A Higher Structure Identity Principle

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(cww B. Ahrens, P. North, M. Shulman)

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Main Idea

Theorem (HoTT Book, Theorem 9.4.16)

For any univalent precategories (=categories) $\mathcal{C}$ and $\mathcal{D}$, the type of categorical equivalences $\mathcal{C} \simeq_{\text{precat}} \mathcal{D}$ is equivalent to $\mathcal{C} =_{\text{UniCat}} \mathcal{D}$

$$(\mathcal{C} \simeq_{\text{precat}} \mathcal{D}) \simeq (\mathcal{C} =_{\text{UniCat}} \mathcal{D})$$
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\[
(\mathcal{C} \equiv_{\text{precat}} \mathcal{D}) \simeq (\mathcal{C} = \text{UniCat} \mathcal{D})
\]

Pre-Theorem

For any univalent models \( \mathcal{M} \) and \( \mathcal{N} \) of an \( \mathcal{L} \)-theory \( \mathcal{T} \), the type of \( \mathcal{L} \)-equivalences \( \mathcal{M} \equiv_{\mathcal{L}} \mathcal{N} \) is equivalent to \( \mathcal{M} = \text{UniMod}(\mathcal{T}) \mathcal{N} \)

\[
(\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}) \simeq (\mathcal{M} = \text{UniMod}(\mathcal{T}) \mathcal{N})
\]
Main Idea

Pre-Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} \simeq_{\text{UniMod}(\mathbb{T})} \mathcal{N}$.

$L$-theory $\mathbb{T}$ = A theory $\mathbb{T}$ over a FOLDS signature
$L$-equivalence = FOLDS $L$-equivalence
univalent model = Model of $\mathbb{T}$ where FOLDS isomorphism is equivalent to identity

UniMod($\mathbb{T}$) = The type of univalent models

The Setting: Two-Level Type Theory (2LTT)
Main Idea

Pre-Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} =_{\text{UniMod}(\mathbb{T})} \mathcal{N}$.

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Pre-Theorem

For any univalent models \( M \) and \( N \) of an \( \mathcal{L} \)-theory \( T \), the type of \( \mathcal{L} \)-equivalences \( M \simeq_{\mathcal{L}} N \) is equivalent to \( M = \text{UniMod}(T) N \).
Main Idea

Pre-Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_\mathcal{L} \mathcal{N}$ is equivalent to $\mathcal{M} = \text{UniMod}(\mathbb{T}) \mathcal{N}$.

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$L$-equivalence $= \text{FOLDS } \mathcal{L}$-equivalence

univalent model $= \text{Model of } \mathbb{T} \text{ where FOLDS isomorphism is equivalent to identity}$

$\text{UniMod}(\mathbb{T}) = \text{The type of univalent models}$

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2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.
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One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$, intensional $=$, propositional truncation $||-||$ and a hierarchy of univalent universes $\mathcal{U}$.
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The other level of 2LTT is the **strict** fragment of **pretypes** which consists of $+^s$, $0^s$, $\mathbb{N}^s$, a strict equality $\equiv$ with UIP and function extensionality, a hierarchy of strict universes $\mathcal{U}^s$. It shares the type constructors $\Pi$, $\Sigma$, $1$ with the fibrant fragment.
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One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of $\Pi, \Sigma, +, 1, 0, \mathbb{N}$, intensional $=$, propositional truncation $\| - \|$ and a hierarchy of univalent universes $U$.

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The rules for the type constructors are the usual ones, and we also have a rule that allows us to consider any fibrant type as a pretype, i.e. the fibrant universes $U$ can be thought of as subuniverses of $U^s$, as well as rules that ensure that $\Sigma$ and $\Pi$ preserve fibrancy, and that the fibrant universes are closed under strict isomorphism.
s-categories

For a pretype $X$, we can write $\text{isfibrant}(X)$ for the pretype $\sum_{Y:U}(Y \equiv X)$.

**Definition (Definition 27, 2LTT)**

A pretype $A$ is **cofibrant** if for any fibration $p: X \to Y$, the induced map $(A \to X) \to (A \to Y)$ is a fibration.

