Turing Categories and Realizability

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Restriction Categories

A *restriction category* is a category in which every map $f : X \to Y$ has a *domain of definition* $\overline{f} : X \to X$ satisfying:

- **[R.1]** $\overline{f} f = f$
- **[R.2]** $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$
- **[R.3]** $\overline{\overline{f}\,g} = \overline{f}\,\overline{g}$
- **[R.4]** $f\overline{g} = \overline{fg}f$

Restriction categories are understood as *categories of partial maps*, where $\overline{f}$ tells us which part of its domain f is defined on.

For example, sets and partial functions form a restriction category, with $\overline{f}(x) = x$ if $f(x) \downarrow$, and $\overline{f}(x) \uparrow$ otherwise.

Each homset in a restriction category is a partial order. For $f, g : X \to Y$ say $f \leq g \Leftrightarrow \overline{f} g = f$.

A map $f : X \to Y$ in a restriction category $\mathbb{X}$ is called *total* in case $\overline{f} = 1_X$. The total maps of a restriction category form a subcategory, $\mathsf{total}(\mathbb{X})$.

Notice that if g is total, then $\overline{f} = \overline{f}\,1 = \overline{f}\,\overline{g} = \overline{f\overline{g}}$. If a restriction category $\mathbb{X}$ has products, the projections are total, so $\overline{f} = \overline{\langle f, 1\rangle} = \overline{\langle f, 1\rangle \pi_1} = \overline{1} = 1$, and the restriction structure is necessarily trivial.

We want limits *and* restriction structure, so we usually work with "restriction limits".

Restriction Categories

A restriction category has *restriction products* in case for every pair A, B of objects there is an object $A \times B$ together with total maps $\pi_0 : A \times B \to A$, $\pi_1 : A \times B \to B$ such that whenever we have maps $f : C \to A$ and $g : C \to B$, there is a unique map $\langle f, g\rangle : C \to A \times B$ with $\langle f, g\rangle \pi_0 = \overline{g} f$ and $\langle f, g\rangle \pi_1 = \overline{f} g$.

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A \arrow[ur, phantom, "\geq", shift right=4] & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ur, phantom, "\leq", shift left=4] & B
\end{tikzcd}$$

A restriction category has a *restriction terminal object*, 1 in case for each object A there is a unique total map $!_A : A \to 1$ such that for all $f : A \to B$, $f!_B \leq !_A$.

A restriction category with both of these is called a *cartesian restriction category*.

A *partial applicative system* in a cartesian restriction category $\mathbb{X}$ consists of an object A and a map $\bullet : A \times A \to A$. (That's it!)

We say a map $f : A \to A$ of $\mathbb{X}$ is A-*computable* in case there is a total map $h : 1 \to A$ such that

$$\begin{tikzcd} A \times A \arrow[r, "\bullet"] & A \\ A \times 1 \simeq A \arrow[u, "1 \times h"] \arrow[ur, "f"'] & \end{tikzcd}$$

A partial applicative system is *combinatory complete* in case the A-computable maps form a cartesian restriction category.

Such a partial applicative system is called a *partial combinatory algebra* (PCA).

A *Turing category* is a cartesian restriction category with a *Turing structure*. That is, a universal object A and a partial applicative system $\bullet : A \times A \to A$ such that every map $f : X \to Y$ is A-computable modulo sections and retractions:

$$\begin{tikzcd}
A \times A \arrow[r, "\bullet"] & A \arrow[r, "r"] & Y \\
A \times 1 \simeq A \arrow[u, "1 \times h"] \arrow[ur, "rfs"'] & & \\
X \arrow[u, "s"] \arrow[uurr, "f"'] & &
\end{tikzcd}$$

Think of a Turing category as a notion of computation. We have a sort of "Gödel numbering" where the universal object plays the role of $\mathbb{N}$, and a version of the parameter theorem and the recursion theorem holds in every Turing category.

For example, the partial recursive functions give a Turing category that embeds into sets and partial functions.

The Turing structure consists of the object $\mathbb{N}$, and the partial applicative structure $\bullet : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $\bullet(m, n) = \phi_n(m)$. Then, if f is the nth partial recursive function, defining $h := \{* \mapsto n\}$ gives

$$\begin{tikzcd}
\mathbb{N} \times \mathbb{N} \arrow[r, "\bullet"] & \mathbb{N} \\
\mathbb{N} \times 1 \simeq \mathbb{N} \arrow[u, "1 \times h"] \arrow[ur, "f"'] &
\end{tikzcd}$$

as required.

