Pseudo-Functors, Principal Bundles, and Torsors

Octoberfest 2017

Michael Lambert

Dalhousie University

28 October 2017

Outline

References

M.E. Descotte, E. J. Dubuc, and M. Szyld.
On the notion of flat 2-functors.
Preprint https://arxiv.org/abs/1610.09429.

S. Mac Lane and I. Moerdijk.
Sheaves in Geometry and Logic.
Springer, Berlin, 1992.

I. Moerdijk.
Classifying Spaces and Classifying Topoi.
Springer Lecture Notes in Mathematics 1616, Berlin, 1995.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q \colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g \colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e \colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Moerdijk's Definition

Let $\mathrm{Sh}(X)$ denote the category of sheaves on a topological space X.

Definition

A $\mathscr{C}$-principal bundle is a functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathscr{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathscr{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathscr{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
3. for parallel arrows $f, g\colon C \rightrightarrows D$ and $q \in Q(C)_x$ for which $Q(f)(q) = Q(g)(q)$, there is an arrow $e\colon B \to C$ with $fe = ge$ and a $z \in Q(B)_x$ such that $Q(e)(z) = q$.

Condition 2. is transitivity and 3. is freeness.

Guiding Question

If Q is instead a pseudo-functor valued in a 2-category, what is a principal bundle?

Case of interest: pseudo-functors $[\mathscr{X}^{op}, \mathfrak{Cat}]$ on a small category $\mathscr{X}$.

Guiding Question

If Q is instead a pseudo-functor valued in a 2-category, what is a principal bundle?

Case of interest: pseudo-functors $[\mathscr{X}^{op}, \mathfrak{Cat}]$ on a small category $\mathscr{X}$.

Guiding Question

If Q is instead a pseudo-functor valued in a 2-category, what is a principal bundle?

Case of interest: pseudo-functors $[\mathscr{X}^{op}, \mathfrak{Cat}]$ on a small category $\mathscr{X}$.

Theorem

There is an isomorphism

$$\mathbf{Prin}(\mathscr{C}) \cong \mathbf{Geom}(\mathrm{Sh}(X), [\mathscr{C}^{op}, \mathbf{Set}]).$$

Any functor $Q \colon \mathscr{C} \to \mathrm{Sh}(X)$ admits a tensor product $- \otimes_{\mathscr{C}} Q$ extension, which preserves finite limits if, and only if, Q is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$ classifies $\mathscr{C}$-principal bundles.

Theorem

There is an isomorphism

$$\mathbf{Prin}(\mathscr{C}) \cong \mathbf{Geom}(\mathrm{Sh}(X), [\mathscr{C}^{op}, \mathbf{Set}]).$$

Any functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ admits a tensor product $- \otimes_{\mathscr{C}} Q$ extension, which preserves finite limits if, and only if, Q is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$ classifies $\mathscr{C}$-principal bundles.

Theorem

There is an isomorphism

$$\mathbf{Prin}(\mathscr{C}) \cong \mathbf{Geom}(\mathrm{Sh}(X), [\mathscr{C}^{op}, \mathbf{Set}]).$$

Any functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ admits a tensor product $- \otimes_{\mathscr{C}} Q$ extension, which preserves finite limits if, and only if, Q is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$ classifies $\mathscr{C}$-principal bundles.

Theorem

There is an isomorphism

$$\mathbf{Prin}(\mathscr{C}) \cong \mathbf{Geom}(\mathrm{Sh}(X), [\mathscr{C}^{op}, \mathbf{Set}]).$$

Any functor $Q\colon \mathscr{C} \to \mathrm{Sh}(X)$ admits a tensor product $- \otimes_{\mathscr{C}} Q$ extension, which preserves finite limits if, and only if, Q is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos $[\mathscr{C}^{op}, \mathbf{Set}]$ classifies $\mathscr{C}$-principal bundles.

Tensor Product of Presheaves

Any functor $Q\colon \mathscr{C} \to \mathscr{E}$ on small $\mathscr{C}$ to a cocomplete topos $\mathscr{E}$ admits a tensor product extension along the Yoneda embedding

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & \mathscr{E}. \\ {[\mathscr{C}^{op}, \mathbf{Set}]} \arrow[ur, dashed, "- \otimes_{\mathscr{C}} Q"'] & \end{tikzcd}$$

The image $P \otimes_{\mathscr{C}} Q$ is defined as a colimit.