**Definition (Definition 7, 2LTT)**

An **s-category** is given by the following data

1. A pretype $C$ of objects
2. For each $x, y: C$ a pretype $C(x, y)$ of arrows
3. For each $x: C$ an arrow $1: C(x, x)$
4. A composition operation $\circ: C(y, z) \to C(x, y) \to C(x, z)$ that is strictly associative and for which $1_x$ is a strict left and right unit.

A s-category is **cofibrant** if its pretypes of objects and arrows are cofibrant.
FOLDS (First-Order Logic with Dependent Sorts)

Invented by Makkai in his 1995 paper.
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The **signatures** $\mathcal{L}$ of FOLDS are (cofibrant) inverse categories with finite fan-out and of finite height.
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The contexts are finite functors $\Gamma : \mathcal{L} \to \textbf{Set}$ and formulas, sentences, sequents etc. in context are defined inductively in the usual way, taking a bit of care with the binding of variables.
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An $\mathcal{L}$-theory $\mathbb{T}$ is a pretype of $\mathcal{L}$-sentences.
An example

\[ \mathcal{L}_{rg} \rightarrow \Gamma \rightarrow \text{Set} \]

\[
\begin{array}{ccc}
2 & 1 & 0 \\
\downarrow & \downarrow & \downarrow \\
i & A & O \\
c & \downarrow & \downarrow & \downarrow \\
d & g & z & \downarrow \\
\end{array}
\]

\[ di = ci \]
An example

\[ \mathcal{L}_{rg} \to \Gamma \to \text{Set} \]

\[
\begin{array}{cccccc}
2 & \quad I \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \\
1 & A & T & T \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & O & i & g & f & \{ f, g \} \\
\end{array}
\]

\[
di = ci
\]

\[
\Gamma = x, y, z: O, f: A(x, y), g: A(z, z), \tau: I(g, z)
\]
An example

\[ \mathcal{L}_{rg} \rightarrow^\Gamma \rightarrow \text{Set} \]

\[
\begin{array}{cccc}
2 & 1 & 0 \\
I & A & O \\
i & c & x \\
\downarrow & \downarrow & \downarrow \\
\iota & g & \{\tau\} \\
\downarrow & \downarrow & \downarrow \\
\tau & f & \{f, g\} \\
\downarrow & \downarrow & \downarrow \\
\tau & c & \{x, y, z\} \\
\downarrow & \downarrow & \downarrow \\
\tau & d & \{x, y, z\} \\
\end{array}
\]

\[ di = ci \]

\[ \Gamma = x, y, z : O, f : A(x, y), g : A(z, z), \tau : l(g, z) \]

\[ \text{Form}(x : O) \quad \forall g : A(z, z). \exists \tau : l(g, z). \top \sim \forall g : A(z, z). l(g, z) \]
Some terminology and notation

\[ r(K) \quad \mathcal{L} \leftarrow K \sqcup \mathcal{L} \quad \partial K = \mathcal{L}(K, -) \]

\[ n = H(\mathcal{L}) \quad R \]

\[ n - 1 \quad A \]

\[ m \quad K \]

\[ 1 \quad X \quad X \leq K \]

\[ 0 \quad O \]
Semantics of FOLDS in 2LTT

We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$.

$D(\mathcal{L}) \equiv R(\mathcal{L})$ but the situation is not that simple.
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We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$. 

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We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$.
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We want to define a type of $\mathcal{L}$-structures $\textbf{Struct}(\mathcal{L})$.

$$\mathcal{O}(\mathcal{L}_{rg}) \quad \mathcal{L}_{rg}$$

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$$ \mathcal{O}(\mathcal{L}_{rg}) \quad \mathcal{L}_{rg} $$

$$ \cdots \sum_{A: O \times O \rightarrow \mathcal{U}} \cdots $$

$$ \sum_{O: \mathcal{U}} \cdots $$

$$ \sum \cdots $$

$$ I \downarrow i $$

$$ A \downarrow c \downarrow d $$

$$ O $$

$$ \ReedyFib(\mathcal{L}_{rg}, \mathcal{U}) $$
We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$.