This example is caled *Kleene's first model* of computation.

Let $A = (A, \bullet)$ be a PCA in the category of sets and partial functions (**ptl**).

An *A-assembly* (X, φ) is a set X together with a function $\varphi : X \to P(A)^*$.

A morphism of A-assemblies $(X\varphi) \to (Y, \psi)$ is a total function $f : X \to Y$ for which there exists a *tracking element* $a \in A$ such that $\forall x \in X.\forall i \in \varphi(x).a \bullet i \downarrow \wedge a \bullet i \in \psi(f(x))$.

Assemblies and their morphisms form a category. Think of an assembly (X, φ) as a "computer representation" of X. For a morphism $f : (X, \varphi) \to (Y, \psi)$, think of each tracking element for f as giving a "computer implementation" of f for the representations.

These categories of assemblies are finitely complete, cartesian closed, and regular.

The forgetful functor that maps each assembly to its underlying set (and simply forgets about tracking elements) is a fibration. In fact, it is a *tripos*.

Every tripos defines a topos. The topos associated with a category of assemblies is called a *realizability topos*.

A realizability topos is something like a foundation for "A-computable mathematics", where $A = (A, \bullet)$ is the PCA we started with.

More General Assemblies

Let $\mathbb{A}$ be a restriction category, $\mathbb{X}$ be a cartesian restriction category, and $F : \mathbb{A} \to \mathbb{X}$ be a restriction functor.

An *assembly* is a restriction idempotent $\varphi : \mathcal{O}(F(A) \times X)$ in $\mathbb{X}$ where A is an object of $\mathbb{A}$, and X is an object of $\mathbb{X}$.

A *morphism of assemblies* $f : \varphi \to \psi$ for $\varphi : \mathcal{O}(F(A) \times X)$, $\psi : \mathcal{O}(F(B) \times Y)$ is a map $f : X \to Y$ of $\mathbb{X}$ which is *tracked* by some map $\gamma : A \to B$ of $\mathbb{A}$. That is

- $\varphi(F(\gamma) \times f) = \varphi(F(\gamma) \times f)\psi$
- $\overline{\varphi(1 \times f)} = \overline{\varphi(F(\gamma) \times f)}$

Assemblies and their morphisms form a restriction category, denoted $\mathsf{asm}(F)$.

More General Assemblies

If $\mathbb{A}$ is the category of A-computable maps for a PCA $(A, \bullet)$ in **ptl** and we take $F : \mathbb{A} \to \textbf{ptl}$ to be the inclusion functor, then $\mathsf{total}(\mathsf{asm}(F))$ is the classical category of assemblies.

There is a forgetful restriction functor $\partial : \mathsf{asm}(F) \to \mathbb{X}$ which maps (X, φ) to $\mathbb{X}$ and maps $f : (X, \varphi) \to (Y, \psi)$ to $f : X \to Y$.

Recall that in the classical case, this functor is the realizability tripos. While we don't expect ∂ to be a tripos for any $\mathbb{A}, \mathbb{X}$, and $F : \mathbb{A} \to \mathbb{X}$, we would at least like ∂ to be a fibration...

…but it isn't!

The problem is that in a fibration $\mathsf{asm}(F) \to \mathbb{X}$, we ask that the prone maps f produce *unique* liftings of maps:

in $\mathsf{asm}(F)$:
$$\begin{tikzcd}
(Z,\chi) \arrow[d, dotted, "\exists! \widetilde{\mathbf{h}}"'] \arrow[dr, "g"] & \\
(X,\varphi) \arrow[r, "f"'] & (Y,\psi)
\end{tikzcd}$$

in $\mathbb{X}$:
$$\begin{tikzcd}
Z \arrow[d, "h"'] \arrow[dr, "g"] & \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

With the forgetful functor $\mathsf{total}(\mathsf{asm}(F)) \to \mathsf{total}(\mathbb{X})$ this works out, but when partial maps are involved there may be many liftings that are not equal. (If $\overline{g}h = g$ and $\widetilde{h}$ is a potential lifting, so is $\overline{g}\,\widetilde{h}$, for example)