Tensor Product of Presheaves

Any functor $Q\colon \mathscr{C} \to \mathscr{E}$ on small $\mathscr{C}$ to a cocomplete topos $\mathscr{E}$ admits a tensor product extension along the Yoneda embedding

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & \mathscr{E}. \\ {[\mathscr{C}^{op}, \mathbf{Set}]} \arrow[ur, dashed, "- \otimes_{\mathscr{C}} Q"'] & \end{tikzcd}$$

The image $P \otimes_{\mathscr{C}} Q$ is defined as a colimit.

Tensor Product of Presheaves

Any functor $Q\colon \mathscr{C} \to \mathscr{E}$ on small $\mathscr{C}$ to a cocomplete topos $\mathscr{E}$ admits a tensor product extension along the Yoneda embedding

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & \mathscr{E}. \\ {[\mathscr{C}^{op}, \mathbf{Set}]} \arrow[ur, dashed, "- \otimes_{\mathscr{C}} Q"'] & \end{tikzcd}$$

The image $P \otimes_{\mathscr{C}} Q$ is defined as a colimit.

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P\otimes_{\mathscr{C}} Q, X)\cong[\mathscr{C}^{op},\mathbf{Set}](P,\mathscr{E}(Q,X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C}\to\mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is “flat.” In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C},\mathscr{E})\simeq\mathbf{Geom}(\mathscr{E},[\mathscr{C}^{op},\mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P\otimes_{\mathscr{C}} Q, X)\cong [\mathscr{C}^{op},\mathbf{Set}](P,\mathscr{E}(Q,X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C}\to\mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C},\mathscr{E})\simeq \mathbf{Geom}(\mathscr{E},[\mathscr{C}^{op},\mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

The functor $-\otimes_{\mathscr{C}} Q$ is one half of a tensor-hom adjunction

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

Theorem

The tensor-functor $-\otimes_{\mathscr{C}} Q$ arising from $Q\colon \mathscr{C} \to \mathscr{E}$ preserves finite limits if, and only if, Q is filtering.

Such a functor Q is "flat." In the case that $\mathscr{E}$ is **Set** the functor Q is flat if and only if its category of elements $\int_{\mathscr{C}} Q$ is filtered.

Theorem

There is an equivalence

$$\mathbf{Flat}(\mathscr{C}, \mathscr{E}) \simeq \mathbf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathbf{Set}]).$$

This is Theorem VII.5.2 of [MLM92].

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}]}. \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Outline of Our Approach

- Start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk's definition to the case of Q by weakening the equalities to isomorphisms.
- Construct an extension

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] & {[\mathscr{X}^{op}, \mathfrak{Cat}].} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed] & \end{tikzcd}$$

- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.

Main Construction

- Start with pseudo-functors $Q\colon \mathscr{C} \to \mathfrak{Cat}$ and $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(C, p, q) \qquad p \in P(C)_0,\ q \in Q(C)_0$$

and arrows $(C, p, q) \to (D, r, s)$ the triples (f, u, v) with

$$f\colon C \to D \qquad u\colon p \to f^*(r) \qquad v\colon f_!(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

where Σ is the set of cartesian morphisms.

Main Construction

- Start with pseudo-functors $Q\colon \mathscr{C} \to \mathfrak{Cat}$ and $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(C, p, q) \qquad p \in P(C)_0,\ q \in Q(C)_0$$

and arrows $(C, p, q) \to (D, r, s)$ the triples (f, u, v) with

$$f\colon C \to D \qquad u\colon p \to f^*(r) \qquad v\colon f_!(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

where Σ is the set of cartesian morphisms.

Main Construction

- Start with pseudo-functors $Q\colon \mathscr{C} \to \mathfrak{Cat}$ and $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(C, p, q) \qquad p \in P(C)_0,\ q \in Q(C)_0$$

and arrows $(C, p, q) \to (D, r, s)$ the triples (f, u, v) with

$$f : C \to D \qquad u : p \to f^*(r) \qquad v : f_!(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

where Σ is the set of cartesian morphisms.

