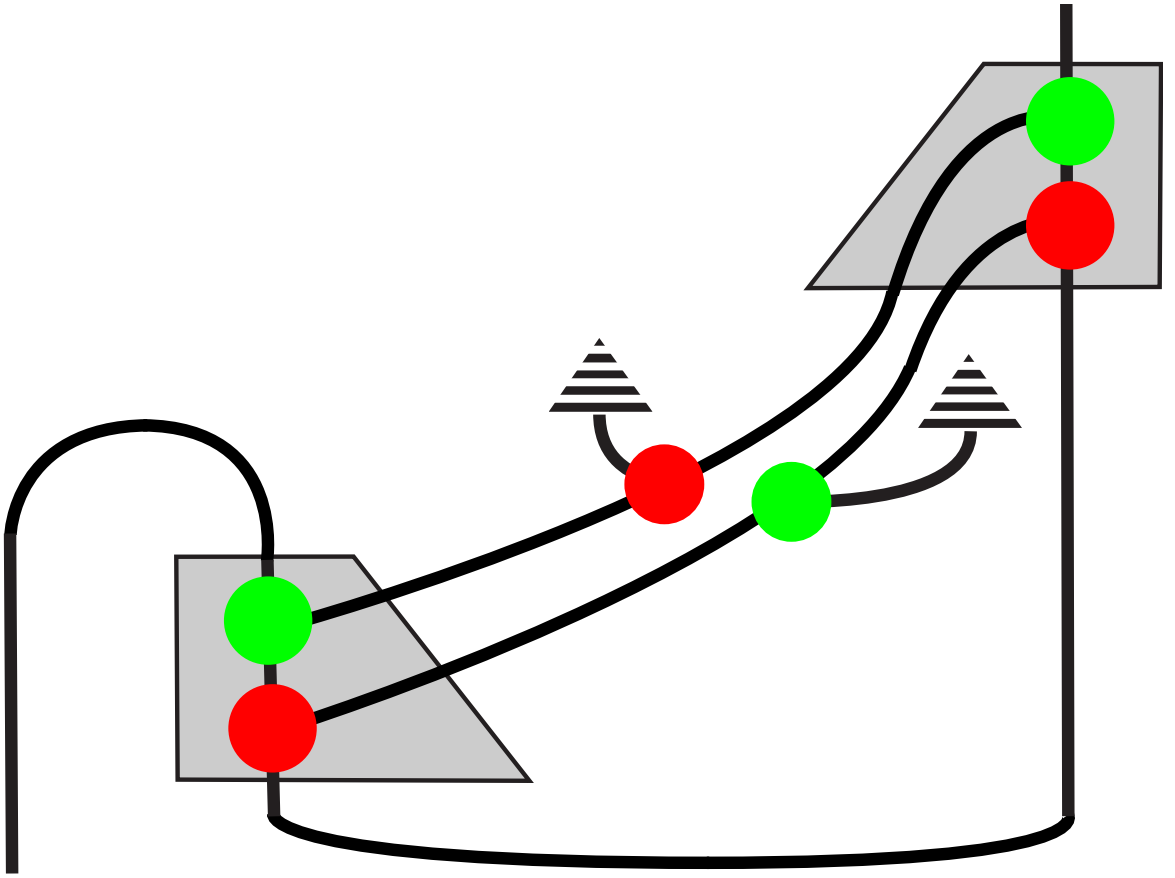
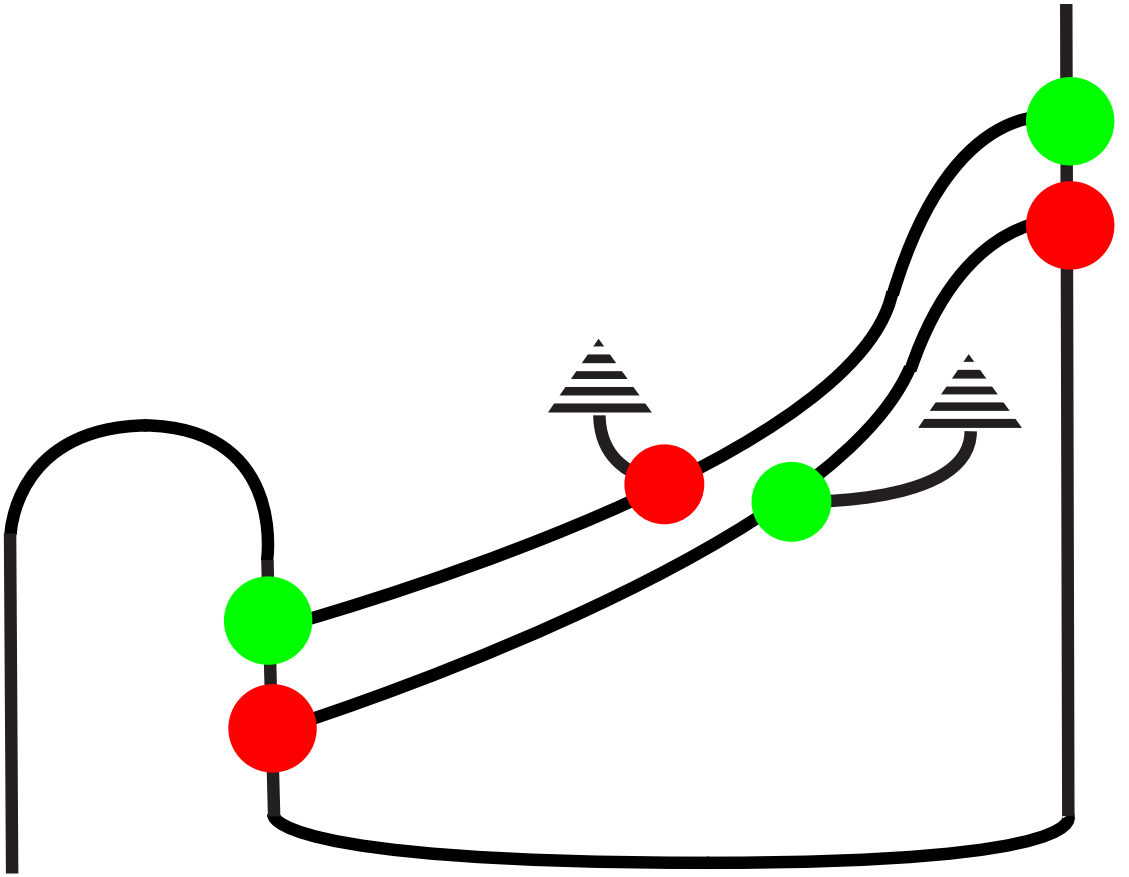
