On Suborbifolds

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References

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Outline

Development of the concept, Part 1

- (Satake, 1956)
  - introduced V-manifolds, built on Riemannian geometry.
  - in terms of atlases with charts involving effective group actions where all fixed point sets have at least codimension two.
  - V-manifolds were merely seen as a generalization of smooth manifolds.
- (Thurston, 1976)
  - introduced orbifolds in a course, unaware of Satake's work.
  - They were given in terms of atlases with charts that involved effective group actions (no codimension requirements).
  - Orbifolds were used to understand the geometry of 3-manifolds.

Development of the concept, Part 2

- (Kawasaki, 1978)
  - developed index theory for orbifolds;
  - introduced the inertia orbifold ;
  - this showed that orbifolds have unique structure that manifolds do not have;
  - showed the importance of non-effective orbifolds.
- (Vafa, Witten et al, 1985) Orbifolds in physics; weighted projective orbifolds carry non-effective group actions.

Development of the concept, Part 2

- (Moerdijk-Pronk, 1995)
  - introduced a representation of orbifolds in terms of étale groupoids with proper diagonal;
  - to develop orbifold homotopy theory and in particular a good notion of map between orbifolds.
- (Chen, Ruan et al, early 2000's) use groupoids to study not necessarily effective orbifolds.
- (Pronk, Scull, Sibih, Tommasini, 2017) introduce two notions of atlas with group actions that do not need to be effective.

The Old Atlas Definition

An (effective) **orbifold** $(X, \mathfrak{U})$ consists of

- a paracompact Hausdorff space X;
- a family $\mathcal{U}$ of open subsets of X which covers X;
- for each $U \in \mathcal{U}$ a triple $(\widetilde{U}, G_U, \varphi_U)$ such that
  - $\widetilde{U} \subseteq \mathbb{R}^n$ is simply connected and open;
  - G_U is a finite group which acts smoothly and effectively on $\widetilde{U}$;
  - $\varphi_U \colon \widetilde{U} \to X$ induces a homeomorphism $\widetilde{U}/G \xrightarrow{\sim} U$.

Furthermore, we require that when $U, V \in \mathcal{U}$ and $x \in U \cap V$ then there is a chart $W_x \in \mathcal{U}$ with $x \in W \subseteq U \cap V$ with atlas embeddings

$$\lambda \colon \widetilde{W} \to \widetilde{U} \text{ and } \lambda' \colon \widetilde{W} \to \widetilde{V}.$$

The Old Atlas Definition

When $U \subseteq V$ are charts in $\mathcal{U}$, an **atlas embedding** $\widetilde{U} \hookrightarrow \widetilde{V}$ is an embedding

$$\lambda \colon \widetilde{U} \to \widetilde{V}$$

such that

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\lambda"] \arrow[d, "\varphi_U"'] & \widetilde{V} \arrow[d, "\varphi_V"] \\ U \arrow[r, phantom, "\subseteq"] & V \end{tikzcd}$$

commutes.

The Bicategory GroupMod

- In order to formalize a more general (not necessarily effective) notion of orbifold atlas, we need the bicategory **GroupMod** of finite groups and bimodules.
- An arrow $\mathrm{M} : G \nrightarrow H$ is a set with a left H action and a compatible right G action.

Atlas Bimodules

- Note that the groups in the classical definition act on the sets of embeddings by composition with the representations of the group elements.
- The individual embeddings from the classical definition will be replaced by bimodules of embeddings in the new definition.
- A bimodule $M\colon G \twoheadrightarrow H$ is called an **atlas bimodule** if
    - the action by H is free and transitive (i.e., M is an H-torsor);
    - the action by G is free.
- An atlas bimodule corresponds to a conjugacy class of injective group homomorphisms $G \to H$.

The New Atlas Definition

A (general) **orbifold** consists of

- a paracompact Hausdorff **space** X;
- a **preorder** $\mathcal{U}$ of open subsets of X such that
  - the elements of $\mathcal{U}$ cover X;
  - the canonical map $\mathcal{U} \to \mathcal{O}(X)$ is monotone;
  - for each pair $U, V \in \mathcal{O}(\mathcal{U})$, the supremum of $\{W \in \mathcal{O}(\mathcal{U}); W \leq U, W \leq V\}$ in $\mathcal{O}(X)$ is $U_i \cap U_j$;
- an **orbifold chart structure** for the elements of $\mathcal{U}$;
- two **pseudo functors** $\mathbf{Abst}, \mathbf{Conc} \colon \mathcal{U} \to \mathbf{GroupMod}$ extending the orbifold chart structures;
- an **oplax transformation** $\rho \colon \mathbf{Abst} \Rightarrow \mathbf{Conc}$ satisfying additional conditions.