Latent Fibrations

Let $p : \mathbb{E} \to \mathbb{B}$ be a restriction functor, and say that a map f of $\mathbb{E}$ is *prone in* p in case whenever we have $hp(f) \geq p(g)$

$$\text{in } \mathbb{E}: \quad \begin{tikzcd} Z \arrow[d, dotted, "\widetilde{h}"'] \arrow[dr, "g"] & \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\geq"] & Y \end{tikzcd} \qquad \text{in } \mathbb{B}: \quad \begin{tikzcd} pZ \arrow[d, "h"'] \arrow[dr, "p(g)"] & \\ pX \arrow[r, "p(f)"'] \arrow[ur, phantom, "\geq"] & pY \end{tikzcd}$$

there is a *minimal lifting* $\widetilde{h}: Z \to X$ such that

- $\widetilde{h}$ is a *candidate lifting*: $\widetilde{h}\, f \geq g$ and $p(\widetilde{h}) \leq h$.
- If k is a candidate lifting, $\widetilde{h} \leq k$.

Now, say that $p : \mathbb{E} \to \mathbb{B}$ is a *latent fibration* in case there is a prone map above every map $f : A \to p(Y)$

Latent Fibrations

Define the *fibre* over and object B of $\mathbb{B}$ to be the category whose objects are the objects X of $\mathbb{E}$ with $pX = B$, and whose maps are the maps f of $\mathbb{E}$ with $p(f) \leq 1_B$.

Then, instead of reindexing functors between the fibres, a cloven latent fibration defines a reindexing *restriction semifunctor* $f^* : \mathbb{E}_B \to \mathbb{E}_A$ for each $f : A \to B$ in the base. (A restriction semifunctor preserves composition and restriction, but need not preserve identities).

If the latent fibration reflects total maps, then our reindexing restriction semifunctors are in fact restriction functors.

Latent Fibrations

The category of restriction categories and restriction functors has pullbacks, and the pullback of a latent fibration along any restriction functor is a latent fibration.

If $p : \mathbb{E} \to \mathbb{B}$ is a latent fibration that reflects total maps, then we obtain a fibration (in the usual sense), $\mathsf{total}(p)$ by pulling back along the inclusion of $\mathsf{total}(\mathbb{B})$ into $\mathbb{B}$:

$$\begin{tikzcd} \mathsf{total}(\mathbb{E}) \arrow[r] \arrow[d, "\mathsf{total}(p)"'] & \mathbb{E} \arrow[d, "p"] \\ \mathsf{total}(\mathbb{B}) \arrow[r, hook] & \mathbb{B} \end{tikzcd}$$

For example, every restriction category $\mathbb{X}$ defines a tripos as follows:

Let $\mathcal{R}(\mathbb{X})$ be the category whose objects are restriction idempotents $e = \overline{e}$ in $\mathbb{X}$, and whose morphisms $f : e \to e'$ are morphisms f of $\mathbb{X}$ with $e \leq \overline{fe'}$. Then the functor that maps objects to their domain/codomain and morphisms to themselves is a latent fibration.

The fibre over X is $\mathcal{O}(X)$, the preorder of all restriction idempotents on X, and so we call this the *domain latent fibration.*

DCCRCs and Triposes

A *discrete cartesian closed restriction category* (DCCRC) is a cartesian closed restriction category in which the diagonal map $\Delta_X : X \to X \times X$ has a partial inverse for each X.

If $\mathbb{X}$ is a DCCRC, then the domain latent fibration $\mathcal{R}(\mathbb{X}) \to \mathbb{X}$ is something along the lines of a "latent tripos". Each fibre $\mathcal{O}(\mathbb{X})$ is a Heyting algebra, it has universal and extistential quantification, and has a generic predicate. Further, $\mathsf{total}(\mathcal{O}) : \mathsf{total}(\mathcal{R}(\mathbb{X})) \to \mathsf{total}(\mathbb{X})$ is a tripos in the usual sense.

The connection between categorical structure in $\mathbb{X}$ and structure in the domain should be fairly deep. For example, we know that the domain latent fibration has "existential quantifiaction" if and only if the base is a range category. It would be nice to know the whole story.