Main Construction

- Start with pseudo-functors $Q\colon \mathscr{C} \to \mathfrak{Cat}$ and $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(C, p, q) \qquad p \in P(C)_0,\ q \in Q(C)_0$$

and arrows $(C, p, q) \to (D, r, s)$ the triples (f, u, v) with

$$f\colon C \to D \qquad u\colon p \to f^*(r) \qquad v\colon f_!(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

where Σ is the set of cartesian morphisms.

Main Construction

- Start with pseudo-functors $Q \colon \mathscr{C} \to \mathfrak{Cat}$ and $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$.
- Set $\Delta(P, Q)$ to be the category with objects triples

$$(C, p, q) \qquad p \in P(C)_0,\ q \in Q(C)_0$$

and arrows $(C, p, q) \to (D, r, s)$ the triples (f, u, v) with

$$f \colon C \to D \qquad u \colon p \to f^*(r) \qquad v \colon f_!(q) \to s.$$

- Take $P \star Q$ to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

where Σ is the set of cartesian morphisms.

Main Construction Continued

- Now start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- For any pseudo-functor $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$, define another $\mathscr{X}^{op} \to \mathfrak{Cat}$ by assigning
$$X \mapsto P \star Q(-)(X)$$
on objects with the induced assignments on arrows and identity cells.
- This yields a 2-functor
$$- \star Q\colon [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

Main Construction Continued

- Now start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- For any pseudo-functor $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$, define another $\mathscr{X}^{op} \to \mathfrak{Cat}$ by assigning

$$X \mapsto P \star Q(-)(X)$$

on objects with the induced assignments on arrows and identity cells.
- This yields a 2-functor

$$- \star Q\colon [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

Main Construction Continued

- Now start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- For any pseudo-functor $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$, define another $\mathscr{X}^{op} \to \mathfrak{Cat}$ by assigning
$$X \mapsto P \star Q(-)(X)$$
on objects with the induced assignments on arrows and identity cells.
- This yields a 2-functor
$$- \star Q\colon [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

Main Construction Continued

- Now start with a pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$.
- For any pseudo-functor $P\colon \mathscr{C}^{op} \to \mathfrak{Cat}$, define another $\mathscr{X}^{op} \to \mathfrak{Cat}$ by assigning
$$X \mapsto P \star Q(-)(X)$$
on objects with the induced assignments on arrows and identity cells.
- This yields a 2-functor
$$- \star Q\colon [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

Tensor-Hom Adjunction

In general, $-\star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op}, \mathfrak{Cat}](Q,-)\colon [\mathscr{X}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op}, \mathfrak{Cat}].$$

Theorem

For any pseudo-functor Q there is an isomorphism of categories

$$[\mathscr{X}^{op}, \mathfrak{Cat}](P\star Q, F) \cong [\mathscr{C}^{op}, \mathfrak{Cat}](P, [\mathscr{X}^{op}, \mathfrak{Cat}](Q,F)).$$

natural in P and F.

Corollary

The pseudo-functor $P\star Q$ gives a computation of the P-weighted pseudo-colimit of Q.

Tensor-Hom Adjunction

In general, $- \star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op}, \mathfrak{Cat}](Q, -)\colon [\mathscr{X}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op}, \mathfrak{Cat}].$$

Theorem

For any pseudo-functor Q there is an isomorphism of categories

$$[\mathscr{X}^{op}, \mathfrak{Cat}](P \star Q, F) \cong [\mathscr{C}^{op}, \mathfrak{Cat}](P, [\mathscr{X}^{op}, \mathfrak{Cat}](Q, F)).$$

natural in P and F.

Corollary

The pseudo-functor $P \star Q$ gives a computation of the P-weighted pseudo-colimit of Q.

Tensor-Hom Adjunction

In general, $-\star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op}, \mathfrak{Cat}](Q, -) \colon [\mathscr{X}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op}, \mathfrak{Cat}].$$

Theorem

For any pseudo-functor Q there is an isomorphism of categories

$$[\mathscr{X}^{op}, \mathfrak{Cat}](P \star Q, F) \cong [\mathscr{C}^{op}, \mathfrak{Cat}](P, [\mathscr{X}^{op}, \mathfrak{Cat}](Q, F)).$$

natural in P and F.

Corollary

The pseudo-functor $P \star Q$ gives a computation of the P-weighted pseudo-colimit of Q.