New Orbifold Chart Structure

An **orbifold chart structure** for $U \in \mathcal{U}$ is a 4-tuple $(\widetilde{U}, G_U, \rho_U, \varphi_U)$ such that

- $\widetilde{U} \subseteq \mathbb{R}^n$ is connected and simply connected;
- G_U is a finite group, with a group homomorphism

$$\rho_U \colon G_U \to \mathrm{Diffeo}(\widetilde{U})$$

  (we will write G_U^{red} for the image of ρ_U)
- $\varphi_U \colon \widetilde{U} \to U \subseteq X$ is the quotient map;

The Pseudo Functors

The two pseudo functors $\mathrm{Abst}, \mathrm{Conc} \colon \mathcal{U} \to \mathrm{GroupMod}$ need to satisfy:

- for any $U \in \mathcal{U}$,
$$\mathrm{Abst}(U) = G_U;$$
- for any μ_{VU}, $\mathrm{Conc}(\mu_{VU})$ is the set of classical atlas embeddings from $\widetilde{U}$ to $\widetilde{V}$ and the group actions are by composition;
- for any arrow $\mu_{VU} \colon U \to V$ in $\mathcal{U}$,
$$\mathrm{Abst}(\mu_{VU}) \colon G_U \nrightarrow G_V$$
is an atlas bimodule.

The Representation Transformation

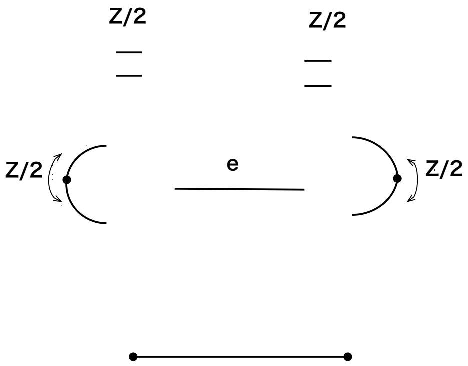
The oplax transformation $\rho\colon \mathrm{Abst} \Rightarrow \mathrm{Conc}$ needs to have the following properties:

- the components ρ_{VU} of ρ correspond to surjective equivariant maps

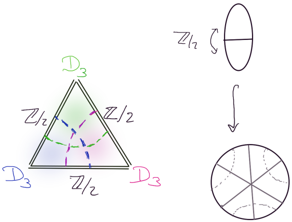
$$\begin{tikzcd} G_U \arrow[r, "\mathrm{Abst}(\mu_{VU})", "/"{marking}] \arrow[d, "\rho_U"'] \arrow[dr, phantom, "\tilde{\rho}_{VU}\Downarrow"] & G_V \arrow[d, "\rho_V"] \\ G_U^{\mathrm{red}} \arrow[r, "\mathrm{Conc}(\mu_{VU})"', "/"{marking}] & G_V^{\mathrm{red}} \end{tikzcd}$$

  i.e., maps $\tilde{\rho}_{VU}\colon \mathrm{Abst}(\mu_{VU}) \to \mathrm{Conc}(\mu_{VU})$ such that
  - $\tilde{\rho}_{VU}(h \cdot \alpha) = \rho_V(h) \cdot \tilde{\rho}_{VU}(\alpha)$ for any $h \in G_V$
  - $\tilde{\rho}_{VU}(\alpha \cdot g) = \tilde{\rho}_{VU}(\alpha) \cdot \rho_U(g)$ for any $g \in G_U$.
- for each μ_{VU}, G_U acts transitively on the fibers of $\tilde{\rho}_{VU}$: whenever $\tilde{\rho}_{VU}(\alpha) = \tilde{\rho}_{VU}(\alpha')$, there is a $g \in G_U$ such that $\alpha \cdot g = \alpha'$.

The Silvered Interval



The Triangular Billiard



The abstract and concrete modules are the same in this case:

$$\{\lambda, r\lambda, s\lambda, rs\lambda, r^2\lambda, r^2 s\lambda\}$$

Thurston's Definition

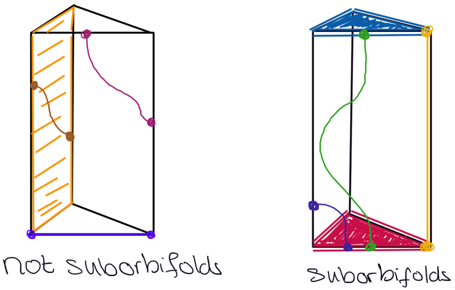
Definition

A d-dimensional suborbifold Q_1 of Q_2 is given by a subspace $X_{Q_1} \subset X_{Q_2}$ locally modeled on $\mathbb{R}^d$ modulo the induced actions of the local groups of Q_2 on invariant d-dimensional subspaces.