DCCRCs and Triposes

Every tripos defines a topos. The objects of the topos are partial equivalence relations in the internal language of the tripos, and the maps are relations that are:

- Strict: $\forall_{xy}(f(x,y) \Rightarrow x \sim x \wedge y \sim y)$
- Relational: $\forall_{xx'yy'}(f(x,y) \wedge x \sim x' \wedge y \sim y' \Rightarrow f(x',y'))$
- Deterministic: $\forall_{xyy'}(f(x,y) \wedge f(x,y') \Rightarrow y \sim y')$
- Total: $\forall_x(x \sim x \Rightarrow \exists_y(f(x,y)))$

If we remove the requirement that maps are total we obtain a *partial topos* instead, and its total map category is the usual topos.

(A partial topos is a DCCRC in which certain "partial monic" maps have partial inverses).

The Realizabililty Tripos

Our forgetful functor $\partial : \mathsf{asm}(F) \to \mathbb{X}$ is a latent fibration.

If $\mathbb{A}$ is a Turing category, $\mathbb{X}$ is a DCCRC, and $F : \mathbb{A} \to \mathbb{X}$ preserves finite restriction products, then $\mathsf{total}(\partial) : \mathsf{total}(\mathsf{asm}(F)) \to \mathsf{total}(\mathbb{X})$ is a tripos.

This captures both the classical realizability tripos, in which $F : \mathbb{A} \hookrightarrow \mathbf{ptl}$ where $\mathbb{A}$ is the computable map category of some PCA in **ptl**, and also later work by Birkedal, in which $F : \mathbb{A} \to \mathbf{ptl}$ where $\mathbb{A}$ is (more or less) a Turing category.

Let's call $\partial : \mathsf{asm}(F) \to \mathbb{X}$ the *realizability latent fibration*. While it's not always a (latent) tripos, it's present for every category $\mathsf{asm}(F)$ of assemblies.

For $F : \mathbb{A} \to \mathbb{X}$ with $\mathbb{X}$ a cartesian restriction category ...

If $\mathbb{A}$ is a cartesian restriction category, F preserves finite restriction products, then $\mathsf{asm}(F)$ is a cartesian restriction category. If in addition $\mathbb{X}$ is discrete, then so is $\mathsf{asm}(F)$.

If $\mathbb{X}$ is a range category then so is $\mathsf{asm}(F)$.

If $\mathbb{A}$ is a weakly cartesian closed restriction category, F preserves finite restriction products, $\mathbb{X}$ is a DCCRC, then $\mathsf{asm}(F)$ is a DCCRC.

If $\mathbb{X}$ is a cartesian restriction category with finite joins, $\mathbb{A}$ is a discrete cartesian restriction category with interleaving, and $F : \mathbb{A} \to \mathbb{X}$ is preserves finite products, then $\mathsf{asm}(F)$ has finite joins. (meaning it has a zero maps and binary joins of compatible parallal maps).

This is particularily interesting because this sort of finite join doesn't make sense in a category with only total maps.

(An *interleaving* of a pair of parallel maps $f, g : A \to B$ is a map $h : A \to B$ with $\overline{f} \leq \overline{h}$, $\overline{g} \leq \overline{h}$, and $h = (h \cap f) \vee (h \cap g)$. This is an abstract characterization of the interleaving of computable functions from recursion theory, but it's a bit hard to see why!)

Future Work

There are many things to do...

We need to figure out exactly what latent fibrations are. (Guess: something like restriction category indexed restriction categories).

Work out "partial categorical logic" for latent fibrations corresponding to (total) categorical logic for fibrations. Fully work out the correspondence between structure in a restriction category and structure in its domain latent fibration.

Characterize the logic of the realizability latent fibration in terms of $\mathbb{A}, \mathbb{X}$, and $F : \mathbb{A} \to \mathbb{X}$. (Birkedal 2002)

Furture Work

Construct a bunch of exotic categories of assemblies, particularly where $\mathbb{X}$ is not a DCCRC, and investigate realizability for categories with less structure than a topos, or with interesting Turing categories (see PTIME, Cockett, Hofstra & Hrubes 2014).

Assemblies over a base ($\mathsf{asm}(F)$,$\mathsf{asm}(G)$ for $F : \mathbb{A} \to X$, $G : \mathbb{B} \to \mathbb{X}$) form a category. What does this category look like? (Specifically, how exactly does the category of Turing categories and simulations (Cockett & Hofstra 2010) relate to assemblies over a base whose categories of realizers are Turing categories?)

Is the construction of $\mathsf{asm}(F)$ a monad somehow? If not, what is it?