# Pseudo-Functors, Principal Bundles, and Torsors Octoberfest 2017

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28 October 2017

## Outline

Introduction: Principal Bundles and Geometric Morphisms

Extending a Pseudo-Functor along the Yoneda Embedding

Properties of Main Construction

Generalizing Principal Bundles

Summary and Conclusion

### References



Preprint https://arxiv.org/abs/1610.09429.



Sheaves in Geometry and Logic.

Springer, Berlin, 1992.



Classifying Spaces and Classifying Topoi.

Springer Lecture Notes in Mathematics 1616, Berlin, 1995.

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- 3. for parallel arrows  $f, g: C \Rightarrow D$  and  $g \in Q(C)_x$  for which Q(f)(q) = Q(g)(q), there is an arrow  $e: B \to C$  with fe = ge and a  $z \in Q(B)_x$  such that Q(e)(z) = q.

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A  $\mathscr{C}$ -principal bundle is a functor  $Q \colon \mathscr{C} \to \operatorname{Sh}(X)$  such that for each point  $x \in X$ 

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Condition 2. is transitivity and 3. is freeness.

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Case of interest: pseudo-functors  $[\mathscr{X}^{op},\mathfrak{Cat}]$  on a small category  $\mathscr{X}$ .

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This is proved in [Moe95].

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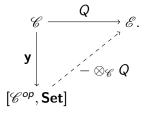
In this sense, the presheaf topos  $[\mathscr{C}^{op}, \mathbf{Set}]$  classifies  $\mathscr{C}$ -principal bundles.

## Tensor Product of Presheaves



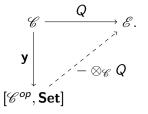
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The image  $P \otimes_{\mathscr{C}} Q$  is defined as a colimit.

$$\mathscr{E}(P \otimes_{\mathscr{C}} Q, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathscr{E}(Q, X)).$$

$$\mathsf{Flat}(\mathscr{C},\mathscr{E}) \simeq \mathsf{Geom}(\mathscr{E}, [\mathscr{C}^{op}, \mathsf{Set}])$$

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Such a functor Q is "flat." In the case that  $\mathscr E$  is **Set** the functor Q is flat

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### Theorem

The tensor-functor  $-\otimes_{\mathscr{C}} Q$  arising from  $Q:\mathscr{C}\to\mathscr{E}$  preserves finite limits if, and only if, Q is filtering.

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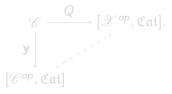
This is Theorem VII.5.2 of [MLM92].

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- Construct an extension



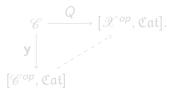
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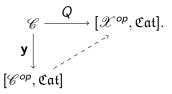
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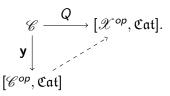
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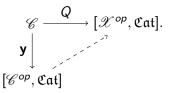
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## Main Construction

- Start with pseudo-functors  $Q: \mathscr{C} \to \mathfrak{Cat}$  and  $P: \mathscr{C}^{op} \to \mathfrak{Cat}$ .
- Set  $\Delta(P,Q)$  to be the category with objects triples

$$(C, p, q)$$
  $p \in P(C)_0, q \in Q(C)_0$ 

$$f: C \to D$$
  $u: p \to f^*(r)$   $v: f_!(q) \to s$ .

Take P \* Q to denote the category of fractions

$$P \star Q := \Delta(P, Q)[\Sigma^{-1}]$$

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where  $\Sigma$  is the set of cartesian morphisms.

- Now start with a pseudo-functor  $Q: \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$ .
- For any pseudo-functor  $P: \mathscr{C}^{op} \to \mathfrak{Cat}$ , define another  $\mathscr{X}^{op} \to \mathfrak{Cat}$  by

$$X \mapsto P \star Q(-)(X)$$

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$$-\star Q: [\mathscr{C}^{op}, \mathfrak{Cat}] \longrightarrow [\mathscr{X}^{op}, \mathfrak{Cat}].$$

$$[\mathscr{X}^{op},\mathfrak{Cat}](Q,-)\colon [\mathscr{X}^{op},\mathfrak{Cat}] \longrightarrow [\mathscr{C}^{op},\mathfrak{Cat}].$$

$$[\mathscr{X}^{op},\mathfrak{Cat}](P\star Q,F)\cong [\mathscr{C}^{op},\mathfrak{Cat}](P,[\mathscr{X}^{op},\mathfrak{Cat}](Q,F)).$$

In general,  $-\star Q$  is a left 2-adjoint. The right adjoint is

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For any pseudo-functor Q there is an isomorphism of categories

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## Corollary

The pseudo-functor  $P \star Q$  gives a computation of the P-weighted pseudo-colimit of Q.