Tensor-Hom Adjunction

In general, $-\star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op},\mathfrak{Cat}](Q,-)\colon [\mathscr{X}^{op},\mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op},\mathfrak{Cat}].$$

Theorem

For any pseudo-functor Q there is an isomorphism of categories

$$[\mathscr{X}^{op},\mathfrak{Cat}](P\star Q,F)\cong [\mathscr{C}^{op},\mathfrak{Cat}](P,[\mathscr{X}^{op},\mathfrak{Cat}](Q,F)).$$

natural in P and F.

Corollary

The pseudo-functor $P\star Q$ gives a computation of the P-weighted pseudo-colimit of Q.

Tensor-Hom Adjunction

In general, $- \star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathscr{X}^{op}, \mathfrak{Cat}](Q, -) \colon [\mathscr{X}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op}, \mathfrak{Cat}].$$

Theorem

For any pseudo-functor Q there is an isomorphism of categories

$$[\mathscr{X}^{op}, \mathfrak{Cat}](P \star Q, F) \cong [\mathscr{C}^{op}, \mathfrak{Cat}](P, [\mathscr{X}^{op}, \mathfrak{Cat}](Q, F)).$$

natural in P and F.

Corollary

The pseudo-functor $P \star Q$ gives a computation of the P-weighted pseudo-colimit of Q.

Further Properties

- For any $C \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$QC \simeq \mathbf{y}C \star Q$$

  pseudo-natural in C.
- So, there is a cell

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq"] & {[\mathscr{X}^{op}, \mathfrak{Cat}]} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed, "- \star Q"'] & \end{tikzcd}$$

  making $- \star Q$ an extension of Q.

Corollary

Any pseudo-functor $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$ is a pseudo-colimit of representable functors.

Further Properties

- For any $C \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$QC \simeq \mathbf{y}C \star Q$$

pseudo-natural in C.

- So, there is a cell

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq"] & {[\mathscr{X}^{op}, \mathfrak{Cat}]} \\
{[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed, "- \star Q"'] &
\end{tikzcd}$$

making $- \star Q$ an extension of Q.

Corollary

Any pseudo-functor $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$ is a pseudo-colimit of representable functors.

Further Properties

- For any $C \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$QC \simeq \mathbf{y}C \star Q$$

  pseudo-natural in C.

- So, there is a cell

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq"] & {[\mathscr{X}^{op}, \mathfrak{Cat}]} \\
{[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed, "- \star Q"'] &
\end{tikzcd}$$

  making $- \star Q$ an extension of Q.

Corollary

Any pseudo-functor $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$ is a pseudo-colimit of representable functors.

Further Properties

- For any $C \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$QC \simeq \mathbf{y}C \star Q$$

  pseudo-natural in C.
- So, there is a cell

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq"] & {[\mathscr{X}^{op}, \mathfrak{Cat}]} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed, "- \star Q"'] & \end{tikzcd}$$

  making $- \star Q$ an extension of Q.

Corollary

Any pseudo-functor $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$ is a pseudo-colimit of representable functors.

Further Properties

- For any $C \in \mathscr{C}_0$, there is a pseudo-natural equivalence

$$QC \simeq \mathbf{y}C \star Q$$

  pseudo-natural in C.
- So, there is a cell

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq"] & {[\mathscr{X}^{op}, \mathfrak{Cat}]} \\ {[\mathscr{C}^{op}, \mathfrak{Cat}]} \arrow[ur, dashed, "- \star Q"'] & \end{tikzcd}$$

  making $- \star Q$ an extension of Q.

Corollary

Any pseudo-functor $P \colon \mathscr{C}^{op} \to \mathfrak{Cat}$ is a pseudo-colimit of representable functors.

Pseudo-Coequalizers

The tensor product $P \otimes_{\mathscr{C}} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

$$P \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} Q \underset{\alpha' \times 1}{\overset{1 \times \alpha}{\rightrightarrows}} P \times_{\mathscr{C}_0} Q \dashrightarrow P \otimes_{\mathscr{C}} Q.$$

Theorem

For pseudo-functors P and Q, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

$$\mathscr{P} \times_{\mathscr{C}} \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{Q} \underset{1 \times \nu}{\overset{\mu \times 1}{\rightrightarrows}} \mathscr{P} \times_{\mathscr{C}} \mathscr{Q} \dashrightarrow P \star Q.$$