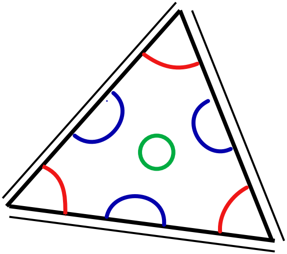
Remark

- This means that for any chart, the preimage of the subspace needs to be a submanifold, invariant under the action of the structure group.
- Furthermore, Thurston only takes the effective part of the actions on the submanifold to obtain the structure group for the suborbifold.

Example 1: The 3-4-6 Prism

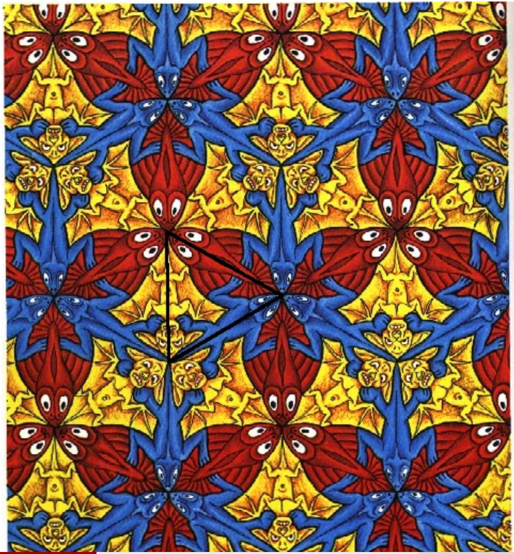


Example 2: The Triangular Billiard



All 1-dimensional closed suborbifolds up to isotopy according to Thurston

A Universal Covering of the Triangular Billiard



Borzellino and Brunsden's Definition

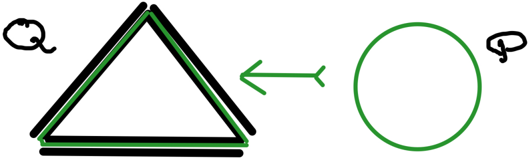
Definition

An (embedded) **suborbifold** $\mathcal{P}$ of an orbifold $\mathcal{Q}$ consists of:

- $X_{\mathcal{P}} \subset X_{\mathcal{Q}}$ subspace;
- For each $x \in X_{\mathcal{P}}$ and neighbourhood W of x in $X_{\mathcal{Q}}$ there is
  - an orbifold chart $(\widetilde{U}, \Gamma_x, \rho_x, \phi_x)$ in $\mathcal{Q}$, $U_x \subset W$,
  - $\Lambda_x \leq \Gamma_x$ and a Λ_x-invariant submanifold $\tilde{V} \subset \tilde{U}$

  such that $(\widetilde{V}_x, \Lambda_x/\Omega_x, \rho_x|_{\Lambda_x}, \psi_x)$ is an orbifold chart for $\mathcal{P}$, where $\Omega_x = \ker(\rho_x) \cap \Lambda_x$);
- $V_x = \psi_x(\widetilde{V}_x/\Lambda_x) = U_x \cap X_{\mathcal{P}}$ is an orbifold chart for x in $\mathcal{P}$.

Example: the Boundary of the Triangular Billiard



$\Gamma_{x,Q} = D_3$ $\quad \Lambda_x = \mathbb{Z}/2$ $\quad \Lambda_x / \Omega_x = 1$

$\Omega_x = \mathbb{Z}/2$

Full and Saturated Suborbifolds

Definition

$\mathcal{P} \subset \mathcal{Q}$ is a **full suborbifold** of $\mathcal{Q}$ if $\Lambda_x = \Gamma_x$ for all $x \in \mathcal{P}$.

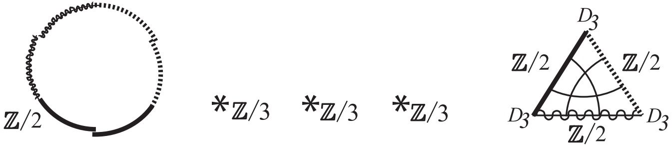
Definition

$\mathcal{P} \subset \mathcal{Q}$ is a **saturated suborbifold** of $\mathcal{Q}$ if for each $x \in \mathcal{P}$ and $\tilde{y} \in \tilde{V}_x$, we have that $(\Gamma_{x,\mathcal{Q}} \cdot \tilde{y}) \cap \widetilde{V}_x = \Lambda_{x,\mathcal{Q}} \cdot \tilde{y}$.