$$\Delta(\mathcal{L}_{rg})$$

$$\ldots \left( \sum_{x} A(x, x) \right) \rightarrow \mathcal{U}$$

$$\ldots \sum A: O \times O \rightarrow U \ldots$$

$$\ldots \sum O: U \ldots$$

$$\sum O: U$$

$$\text{ReedyFib}(\mathcal{L}_{rg}, \mathcal{U})$$

We would like $\Delta(\mathcal{L}) \equiv \mathcal{R}(\mathcal{L})$ but the situation is not that simple.

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Semantics of FOLDS in 2LTT

We want to define a type of $\mathcal{L}$-structures $\text{Struct}(\mathcal{L})$.

$$\mathcal{O}(\mathcal{L}_{\text{rg}})$$

$$\mathcal{L}_{\text{rg}}$$

$$\mathcal{K}(\mathcal{L}_{\text{rg}})$$

$$\ldots \left( \sum_x A(x, x) \right) \rightarrow \mathcal{U}$$

$$\ldots \sum_{A: O \times O \rightarrow \mathcal{U}} \ldots$$

$$\sum_{O: \mathcal{U}} \ldots$$

$$\sum_{O: \mathcal{U}} \ldots$$

$$I$$

$$i$$

$$A$$

$$A$$

$$c$$

$$d$$

$$O$$

$$O$$

$$O$$

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Semantics of FOLDS in 2LTT

We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$.

\[ D(\mathcal{L}_{rg}) \quad \mathcal{L}_{rg} \quad R(\mathcal{L}_{rg}) \]

\[ \ldots \left( \sum_{x: O} A(x, x) \right) \rightarrow U \]

\[ \ldots \sum_{A: O \times O \rightarrow U} \ldots \]

\[ \sum_{O: U} \ldots \]

\[ \text{ReedyFib}(\mathcal{L}_{rg}, U) \]
We want to define a type of $\mathcal{L}$-structures $\text{Struc}(\mathcal{L})$.

We would like $\mathcal{D}(\mathcal{L}) \equiv \mathcal{R}(\mathcal{L})$ but the situation is not that simple.
Semantics of FOLDS in 2LTT

\[ \lim \left( A/\mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right) \]

Theorem \( D(L) \cong R(L) \) as s-categories.

We define the type of \( L \)-structures as \( \text{Struc}(L) = D(L) \) but we will use the equivalence of the above theorem to transfer constructions from \( R(L) \).

Similarly, we denote by \( \text{Mod}(T) \) the type of \( L \)-structures satisfying all the sentences of \( T \).
Semantics of FOLDS in 2LTT

\[
\begin{align*}
F_K & \\
& \downarrow \\
\lim \left( A/\mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right) & \\
\downarrow K & \\
\mathcal{U} &
\end{align*}
\]

Theorem

\[ \mathcal{D}(\mathcal{L}) \simeq \mathcal{R}(\mathcal{L}) \text{ as s-categories}. \]
Semantics of FOLDS in 2LTT

\[
\begin{align*}
F_K & \quad \Rightarrow \\
\lim_{M_A^F} & \quad \text{lim} \left( A/\mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} U \right) \\
K & \quad \Rightarrow \\
U & \quad \Rightarrow
\end{align*}
\]

Theorem

\[\mathcal{D}(\mathcal{L}) \simeq \mathcal{R}(\mathcal{L})\text{ as s-categories.}\]

We define the type of \(\mathcal{L}\)-structures as \(\text{Struc}(\mathcal{L}) = \mathcal{D}(\mathcal{L})\) but we will use the equivalence of the above theorem to transfer constructions from \(\mathcal{R}(\mathcal{L})\).
Semantics of FOLDS in 2LTT

\[
\begin{array}{c}
F_K \\
\downarrow \\
M^F_A \\
\downarrow \\
K \\
\downarrow \\
U
\end{array}
\quad \text{lim} \left( A//\mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right)
\quad \begin{array}{c}
K \\
\downarrow \\
A \\
\vdots \\
\vdots
\end{array}
\]

**Theorem**

\[ \mathcal{D}(\mathcal{L}) \simeq \mathcal{R}(\mathcal{L}) \text{ as s-categories.} \]

We define the type of \( \mathcal{L} \)-structures as \( \text{Struc}(\mathcal{L}) = \mathcal{D}(\mathcal{L}) \) but we will use the equivalence of the above theorem to transfer constructions from \( \mathcal{R}(\mathcal{L}) \).