Pseudo-Coequalizers

The tensor product $P \otimes_{\mathscr{C}} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

$$P \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} Q \overset{1 \times \alpha}{\underset{\alpha' \times 1}{\rightrightarrows}} P \times_{\mathscr{C}_0} Q \dashrightarrow P \otimes_{\mathscr{C}} Q.$$

Theorem

For pseudo-functors P and Q, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

$$\mathscr{P} \times_{\mathscr{C}} \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{Q} \overset{\mu \times 1}{\underset{1 \times \nu}{\rightrightarrows}} \mathscr{P} \times_{\mathscr{C}} \mathscr{Q} \dashrightarrow P \star Q.$$

Pseudo-Coequalizers

The tensor product $P \otimes_{\mathscr{C}} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

$$P \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} Q \underset{\alpha' \times 1}{\overset{1 \times \alpha}{\rightrightarrows}} P \times_{\mathscr{C}_0} Q \dashrightarrow P \otimes_{\mathscr{C}} Q.$$

Theorem

For pseudo-functors P and Q, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

$$\mathscr{P} \times_{\mathscr{C}} \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{Q} \underset{1 \times \nu}{\overset{\mu \times 1}{\rightrightarrows}} \mathscr{P} \times_{\mathscr{C}} \mathscr{Q} \dashrightarrow P \star Q.$$

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_!y \cong q$ and $g_!y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_!q \cong r \qquad v\colon g_!q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_!y \cong q$ making the arrows

$$(fh)_!y \xrightarrow[\cong]{} f_!h_!(y) \xrightarrow[\cong]{f_!w} f_!q \xrightarrow[\cong]{u} r \qquad (gh)_!y \xrightarrow[\cong]{} g_!h_!(y) \xrightarrow[\cong]{g_!w} g_!q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_!y \cong q$ and $g_!y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_!q \cong r \qquad v\colon g_!q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_!y \cong q$ making the arrows

$$(fh)_!y \xrightarrow[\cong]{} f_!h_!(y) \xrightarrow[\cong]{f_!w} f_!q \xrightarrow[\cong]{u} r \qquad (gh)_!y \xrightarrow[\cong]{} g_!h_!(y) \xrightarrow[\cong]{g_!w} g_!q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_! y \cong q$ and $g_! y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_! q \cong r \qquad v\colon g_! q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_! y \cong q$ making the arrows

$$(fh)_! y \xrightarrow[\cong]{} f_! h_!(y) \xrightarrow[\cong]{f_! w} f_! q \xrightarrow[\cong]{u} r \qquad (gh)_! y \xrightarrow[\cong]{} g_! h_!(y) \xrightarrow[\cong]{g_! w} g_! q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_!y \cong q$ and $g_!y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_!q \cong r \qquad v\colon g_!q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_!y \cong q$ making the arrows

$$(fh)_!y \xrightarrow[\cong]{} f_!h_!(y) \xrightarrow[\cong]{f_!w} f_!q \xrightarrow[\cong]{u} r \qquad (gh)_!y \xrightarrow[\cong]{} g_!h_!(y) \xrightarrow[\cong]{g_!w} g_!q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_! y \cong q$ and $g_! y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_! q \cong r \qquad v\colon g_! q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_! y \cong q$ making the arrows

$$(fh)_! y \xrightarrow[\cong]{} f_! h_!(y) \xrightarrow[\cong]{f_! w} f_! q \xrightarrow[\cong]{u} r \qquad (gh)_! y \xrightarrow[\cong]{} g_! h_!(y) \xrightarrow[\cong]{g_! w} g_! q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_! y \cong q$ and $g_! y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_! q \cong r \qquad v\colon g_! q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_! y \cong q$ making the arrows

$$(fh)_! y \xrightarrow[\cong]{} f_! h_!(y) \xrightarrow[\cong]{f_! w} f_! q \xrightarrow[\cong]{u} r \qquad (gh)_! y \xrightarrow[\cong]{} g_! h_!(y) \xrightarrow[\cong]{g_! w} g_! q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Definition

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ provided that for each $X \in \mathscr{X}_0$, each $Q(C)(X)$ is in $\mathfrak{Grpd}$ and