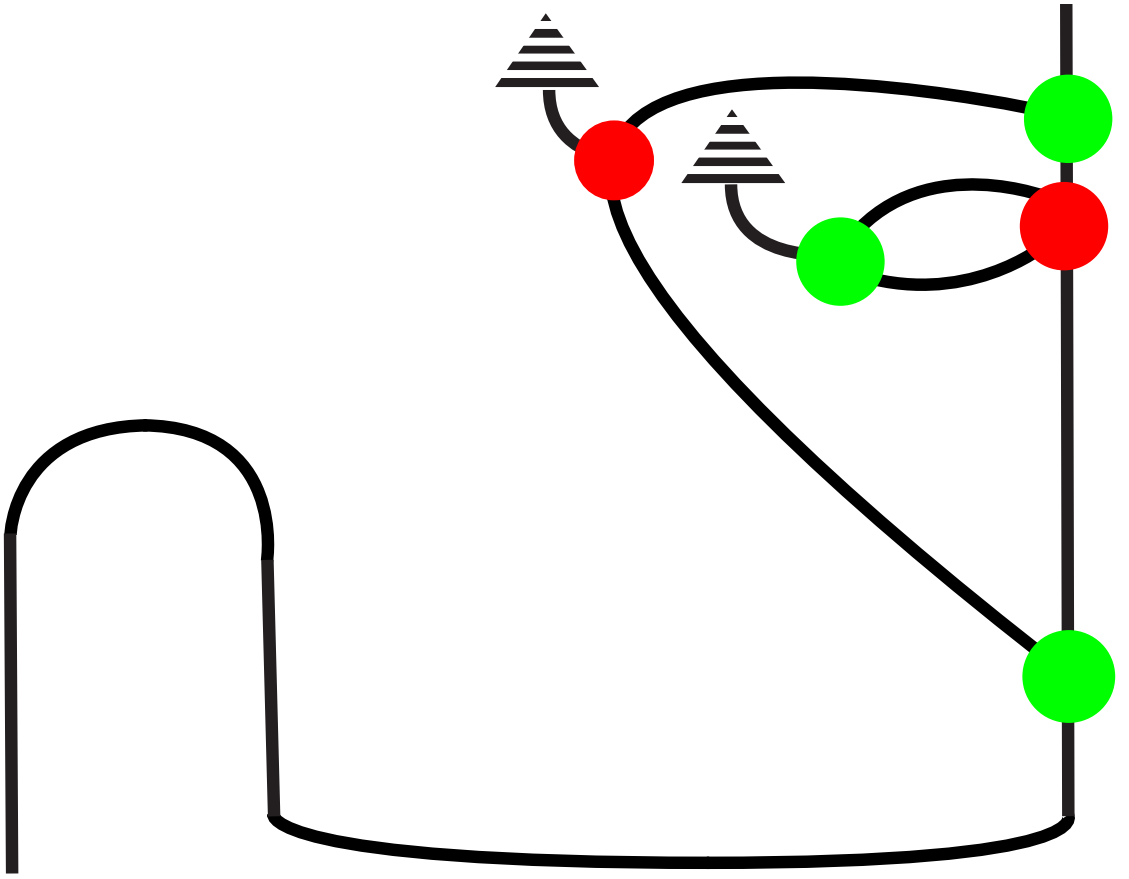