• For any  $C \in \mathcal{C}_0$ , there is a pseudo-natural equivalence

$$QC \simeq yC \star Q$$

So, there is a cell

$$\begin{array}{c|c} \mathscr{C} & \stackrel{Q}{\longrightarrow} [\mathscr{X}^{op}, \mathfrak{Cat}] \\ \mathbf{y} & \stackrel{\simeq}{\longrightarrow} - \star Q \\ \mathscr{C}^{op}, \mathfrak{Cat}] \end{array}$$

• For any  $C \in \mathcal{C}_0$ , there is a pseudo-natural equivalence

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## Corollary

Any pseudo-functor  $P: \mathscr{C}^{op} \to \mathfrak{Cat}$  is a pseudo-colimit of representable functors.

# Pseudo-Coequalizers

$$P \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} Q \xrightarrow{1 \times \alpha} P \times_{\mathscr{C}_0} Q \xrightarrow{---} P \otimes_{\mathscr{C}} Q$$

$$\mathscr{P} \times_{\mathscr{C}} \mathscr{C}^2 \times_{\mathscr{C}} \mathscr{Q} \xrightarrow{\underset{1 \times \nu}{\longrightarrow}} \mathscr{P} \times_{\mathscr{C}} \mathscr{Q} \xrightarrow{---} P \star Q.$$

# Pseudo-Coequalizers

The tensor product  $P \otimes_{\mathscr{C}} Q$  of ordinary presheaves fits into a coequalizer diagram of the form

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### Theorem

For pseudo-functors P and Q, the category of fractions  $P \star Q$  fits into a pseudo-coequalizer diagram

$$\mathscr{P} \times_{\mathscr{C}} \mathscr{C}^{2} \times_{\mathscr{C}} \mathscr{Q} \xrightarrow{\prod_{1 \times \nu}} \mathscr{P} \times_{\mathscr{C}} \mathscr{Q} \xrightarrow{---} P \star Q.$$

$$u: f_! q \cong r$$
  $v: g_! q \cong r$ 

$$(fh)_! y \xrightarrow{\cong} f_! h_!(y) \xrightarrow{f_! w} f_! q \xrightarrow{\underline{u}} r \qquad (gh)_! y \xrightarrow{\cong} g_! h_!(y) \xrightarrow{\underline{g}_! w} g_! q \xrightarrow{\underline{v}} r$$

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A pseudo-functor  $Q: \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$  is a  $\mathscr{C}$ -principal bundle over  $\mathscr{X}$ provided that for each  $X \in \mathcal{X}_0$ , each Q(C)(X) is in  $\mathfrak{Grpd}$  and

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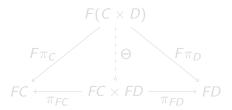
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- When a  $\mathscr{C}$ -principal bundle  $Q:\mathscr{C}\to\mathfrak{Cat}$  takes sets as values, it is essentially just a flat **Set**-valued functor.

# Set-Up for Statement of Main Result

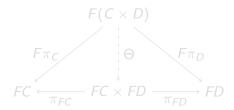
- Weighted pseudo-limits can be constructed from finite products,
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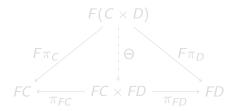
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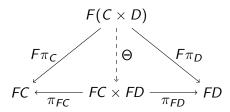
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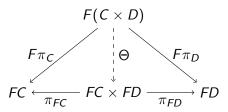
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#### Theorem

A pseudo-functor  $Q: \mathscr{C} \to [\mathscr{X}^{op}, \mathfrak{Cat}]$  is a  $\mathscr{C}$ -principal bundle over  $\mathscr{X}$  if, and only if, the extension  $-\star Q$  preserves all finite weighted pseudo-limits.

- Can reduce to the case where  $\mathscr X$  is 1.
- The definition implies that  $1 \star Q \simeq 1$ . From this it can be seen that
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- Let \( \mathfrak{\Pi}\) in (\( \mathfrak{C} \)) denote the 2-category of \( \mathfrak{C}\)-principal bundles.
- Let  $\mathfrak{Hom}(\mathfrak{Cat}, [\mathscr{C}^{op}, \mathfrak{Cat}])$  denote the 2-category of 2-adjunctions

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#### Theorem

There is a 2-categorical equivalence

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For Cat-valued pseudo-functors P and Q as above define

$$P \otimes_{\mathscr{C}} Q := \mathscr{C}^{op}(-,-) \star P \times Q.$$

- So,  $P \otimes_{\mathscr{C}} Q$  has as objects triples (f, p, q) for  $f: C \to D$  with  $p \in PD$
- There is an equivalence of categories

$$\operatorname{\mathfrak{C}at}(P\otimes_{\operatorname{\mathscr{C}}} Q, \operatorname{\mathscr{A}})\simeq [\operatorname{\mathscr{C}^{op}},\operatorname{\mathfrak{C}at}](P,\operatorname{\mathfrak{C}at}(Q,\operatorname{\mathscr{A}}))$$

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• But in addition  $-\otimes_{\mathscr{C}} Q$  is functorial and gives a computation of the left biadjoint of  $\mathfrak{Cat}(Q, -)$ .

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