1. there is $C \in \mathscr{C}_0$ such that $Q(C)(X)$ is nonempty;
2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathscr{C}$ and $y \in Q(E)(X)_0$ such that $f_!y \cong q$ and $g_!y \cong r$;
3. and given two arrows $f, g\colon C \rightrightarrows D$ of $\mathscr{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u\colon f_!q \cong r \qquad v\colon g_!q \cong r$$

of $Q(D)(X)$, there is an arrow $h\colon E \to C$ equalizing f and g with an object $y \in Q(E)(X)$ and isomorphism $w\colon h_!y \cong q$ making the arrows

$$(fh)_!y \xrightarrow[\cong]{} f_!h_!(y) \xrightarrow[\cong]{f_!w} f_!q \xrightarrow[\cong]{u} r \qquad (gh)_!y \xrightarrow[\cong]{} g_!h_!(y) \xrightarrow[\cong]{g_!w} g_!q \xrightarrow[\cong]{v} r$$

equal in $Q(D)(X)$.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion
$$\int_{\mathscr{C}} Q(-)(X)$$
is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q \colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat **Set**-valued functor.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion
$$\int_{\mathscr{C}} Q(-)(X)$$
is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q \colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat **Set**-valued functor.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion
$$\int_{\mathscr{C}} Q(-)(X)$$
is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q\colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat **Set**-valued functor.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion
$$\int_{\mathscr{C}} Q(-)(X)$$
is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q\colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat $\mathbf{Set}$-valued functor.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion

$$\int_{\mathscr{C}} Q(-)(X)$$

  is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q\colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat **Set**-valued functor.

Remarks

- The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathscr{X}_0$, the Grothendieck completion
$$\int_{\mathscr{C}} Q(-)(X)$$
is filtered.
- In the case $\mathscr{X} = \mathbf{1}$, the construction $P \star Q$ admits a right calculus of fractions if $\int_{\mathscr{C}} Q$ is filtered.
- The fibers of Q are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.
- When a $\mathscr{C}$-principal bundle $Q \colon \mathscr{C} \to \mathfrak{Cat}$ takes sets as values, it is essentially just a flat **Set**-valued functor.

Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with **2**.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

$$\begin{tikzcd} & F(C \times D) \arrow[dl, "F\pi_C"'] \arrow[d, dashed, "\Theta"] \arrow[dr, "F\pi_D"] & \\ FC & FC \times FD \arrow[l, "\pi_{FC}"] \arrow[r, "\pi_{FD}"'] & FD \end{tikzcd}$$

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.

Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with **2**.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

$$\begin{tikzcd} & F(C \times D) \arrow[dl, "F\pi_C"'] \arrow[d, dashed, "\Theta"] \arrow[dr, "F\pi_D"] & \\ FC & FC \times FD \arrow[l, "\pi_{FC}"] \arrow[r, "\pi_{FD}"'] & FD \end{tikzcd}$$

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.

Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with $\mathbf{2}$.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

$$\begin{tikzcd} & F(C \times D) \arrow[dl, "F\pi_C"'] \arrow[d, dashed, "\Theta"] \arrow[dr, "F\pi_D"] & \\ FC & FC \times FD \arrow[l, "\pi_{FC}"] \arrow[r, "\pi_{FD}"'] & FD \end{tikzcd}$$

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.

Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with $\mathbf{2}$.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

$$\begin{tikzcd} & F(C \times D) \arrow[dl, "F\pi_C"'] \arrow[d, dashed, "\Theta"] \arrow[dr, "F\pi_D"] & \\ FC & FC \times FD \arrow[l, "\pi_{FC}"] \arrow[r, "\pi_{FD}"'] & FD \end{tikzcd}$$

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.

Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with $\mathbf{2}$.
- For F valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

$$\begin{tikzcd} & F(C \times D) \arrow[dl, "F\pi_C"'] \arrow[d, dashed, "\Theta"] \arrow[dr, "F\pi_D"] & \\ FC & FC \times FD \arrow[l, "\pi_{FC}"] \arrow[r, "\pi_{FD}"'] & FD \end{tikzcd}$$

- Say that a pseudo-functor (valued in $[\mathscr{X}^{op}, \mathfrak{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.

Main Result

Theorem

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ if, and only if, the extension $- \star Q$ preserves all finite weighted pseudo-limits.

Main Result

Theorem

A pseudo-functor $Q\colon \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ is a $\mathscr{C}$-principal bundle over $\mathscr{X}$ if, and only if, the extension $- \star Q$ preserves all finite weighted pseudo-limits.

