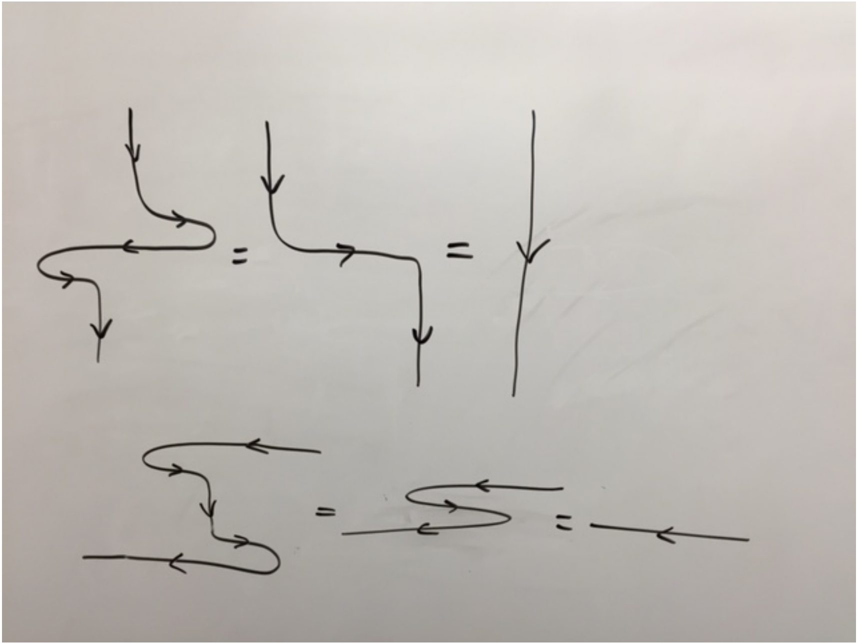
String Diagrams for (Virtual) Proarrow Equipments

David Jaz Myers

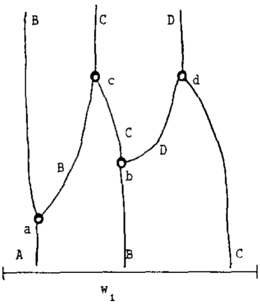
July 22, 2017



Theorem (Joyal and Street)

The graphical calculus for monoidal categories is sound.

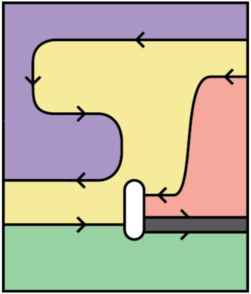
For any deformation $h : \Gamma \times [0,1] \to [a,b] \times [c,d]$ *of diagrams, the value of* $h(-,0)$ *equals that of* $h(-,1)$.



Theorem (M.)

The graphical calculi for double categories and equipments are sound.

For any deformation $h : \Gamma \times [0,1] \to [a,b] \times [c,d]$ *of diagrams, the value of* $h(-,0)$ *equals that of* $h(-,1)$.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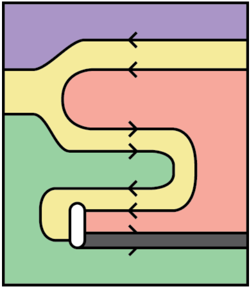
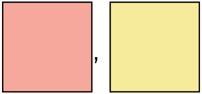
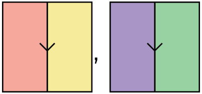
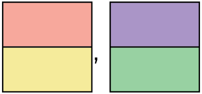
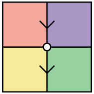


Theorem (M.)

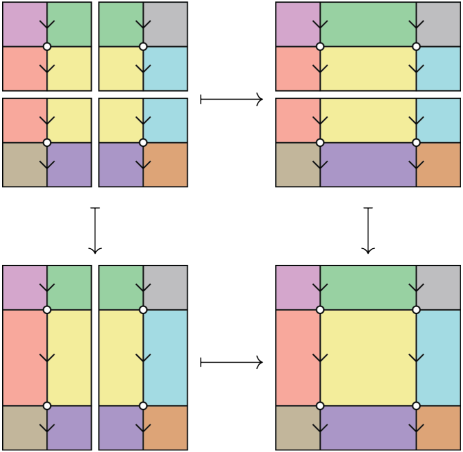
There is a canonical (Yoneda-style) embedding $|\cdot| : \mathcal{E} \to \mathcal{E}\text{-}\mathbf{Cat}$ *of a virtual equipment into the virtual equipment of categories enriched in it, which is full on arrows and coreflective on proarrows.*

What is a Double Category

A **double category** is a category internal to the category of categories.