Similarly, we denote by \( \text{Mod}(\mathcal{T}) \) the type of \( \mathcal{L} \)-structures satisfying all the sentences of \( \mathcal{T} \).
The $\mathcal{L}_{\text{cat}}$-theory $\mathbb{T}_{\text{cat}}$

\[
\begin{array}{c}
\text{2} & \circ & l & E_A \\
\text{1} & t_2 & A & s \\
\text{0} & t_0 & c & d \\
\end{array}
\]

\[
dt_0 = dt_2 \quad ct_1 = ct_2 \quad dt_1 = ct_0
\]

\[
ds = dt \quad cs = ct
\]

\[
\text{ci} = \text{di}
\]
The $\mathcal{L}_{\text{cat}}$-theory $\mathbb{T}_{\text{cat}}$

\[ 
\begin{array}{cccc}
2 & 1 & 0 \\
\circ & \circ & \\
E_A & i & A \\
i & t & c \\
\end{array}
\]

\[ d t_0 = d t_2 \quad c t_1 = c t_2 \quad d t_1 = c t_0 \]
\[ d s = d t \quad c s = c t \]
\[ c i = d i \]

$\mathbb{T}_{\text{cat}}$ is the $\mathcal{L}_{\text{cat}}$-theory with the usual axioms of category theory expressed in relational form using $E_A$ as the equality on arrows.
The $\mathcal{L}_{\text{cat}}$-theory $\mathbf{T}_{\text{cat}}$

\[
\begin{array}{ccc}
2 & \circ & I \\
\downarrow & & \downarrow \\
1 & t_0 & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
0 & t_1 & s & \downarrow & \downarrow \\
\circ & t_2 & \downarrow & \downarrow \\
\circ & \circ & \circ & \circ & \circ
\end{array}
\]

$E_A$

$dt_0 = dt_2 \quad ct_1 = ct_2 \quad dt_1 = ct_0$

$ds = dt \quad cs = ct$

$ci = di$

$\mathbf{T}_{\text{cat}}$ is the $\mathcal{L}_{\text{cat}}$-theory with the usual axioms of category theory expressed in relational form using $E_A$ as the equality on arrows.

**Theorem**

If $E_A$ is interpreted as the identity type on $A$ then

\[
\text{Mod}(\mathbf{T}_{\text{cat}}) \simeq \sum_{O : \mathcal{U}} \sum_{A : O \to O \to \mathcal{U}} \circ : \sum_{x,y,z : O} A(x,y) \to A(y,z) \to A(x,z) (\ldots \ldots)
\]

$\circ^*: A(x,y) \to A(x,z)$
Generalized Isomorphism?

Let $\mathcal{M} : \text{Mod}(\mathbb{T})$
Generalized Isomorphism?

Let $\mathcal{M} : \text{Mod}(\mathbb{T})$

\[\begin{array}{ccc}
\mathcal{M}_Q & \rightarrow & \mathcal{M}_R \\
\downarrow & & \downarrow \\
\mathcal{M}_A & \rightarrow & \mathcal{M}_K' \\
\downarrow & & \downarrow \\
\mathcal{M}_K & \rightarrow & \mathcal{M}_O
\end{array}\]

$q \quad r \quad ???$

\[\begin{array}{ccc}
a & \rightarrow & b \\
\downarrow & & \downarrow \\
x & \rightarrow & \partial x
\end{array}\]

