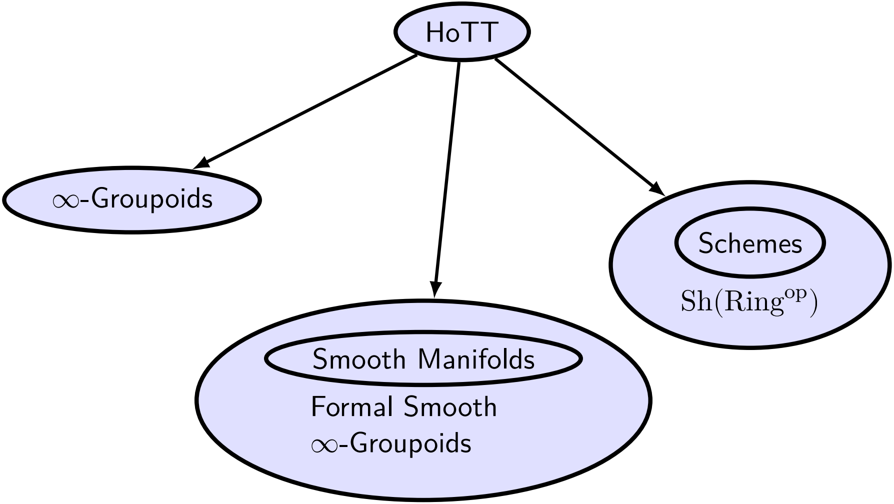
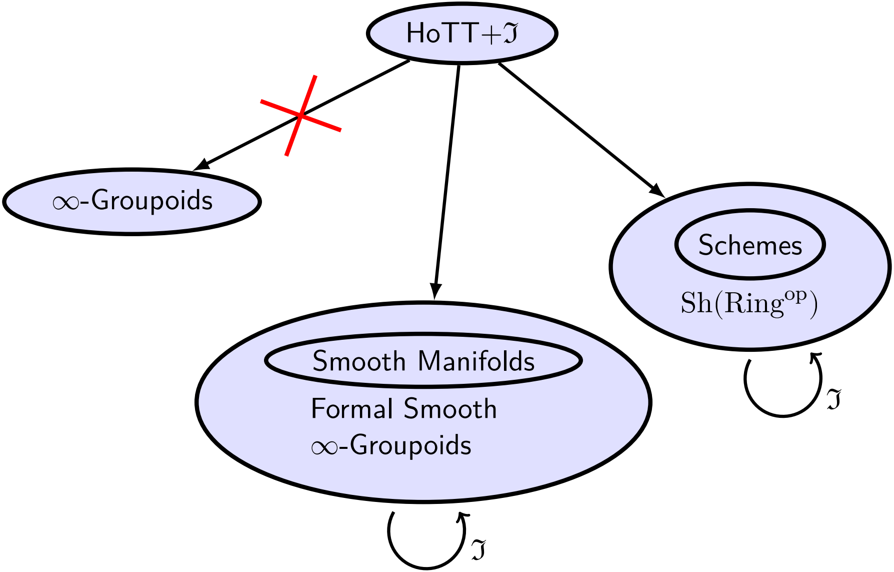
Differential Geometry via Modal HoTT

Felix Wellen


HoTT
∞-Groupoids
Schemes
Sh(Ring^op)
Smooth Manifolds
Formal Smooth
∞-Groupoids

Add a modality $\mathfrak{I}$...



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$\mathfrak{I}$ is a left and right adjoint idempotent monad known by the names *coreduction*, *deRham-stack* and *infinitesimal shape*.

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(iii) For any type A, $\Im A$ is coreduced.

(iv) For any $B : \Im A \to \mathcal{U}$, such that $\prod_{a : \Im A} B(a)$ is coreduced, a map $s : \prod_{a : \Im A} B(a)$ is defined by $s_0 : \prod_{a : A} B(\iota_A(a))$.

(v) Coreduced types have coreduced identity types.

(iv) may be specialized to:
For any coreduced B

$$\_ \circ \iota_A : (\Im A \to B) \to (A \to B)$$

is an equivalence.

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Let A be a type.

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(c) The type

$$\mathrm{T}_\infty A :\equiv \sum_{a:A} \mathbb{D}_a$$

is called the *formal disk bundle of* A.

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(c) If f is an equivalence, then for $a : A$,

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is an equivalence.

Towards a theorem

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Definition

A type V is called *homogeneous* if

(i) There is a point $e : V$.

(ii) For any $x : V$, there is an equivalence

$$\psi_x : V \to V$$

such that $\psi_x(e) = x$.

Examples

(a) Groups

(b) Pseudogroups

(c) Loopspaces

(d) Connected H-spaces

The triviality theorem

Theorem

Let V be a homogeneous type and $\mathbb{D}_e$ the formal disk at its point. Then the following commutes:

$$\begin{tikzcd} T_\infty V \arrow[rr, "\simeq"] \arrow[dr] & & V \times \mathbb{D}_e \arrow[dl] \\ & V & \end{tikzcd}$$

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$\mathbb{D}_{\psi_x(e)}$ and $\mathbb{D}_x$ are equivalent by transport along $\psi_x(e) = x$.
So, for any $x : V$ there is an equivalence $\varphi_x\colon \mathbb{D}_e \to \mathbb{D}_x$.
$(x, d) \mapsto (x, \phi^{-1}(d))\colon \left(\sum_{x:V} \mathbb{D}_x\right) \to V \times \mathbb{D}_e$ is an equivalence.

Thank you for your attention!

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A map $f\colon A \to B$ is called *formally étale* if the naturality square

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & \Im A \arrow[d, "\Im f"] \\ B \arrow[r, "\iota_B"'] & \Im B \end{tikzcd}$$

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Remark

For smooth manifolds formally étale maps correspond to local diffeomorphisms.

For noetherian schemes, they correspond to étale maps.

Structured spaces

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Definition

Let V be a homogeneous type. A type M is called a V-Manifold, if there is a span of formally étale maps

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Theorem (needs Univalence)

Any V-Manifold has a locally trivial formal disk bundle witnessed by a classifying map

$$\chi_M \colon M \to \mathrm{BAut}(\mathbb{D}_e)$$

Cartan Geometry

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If we have a delooping BG of a group G with a map $\varphi\colon \mathrm{BG} \to \mathrm{BAut}(\mathbb{D}_e)$, we can ask if there is a lift:

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For example, such a lift for $G = O(n)$ is a Pseudo-Riemannian structure on M.

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to get the site of *formal cartesian spaces*

$$\{\underbrace{\mathcal{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V)}_{=:(\mathbb{R}^n \times \mathbb{D}_V)^{\mathrm{op}}} \mid n \in \mathbb{N} \text{ and } V \text{ nilpotent, } \dim_{\mathbb{R}} V < \infty\}^{\mathrm{op}}$$

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On representables $\_\__{\mathrm{red}}$ is given by reduction of $\mathbb{R}$-algebras:

$$(\mathbb{R}^n \times \mathbb{D}_V)_{\mathrm{red}} = \mathbb{R}^n$$

Differential Cohesive Toposes

$$\begin{array}{ccccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \cup & & \cup & & \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

$\Im$, $\int$ and $\sharp$ are reflections.
$\Re$, $\&$ and $\flat$ are coreflections.
$\int$ and $\Re$ preserve finite products.