The Inertia Orbifold of the Triangular Billiard

Let $(X, \mathfrak{U})$ be an orbifold. Its **inertia orbifold** is defined as follows

- Its charts are the fixed point sets of the actions of the elements of the structure groups on the charts of X.
- The embedding modules between them are formed by restrictions of the modules of $(\mathfrak{U})$.



Adem, Leida and Ruan's Definition

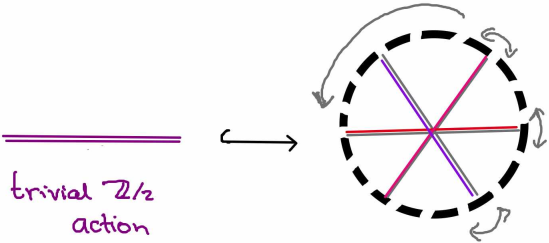
This definition was given for étale orbifold groupoids:

Definition

A homomorphism of orbifold groupoids $\phi \colon \mathcal{G} \to \mathcal{H}$ is an **embedding** if the following are satisfied:

- $\phi_0 \colon \mathcal{G}_0 \to \mathcal{H}_0$ is an immersion.
- Let $x \in \mathrm{im}(\phi_0) \subset \mathcal{H}_0$ with neighbourhood U_x such that $\mathcal{H}|U_x \cong H_x \ltimes U_x$. Then the $\mathcal{G}$-action on $\phi_0^{-1}(x)$ is transitive, and there exists an open neighbourhood $V_y \subseteq \mathcal{G}_0$ for each $y \in \phi_0^{-1}(x)$ such that $\mathcal{G}|_{V_y} \cong G_y \ltimes V_y$ and
$$\mathcal{G}|_{\phi_0^{-1}(U_x)} \cong H_x \ltimes (H_x/\phi_1(G_y) \times V_y).$$
- $|\phi| \colon |\mathcal{H}| \to |\mathcal{G}|$ is proper.

New Embeddings



Cho, Hong and Shin's Definition

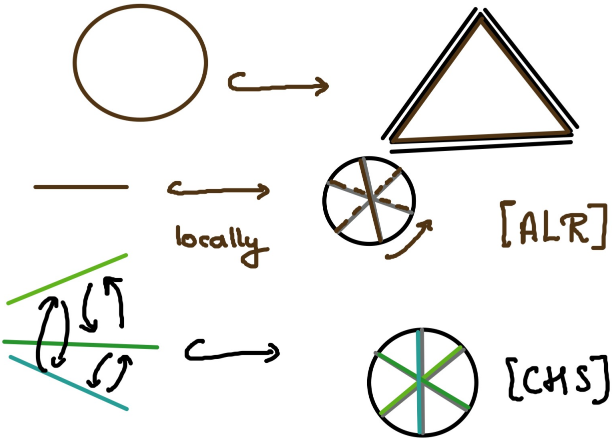
Definition

A homomorphism of orbifold groupoids $\varphi\colon \mathcal{H} \to \mathcal{G}$ is an **embedding** if the following conditions are satisfied:

1. $\varphi_0\colon \mathcal{H}_0 \to \mathcal{G}_0$ is an immersion
2. Let $x \in \mathrm{Im}\,(\varphi_0) \subset \mathcal{G}_0$ and let U_x be a neighbourhood such that $\mathcal{G}|_{U_x} \cong \mathcal{G}_x \ltimes U_x$. Then, the $\mathcal{H}$-action on $\varphi_0^{-1}(x)$ is transitive, and there exists an open neighborhood $V_y \subset \mathcal{H}_0$ for each $y \in \varphi_0^{-1}(x)$ such that $\mathcal{H}|_{V_y} \cong \mathcal{H}_y \ltimes V_y$ and
$$\mathcal{H}|_{\varphi_0^{-1}(U_x)} \cong \mathcal{G}_x \ltimes (\mathcal{G}_x \times_{\mathcal{H}_y} V_y)$$
3. $|\varphi|\colon |\mathcal{G}| \to |\mathcal{H}|$ is proper and injective.

$\mathcal{H}$ together with φ is called an orbifold embedding of $\mathcal{G}$.