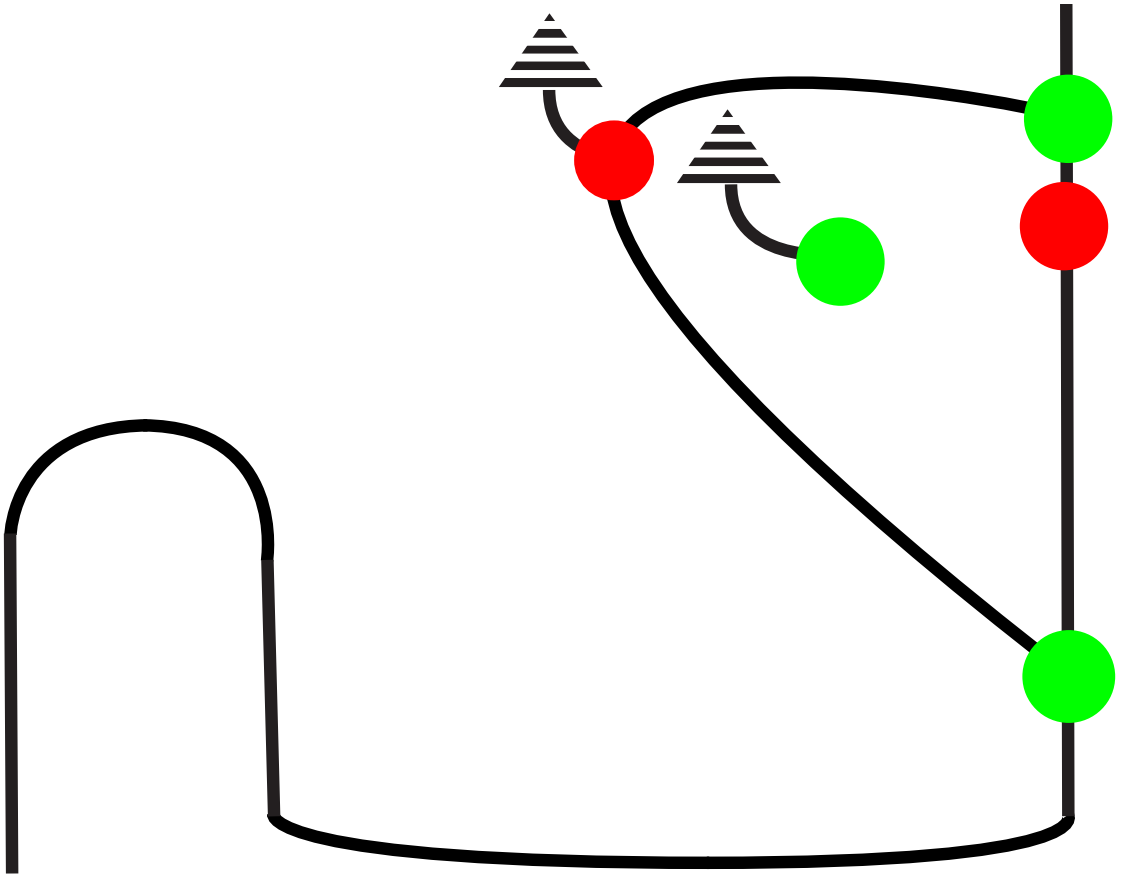
Coecke & Duncan (2008) *Interacting quantum observables*. arXiv:0906.4725