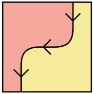
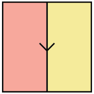
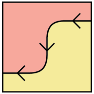
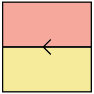
- Objects A, B,. . ., ,. . .
- Arrows $f : A \to B$,. . . , ,. . .
- Proarrows $J : A \nrightarrow B$, . . . , ,. . .
- 2-cells $\begin{tikzcd} A \arrow[r] \arrow[d, "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "|" marking] \\ C \arrow[r] & D \end{tikzcd}$,. . . , ,. . .

What is a Double Category

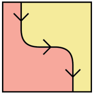
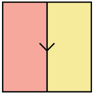
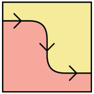
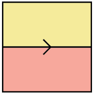


Companions and Conjoints

An arrow  has a *companion* if there is a proarrow  together with two 2-cells  and  such that

 =  and  = .

Similarly,  is said to have a *conjoint* if there is a proarrow  together with two 2-cells  and  such that

 =  and  = .

Proarrow Equipments

Definition

A **proarrow equipment** is a double category where every arrow has a conjoint and a companion.

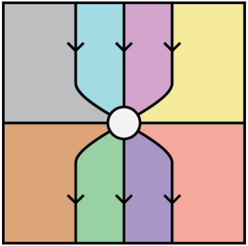
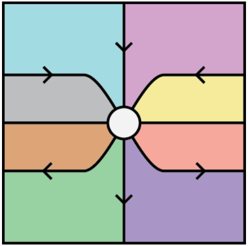
Examples

- Sets, Functions, Relations.
- Rings, Homomorphisms, Bimodules.
- Categories, Functors, Profunctors.
- Enriched Categories, Enriched Functors, Enriched Profunctors, etc.

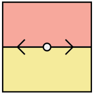
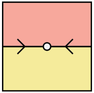
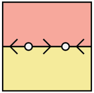
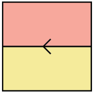
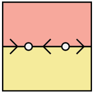
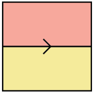
Spider Lemma

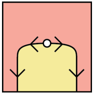
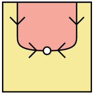
Lemma (Spider Lemma)

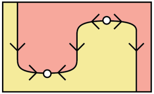
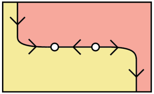
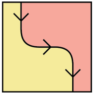
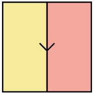
In an equipment, we can bend arrows. More formally, there is a bijective correspondence between diagrams of form of the left, and diagrams of the form of the right:

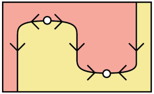
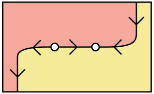
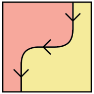
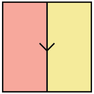
 $\approx$ .

Hom-Set to Zig-Zag Adjunctions

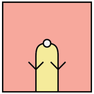
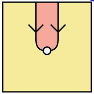
Given arrows  and , with an isomorphism  with inverse :  =  and  = .

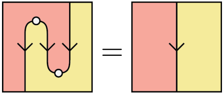
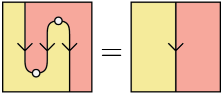
Bend to  and , then

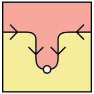
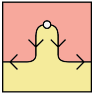
 =  =  = ,

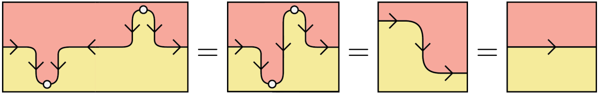
 =  =  = .

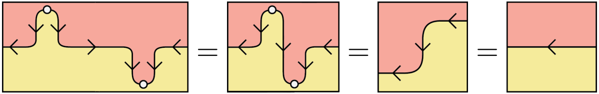
Zig-Zag to Hom-Set Adjunctions

Given  and , satisfying

 and .