$a \cong b?$

$\partial a = \partial b$
FOLDS $\mathcal{L}$-equivalence

$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} = \text{df } \sum \langle P, m, n \rangle \left( \begin{array}{c} m \text{ v.s.} \vspace{0.5cm} \begin{array}{c} \mathcal{M} \\ \text{v.s.} \end{array} \\ n \text{ v.s.} \\ \mathcal{N} \end{array} \right)$
**FOLDS $\mathcal{L}$-equivalence**

\[
\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad \overset{=\text{df}}{=} \quad \sum_{\langle P, m, n \rangle} \left( \begin{array}{c}
P \\
m \ & \overset{\text{v.s.}}{\downarrow} \\
\mathcal{M} \\ \\
\mathcal{N} \\
& \overset{\text{v.s.}}{\downarrow} \\
& n \\
\end{array} \right)
\]

**Definition**

\[
\text{isverysurjective}(m) = \text{df} \prod_{K: \mathcal{L}} \text{issurjective}(\mathcal{P}_K \rightarrow \mathcal{M}^\mathcal{P}_K \times \mathcal{M}^\mathcal{M}_K \mathcal{M}_K)
\]

**Theorem (Makkai, 1995)**

*If $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ then $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$*
**FOLDS pre-isomorphism**

Fix $\mathcal{M} : \text{Mod}(\mathcal{T})$ and $\mathcal{K} : \mathcal{L}$. Let $a, b : \mathcal{M}_\mathcal{K}$.

**Definition (Pre-isomorphism)**

A **pre-isomorphism** from $a$ to $b$ is given by the following cospan of spans

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow s \\
\mathcal{P} \\
\downarrow \downarrow m & t & n \\
\mathcal{M} & \partial \mathcal{K} & \mathcal{M} \\
\end{array}
\]

where $\langle m, \mathcal{P}, n \rangle$ is a FOLDS equivalence.

**Theorem (Makkai, 1995)**

If $a$ is pre-isomorphic to $b$ then $\mathcal{M} \models \phi[a] \iff \mathcal{M} \models \phi[b]$
FOLDS isomorphism

Fix \( \mathcal{M} : \text{Mod}(\mathbb{T}) \) and \( K : \mathcal{L} \). Let \( a, b : \mathcal{M}_K \).

**Definition (FOLDS isomorphism)**

A **FOLDS isomorphism** is a pre-isomorphism \( \langle m, \mathcal{P}, n \rangle \) such that:

1. For any \( f : K \to A \) we have

   \[
   \mathcal{M} \xleftarrow{m} \mathcal{P} \xrightarrow{n} \mathcal{M} \\
   [\text{id}, \mathcal{M}_f(a)] \sim \mathcal{M} \amalg \partial A \leftrightarrow \mathcal{M} \\
   [\text{id}, \mathcal{M}_f(b)]
   \]

2. For any \( A \succ K \) we have

   \[
   \mathcal{M}_A \times_{\mathcal{M}_A^\mathcal{M}} \mathcal{M}_A^\mathcal{P} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{\mathcal{M}_A^\mathcal{M}} \mathcal{M}_A^\mathcal{P}
   \]

We write \( a \cong b \) for the type of FOLDS isomorphisms.
FOLDS isomorphism

Fix $\mathcal{M} : \text{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

**Definition (FOLDS isomorphism)**

A **FOLDS isomorphism** is a pre-isomorphism $\langle m, \mathcal{P}, n \rangle$ such that:

1. For any $f : K \to A$ we have

$$
\begin{array}{c}
\mathcal{M} \\
\mathcal{M} \sqcup \partial A
\end{array}
\xymatrix{ M \ar[r]^m & \mathcal{P} \ar[r]^n \ar@{->}@/^/[d]^\sim & M \\
\mathcal{M} \sqcup \partial A \ar@{->}@/_/[u]_{[\text{id}, \mathcal{M}_f(a)]} \ar@{->}@/^/[u]_{[\text{id}, \mathcal{M}_f(b)]}}
$$

2. For any $A > K$ we have $\mathcal{M}_A \times_{\mathcal{M}_A^\mathcal{M}} \mathcal{M}_A^\mathcal{P} \xymatrix{ \sim & \mathcal{P}_A \ar[r] \ar[l] & \mathcal{M}_A \times_{\mathcal{M}_A^\mathcal{M}} \mathcal{M}_A^\mathcal{P}}$

We write $a \sim b$ for the type of FOLDS isomorphisms.