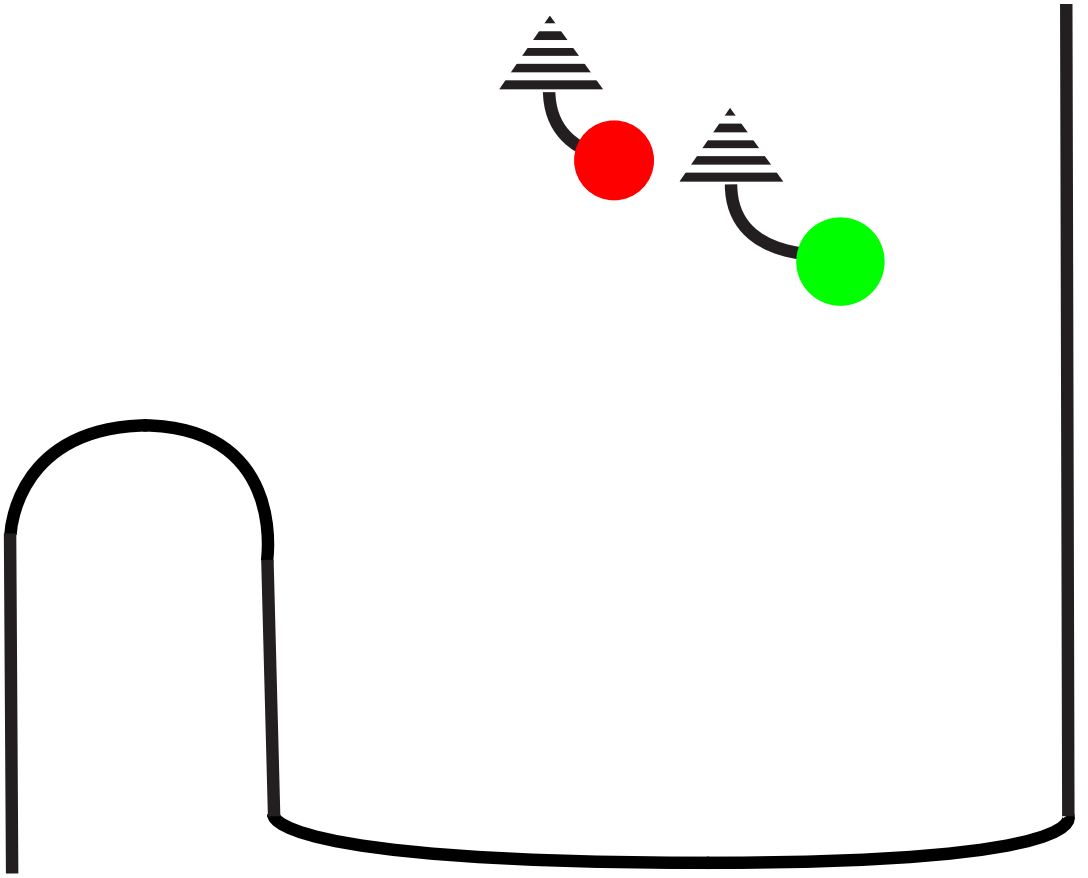


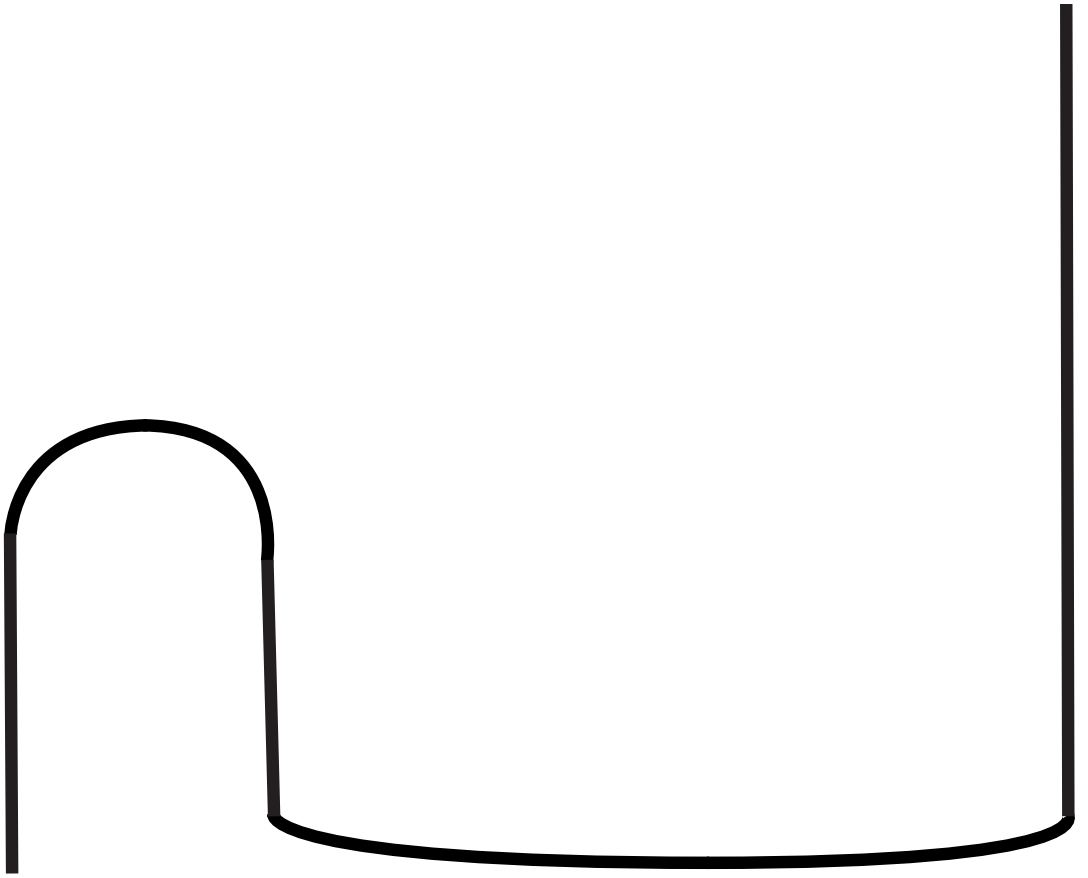






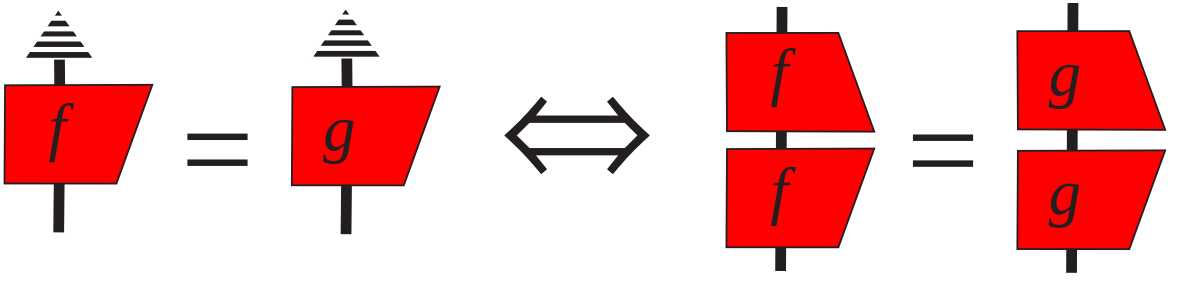






— *environment* —

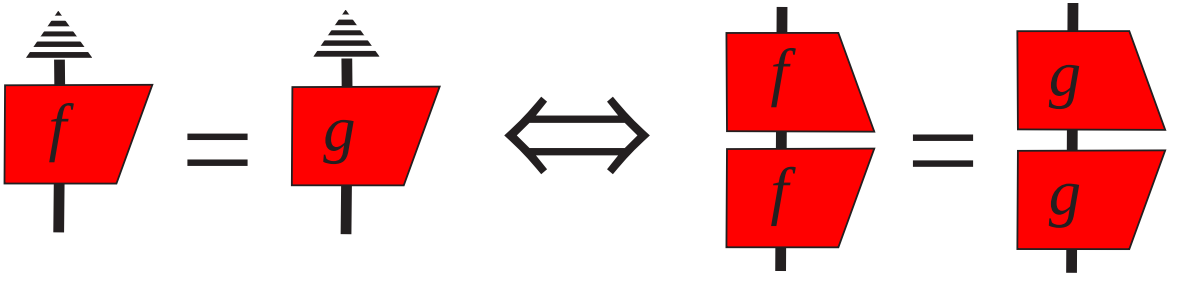
‘*ground*’ = 