Bend to  and , then

,

,

Enriching in a Virtual Equipment

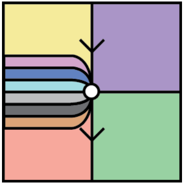
- Lawvere ('73):
  - Not only are the most fundamental structures of mathematics organized in categories,
  - They are in many cases (enriched) categories themselves.
- With the graphical calculus, we can show that so long as our objects form a virtual equipment, then they are enriched categories of a sort.

Theorem (M.)

There is a Yoneda-style embedding $|\cdot| : \mathcal{E} \to \mathcal{E}\text{-}\mathbf{Cat}$ *of a virtual equipment into the virtual equipment of categories enriched in it.*

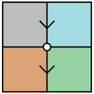
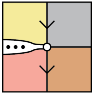
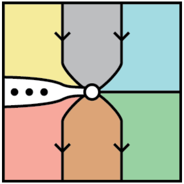
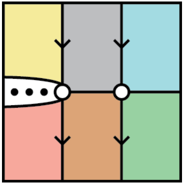
Enrichment and Virtual Equipments

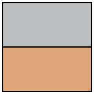
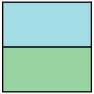
- Composing proarrows requires taking a colimit in the base category.
- But what if the base category is not suitably cocomplete?
- Then we use "virtual equipments" instead.



Restrictions

Definition

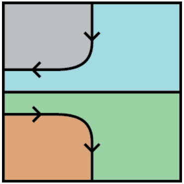
A cell  is called *cartesian* if for any , there exists a unique  so that  = .

We call  the **restriction** of  along  and .

Restrictions

Definition
A **Virtual Equipment** is a virtual double category with all restrictions (and a unit condition).

Lemma (Cruttwell and Shulman)
In a virtual equipment, every restriction is of the form



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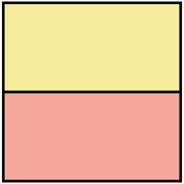
Enriching in a Virtual Equipment

The main difference between enriching in a virtual equipment and enriching in a monoidal category is **extent**:

$\mathcal{C}$ a $\mathcal{V}$-category means: $\forall A, B \in \mathcal{C}_0, \quad \mathcal{C}(A, B) \in \mathcal{V}$,

$\mathcal{C}$ a $\mathcal{E}$-category means: $\begin{cases} \forall A \in \mathcal{C}_0 & \mathcal{C}(A) \in \mathcal{E} \\ \forall A, B \in C_0 & \mathcal{C}(A) \xrightarrow{\mathcal{C}(A,B)} \mathcal{C}(B) \end{cases}$



Examples of Enrichment in a Virtual Equipment

With a single object:

- In Sets and Spans: Categories.
- In Rings and Bimodules: Algebras.
- In Enriched Cats and Profunctors: Arrows.
- Multicategories, Many-sorted Lawvere theories, Virtual double categories, etc.

With many objects:

- In Sets and Spans: Smooth paths in a manifold.

Conjecture (M.)

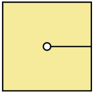
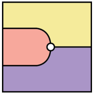
There is a full and faithful functor

$$\mathbf{Kleisli}(\mathbf{Jet}) \hookrightarrow \mathbf{Span\text{-}Cat}.$$

sending a smooth manifold to its category of smooth paths.

Enriching in a Virtual Equipment

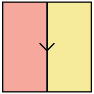
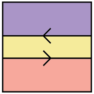
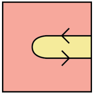
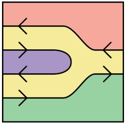
A category $\mathcal{C}$ enriched in a virtual equipment $\mathcal{E}$ consists of the following data:

- A class of objects $\mathcal{C}_0$, with each object $A \in \mathcal{C}_0$ associated with an object $\mathcal{C}(A) =$  in $\mathcal{E}$ called its *extent*.
- For each pair of objects  and  in $\mathcal{C}_0$, a proarrow $\mathcal{C}($, $) =$  in $\mathcal{E}$.
- For each object  in $\mathcal{C}_0$, a 2-cell **id** $=$  called the *identity*.
- For each triple of objects , , , a 2-cell  called *composition*.