**Lemma**

$\sim : \mathcal{M}_K \to \mathcal{M}_K \to \mathcal{U}$ is reflexive.
FOLDS isomorphism

Fix $\mathcal{M} : \text{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : M_K$.

**Definition (FOLDS isomorphism)**

A **FOLDS isomorphism** is a pre-isomorphism $\langle m, P, n \rangle$ such that:

1. For any $f : K \to A$ we have $M \xleftarrow{m} P \xrightarrow{n} M$

   \[
   \begin{array}{c}
   \mathcal{M} \xleftarrow{m} \mathcal{P} \xrightarrow{n} \mathcal{M} \\
   \downarrow \sim \\
   \mathcal{M} \cap \partial A \\
   \end{array}
   \]

   $[\text{id}, M_f(a)] \sim [\text{id}, M_f(b)]$

2. For any $A > K$ we have $\mathcal{M}_A \times_{M_A^P} M_A^P \leftarrow \mathcal{P}_A \rightarrow \mathcal{M}_A \times_{M_A^P} M_A^P$

We write $a \cong b$ for the type of FOLDS isomorphisms.

**Lemma**

$\cong : \mathcal{M}_K \to \mathcal{M}_K \to \mathcal{U}$ is reflexive.

**Corollary**

$idtoiso_{a,b} : a =_{M_K} b \rightarrow a \cong b$
Univalent Models

Fix $\mathcal{M} : \text{Mod}(\mathbb{T})$

**Definition (Univalence for $\mathcal{M}$)**

- $K$-univalent
  
  $\text{univ}_K(\mathcal{M}) = \text{df} \prod_{a, b : \mathcal{M}_K} \text{isequiv}(\text{idtoiso}_{a, b})$

- $m$-univalent
  
  $\text{univ}_m(\mathcal{M}) = \text{df} \prod_{K : \mathcal{L} \geq m} \text{univ}_K(\mathcal{M})$

- univalent
  
  $\text{univ}(\mathcal{M}) = \text{df} \prod_{K : \mathcal{L}} \text{univ}_K(\mathcal{M})$

**Definition (Type of Univalent Models)**

- $\text{UniMod}_m(\mathbb{T}) = \text{df} \sum_{\mathcal{M} : \text{Mod}(\mathbb{T})} \text{univ}_m(\mathcal{M})$

- $\text{UniMod}(\mathbb{T}) = \text{df} \sum_{\mathcal{M} : \text{Mod}(\mathbb{T})} \text{univ}(\mathcal{M})$
Some Results

Theorem

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**Corollary**

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Let \( H(\mathcal{L}) \geq n \geq m, K : \mathcal{L}^{=n} \) and \( \mathcal{M} : \text{UniMod}_m(\mathcal{T}) \). Then \( \mathcal{M}_K \) is an \( m \)-type.
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Theorem

Let \( \mathcal{T}_{cat} \) be the \( \mathcal{L}_{cat} \)-theory of categories. Then we have:

\[
\text{UniMod}_1(\mathcal{T}_{cat}) \cong \text{PreCat} \\
\text{UniMod}(\mathcal{T}_{cat}) \cong \text{UniCat}
\]
We began with:

**Theorem**

*For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} = \text{UniMod}(\mathbb{T}) \mathcal{N}$.***
A Higher Structure Identity Principle

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**Theorem**

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And now we can obtain the precise version:

**Theorem (A Higher Structure Identity Principle, in progress)**

For any $\mathcal{M}, \mathcal{N} : \text{UniMod}(\mathbb{T})$ for a FOLDS $\mathcal{L}$-theory $\mathbb{T}$ we have

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \iff \mathcal{M} = _{\text{UniMod}(\mathbb{T})} \mathcal{N}$$
Thank you