Coecke & Perdrix (2010) *Environment and class. chan. ...* arXiv:1004.1598

— *environment* —

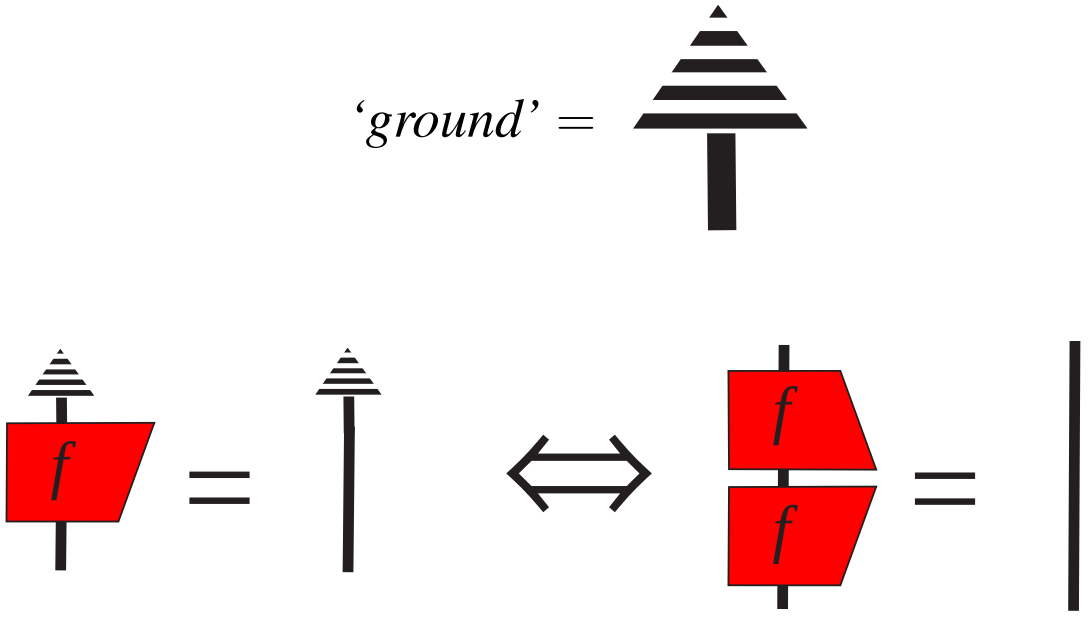
'ground' = 



Thm. $\Rightarrow$ mixed states, CP maps, class. probs in **Hilb**.