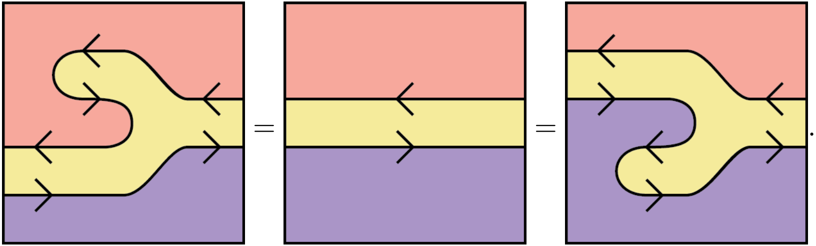
Defining the "Yoneda" Embedding

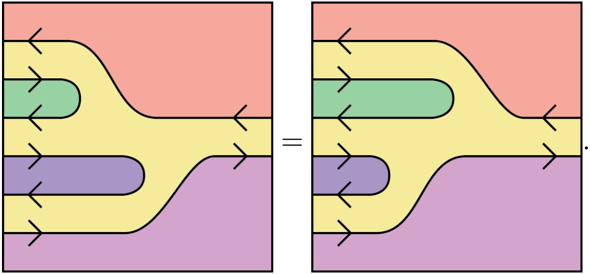
For an object  of $\mathcal{E}$, we define its representative to be

|| :=

- Objects are vertical arrows , with each object's extent being its domain.
- Between objects  and , a hom-object .
- For object , an identity arrow .
- For each composable triple, a composition arrow .

Defining the "Yoneda" Embedding





Properties of "Yoneda" Embedding

Proposition (M.)

The "Yoneda" embedding $|\cdot| : \mathcal{E} \to \mathcal{E}\text{-}\mathbf{Cat}$

- *is full on 2-cells (and therefore faithful on arrows and proarrows);*
- *is full on arrows;*
- *is coreflective on proarrows;*
- *preserves composition;*
- *reflects Morita equivalence.*

Conjecture (M.)

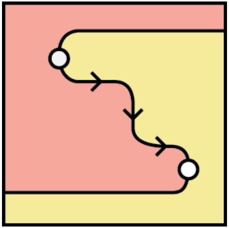
For a "fibrantly enriched" $\mathcal{E}$-category $\mathcal{C}$, denote by $\mathcal{C}[A]$ the full subcategory of $\mathcal{C}$ whose objects have extent A. Then

$$\mathcal{E}\text{-}\mathbf{Cat}(|A|, \mathcal{C}) \simeq \mathcal{C}[A].$$

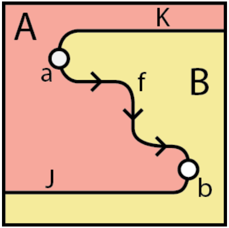
Soundness of Graphical Calculi

- Similar to the proof of Joyal and Street for monoidal categories.
- But using the tile-order machinery of Dawson and Paré to handle the two sorts of composition.

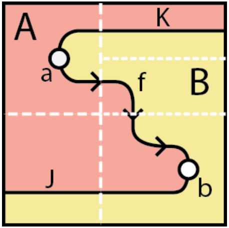
Take a Diagram,



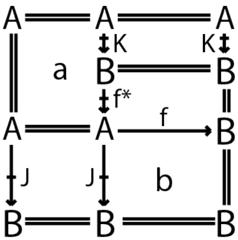
Label it,



Tile it,



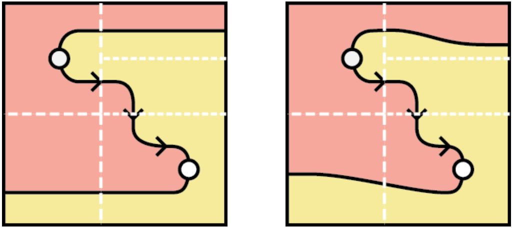
Turn it into the usual notation,



Compose it.

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "J"] \arrow[dr, phantom, "\mathrm{a.u.b}"] & A \arrow[d, "K"'] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

Tilings are stable under small deformations.



In Conclusion

Equipments are fundamental and useful objects

1. for combining “scalar” arrows and “linear” proarrows, and
2. as a setting for formal (enriched, internal, higher) category theory.

I hope that the string diagrams can make working with them easier!

References

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