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Remarks on the Proof

- Can reduce to the case where $\mathscr{X}$ is $\mathbf{1}$.
- The definition implies that $\mathbf{1} \star Q \simeq \mathbf{1}$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathfrak{Grpd}$ corresponds to cotensors with $\mathbf{2}$; nontriviality corresponds to $\mathbf{1}$; transitivity to binary products; and freeness to equalizers.
- Proof of sufficiency uses only representables, more-or-less replicating the proof that flat implies filtered in VII.6.3 of [MLM92].

Pseudo-Functors Classify Generalized Principal Bundles

- Let $\mathfrak{Prin}(\mathscr{C})$ denote the 2-category of $\mathscr{C}$-principal bundles.
- Let $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$ denote the 2-category of 2-adjunctions

$$[\mathscr{C}^{op}, \mathfrak{Cat}] \rightleftarrows \mathfrak{Cat}$$

whose left adjoints preserve finite limits.

Theorem

There is a 2-categorical equivalence

$$\mathfrak{Prin}(\mathscr{C}) \simeq \mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}]).$$

Pseudo-Functors Classify Generalized Principal Bundles

- Let $\mathfrak{Prin}(\mathscr{C})$ denote the 2-category of $\mathscr{C}$-principal bundles.
- Let $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$ denote the 2-category of 2-adjunctions

$$[\mathscr{C}^{op}, \mathfrak{Cat}] \rightleftarrows \mathfrak{Cat}$$

whose left adjoints preserve finite limits.

Theorem

There is a 2-categorical equivalence

$$\mathfrak{Prin}(\mathscr{C}) \simeq \mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}]).$$

Pseudo-Functors Classify Generalized Principal Bundles

- Let $\mathfrak{Prin}(\mathscr{C})$ denote the 2-category of $\mathscr{C}$-principal bundles.
- Let $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$ denote the 2-category of 2-adjunctions

$$[\mathscr{C}^{op}, \mathfrak{Cat}] \rightleftarrows \mathfrak{Cat}$$

whose left adjoints preserve finite limits.

Theorem

There is a 2-categorical equivalence

$$\mathfrak{Prin}(\mathscr{C}) \simeq \mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}]).$$

Pseudo-Functors Classify Generalized Principal Bundles

- Let $\mathfrak{Prin}(\mathscr{C})$ denote the 2-category of $\mathscr{C}$-principal bundles.
- Let $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$ denote the 2-category of 2-adjunctions

$$[\mathscr{C}^{op}, \mathfrak{Cat}] \rightleftarrows \mathfrak{Cat}$$

whose left adjoints preserve finite limits.

Theorem

There is a 2-categorical equivalence

$$\mathfrak{Prin}(\mathscr{C}) \simeq \mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}]).$$

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f \colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v) \colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u \colon k^*p \to r$ and $v \colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

  exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.
- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f \colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v) \colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u \colon k^*p \to r$ and $v \colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.
- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f \colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v) \colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u \colon k^* p \to r$ and $v \colon h_! q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.

- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f\colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v)\colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u\colon k^*p \to r$ and $v\colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.
- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f \colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v) \colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u \colon k^*p \to r$ and $v \colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.

- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f \colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v) \colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u \colon k^*p \to r$ and $v \colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

  exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.
- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

What is the tensor product, then?

- For $\mathfrak{Cat}$-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So, $P \otimes_{\mathscr{C}} Q$ has as objects triples (f, p, q) for $f\colon C \to D$ with $p \in PD$ and $q \in QC$ and as arrows those $(h, k, u, v)\colon (f, p, q) \to (g, r, s)$ with $f = kgh$ and $u\colon k^*p \to r$ and $v\colon h_!q \to s$.
- There is an equivalence of categories

$$\mathfrak{Cat}(P \otimes_{\mathscr{C}} Q, \mathscr{A}) \simeq [\mathscr{C}^{op}, \mathfrak{Cat}](P, \mathfrak{Cat}(Q, \mathscr{A}))$$

  exhibiting $P \otimes_{\mathscr{C}} Q$ as the bicolimit of Q weighted by P.
- But in addition $- \otimes_{\mathscr{C}} Q$ is functorial and gives a computation of the left biadjoint of $\mathfrak{Cat}(Q, -)$.

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!

A Brief Recap

- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk's definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors "classify" principal bundles.
- Thank you for your attention!