Example: The Boundary of the Billiard



A Module Definition of Suborbifold

An **orbifold embedding** $(A, \mathfrak{W}) \to (X, \mathfrak{U})$ is given by

- A subspace $j_A \colon A \hookrightarrow X$.
-
$$\begin{tikzcd}
\mathcal{O}(\mathcal{W}) \arrow[rr, "J"] \arrow[dr, bend right, "\mathrm{Conc}_{\mathfrak{W}}"'] \arrow[dr, "\mathrm{Abst}_{\mathfrak{W}}"] & & \mathcal{O}(\mathcal{U}) \arrow[dl, "\mathrm{Abst}_{\mathfrak{U}}"'] \arrow[dl, bend left, "\mathrm{Conc}_{\mathfrak{U}}"] \\
& \textbf{GroupMod} &
\end{tikzcd}$$
-
$$\begin{tikzcd}
\mathrm{Abst}_{\mathfrak{W}} \arrow[r, "\alpha^J"] \arrow[d, "\rho_{\mathfrak{W}}"'] \arrow[dr, phantom, "\downarrow R"] & \mathrm{Abst}_{\mathfrak{U}} J \arrow[d, "\rho_{\mathfrak{U}} J"] \\
\mathrm{Conc}_{\mathfrak{W}} \arrow[r, "\gamma^J"'] & \mathrm{Conc}_{\mathfrak{U}} J
\end{tikzcd}$$
    - the components $(\alpha^J)_W$ are atlas modules.
    - the components $(\gamma^J)_W$ are sets of embeddings that cover the embedding of the quotient spaces.

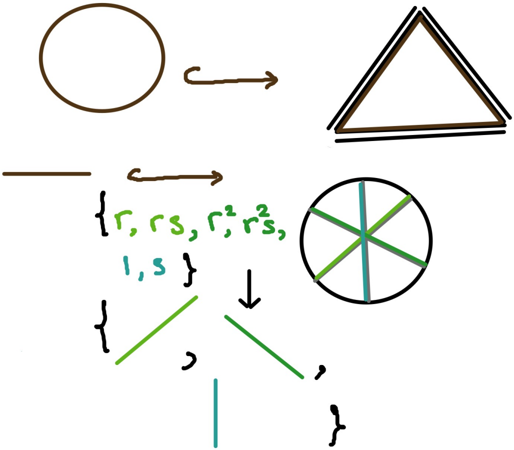
A Module Definition of Suborbifold

The 3-cell R has components

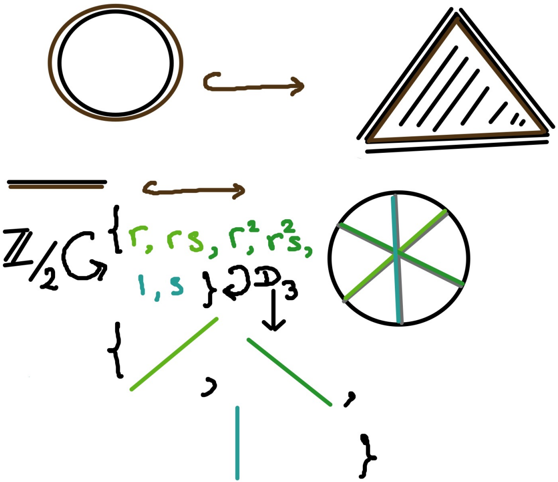
$$\begin{tikzcd}
G_W \arrow[r, "\alpha^J_W"] \arrow[d, "\rho_W"'] \arrow[dr, phantom, "R_W"] & G_{JW} \arrow[d, "\rho_{JW}"] \\
G^{\mathrm{red}}_W \arrow[r, "\gamma^J_W"'] & G^{\mathrm{red}}_{JW}
\end{tikzcd}$$

represented by surjective equivariant functions $\tilde{R}_W \colon \alpha^J_W \to \gamma^J_W$ such that G_W acts transitively on the fibers.
All other components of this 3-cell need to be isomorphisms.

Example: The Boundary of the Billiard



Example: The Boundary of the Billiard



Conjectures

- Each suborbifold in our sense has an atlas refinement such that for every pair of corresponding charts the CHS condition is satisfied. If you want this to be completely Morita-invariant, you need to work with partial maps, i.e. restriction categories.
- The canonical way to include effective suborbifolds or effective orbifolds in this frame work, is to add the abstract structure layer given by the groups keeping the suborbifold invariant, and work with stronger notiion of embedding. This way we can provide embedding maps for all saturated Borzellino-Brunsden suborbifolds. (This takes away to need for split extensions of Ω_x.)