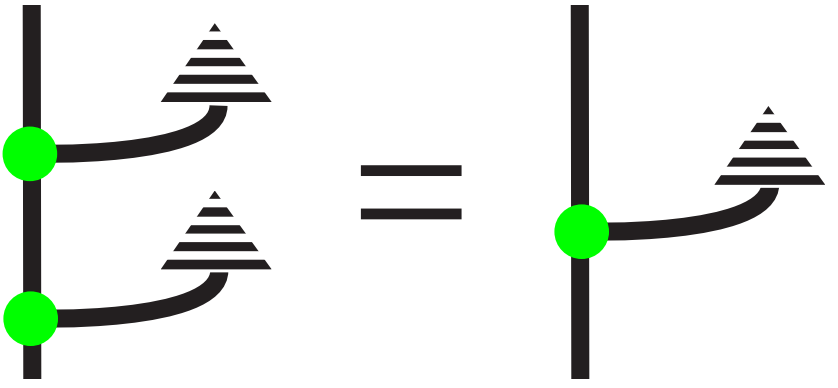
Coecke & Perdrix (2010) *Environment and class. chan. ...* arXiv:1004.1598

— *environment* —

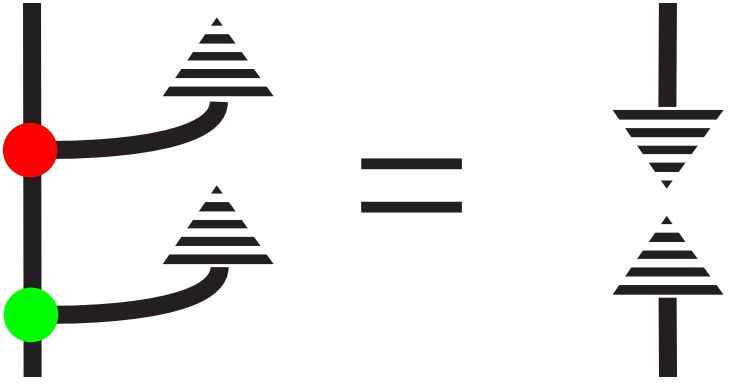


Coecke & Perdrix (2010) *Environment and class. chan. ...* arXiv:1004.1598

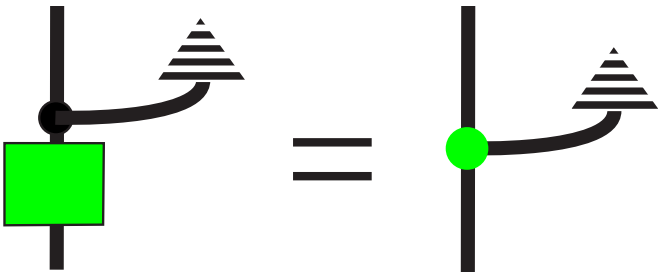
Prop 1:



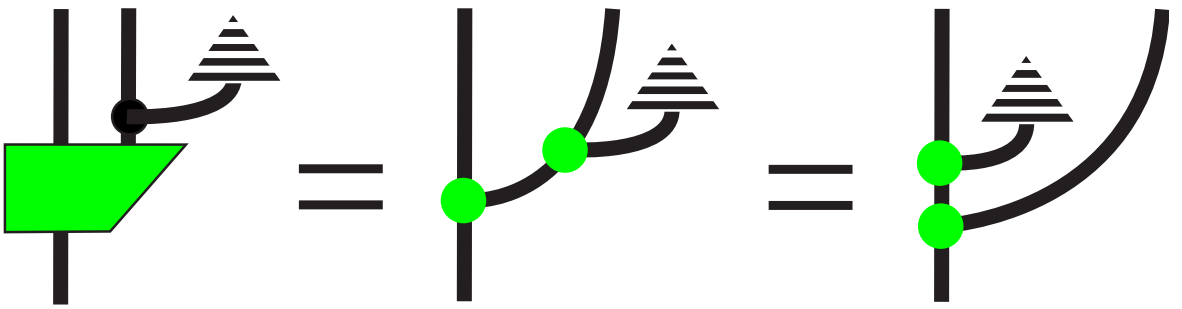
Prop 2:



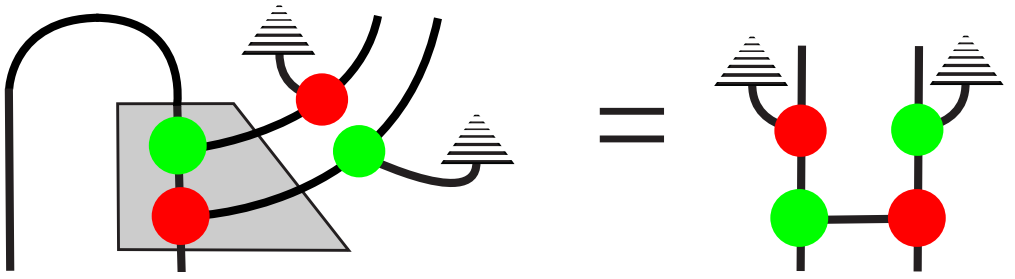
Destructive measurement:



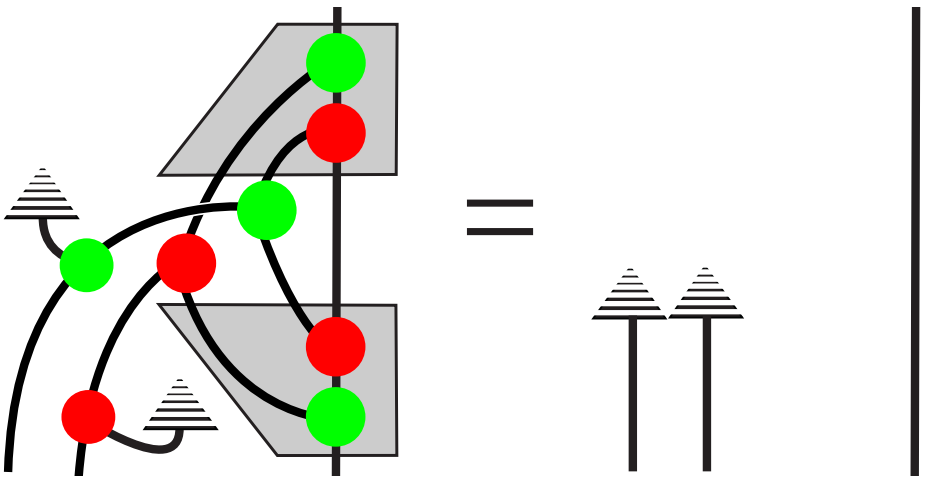
Non-destructive measurement:



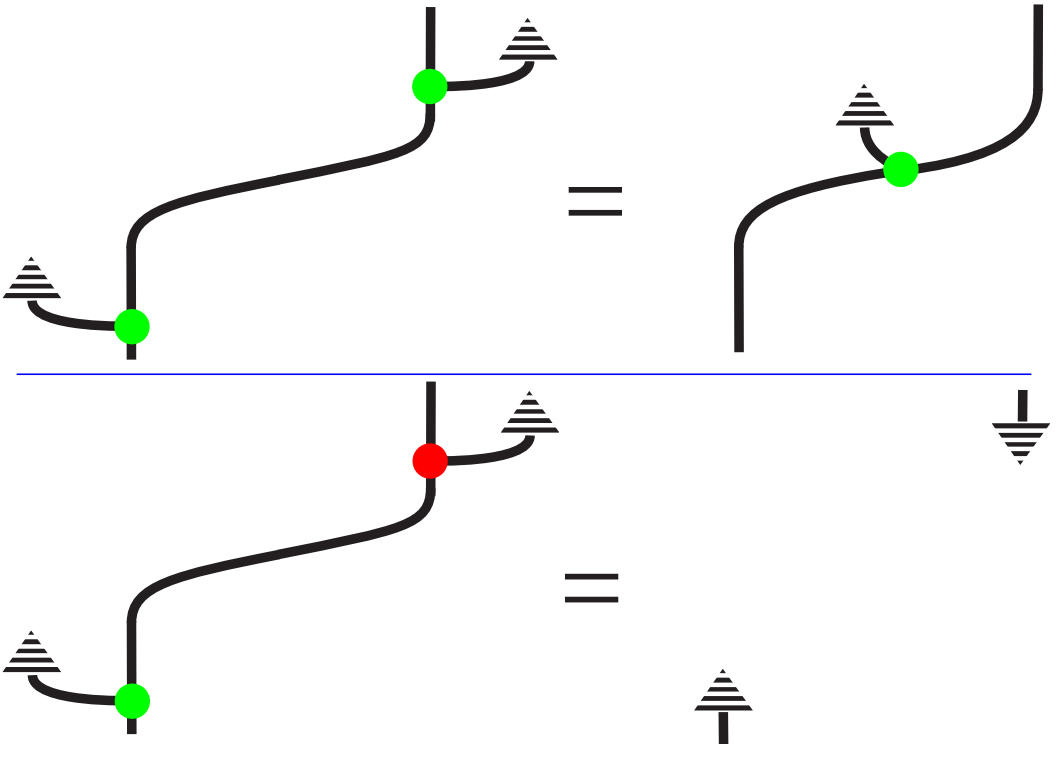
Indeed measurement:



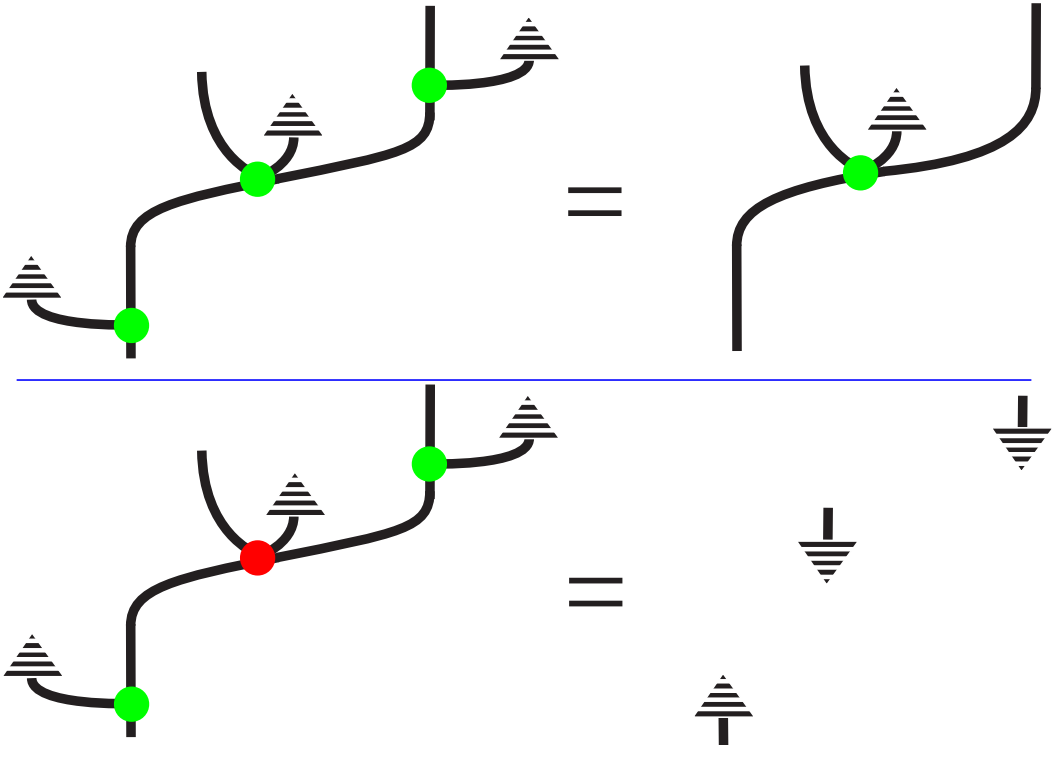
Indeed controlled unitary:



— *key distribution* —

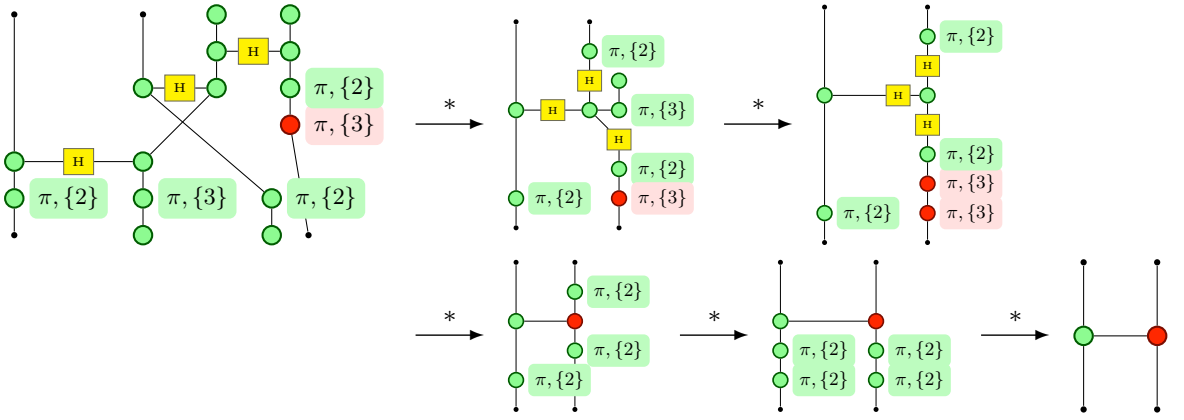


— *key distribution* —



— *phase groups and universality for QC*—

Translation to circuits and determinism for so-called **measurement based quantum computations**:



Ross Duncan & Simon Perdrix (2010) Rewriting measurement-based quantum computations with generalised flow. ICALP'10.

— *phase groups and quantum non-locality*—

Toy qubits vs. true quantum theory in one language:

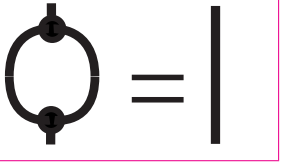
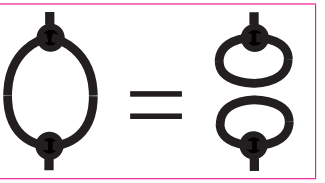
$$\frac{\text{Spekkens' qubit QM}}{\text{stabilizer qubit QM}} = \frac{Z_2 \times Z_2}{Z_4} = \frac{\text{local}}{\text{non-local}}$$

Bob Coecke, Bill Edwards & Rob Spekkens (2010) Phase groups and the origin of non-locality for qubits. arXiv:1003.5005

— *entanglement classification* —

Tripartite SLOCC-classes as comm. Frobenius algs:

$$\frac{GHZ = |000\rangle + |111\rangle}{W = |001\rangle + |010\rangle + |100\rangle} = \frac{\text{'special' CFAs}}{\text{'anti-special' CFAs}}$$

$$= \frac{\boxed{\text{  }}}{\boxed{\text{  }}}$$

Coecke & Aleks Kissinger (2010) The compositional structure of multipartite quantum entanglement. ICALP'10. arXiv:1002.2540

PUNCHLINE ON LOGIC

Taking (de)composition as primitive (vs. an individual entities' properties) we get very far with very little!

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Overall stance: Quantum 'richness' is not going to be understood by 'weakening' standard logical tools, but by considering radically different (classical?) ones.

Logic indeed (vs. quantum (non-)logic): Automation is demonstrated via the `quantomatic` software.

Dixon, Duncan & Kissinger. http://dream.inf.ed.ac.uk/projects/quantomatic/