Pseudo-Functors, Principal Bundles, and Torsors
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Michael Lambert

Dalhousie University

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Outline

Introduction: Principal Bundles and Geometric Morphisms

Extending a Pseudo-Functor along the Yoneda Embedding

Properties of Main Construction

Generalizing Principal Bundles

Summary and Conclusion
References

M.E. Descotte, E. J. Dubuc, and M. Szyld.  
On the notion of flat 2-functors.  

S. Mac Lane and I. Moerdijk.  
*Sheaves in Geometry and Logic*.  

I. Moerdijk.  
*Classifying Spaces and Classifying Topoi*.  
Moerdijk’s Definition

Let $\mathsf{Sh}(X)$ denote the category of sheaves on a topological space $X$.

**Definition**

A $\mathcal{C}$-principal bundle is a functor $Q: \mathcal{C} \to \mathsf{Sh}(X)$ such that for each point $x \in X$

1. there is a $C \in \mathcal{C}_0$ for which the stalk $Q(C)_x \neq \emptyset$;
2. for any $q \in Q(C)_x$ and $r \in Q(D)_x$ there is a $D \in \mathcal{C}_0$, a span $C \xleftarrow{f} B \xrightarrow{g} D$ in $\mathcal{C}$ and a $z \in Q(B)_x$ such that $Q(f)(z) = q$ and $Q(g)(z) = r$; and
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If $Q$ is instead a pseudo-functor valued in a 2-category, what is a principal bundle?

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Theorem

There is an isomorphism

\[ \text{Prin}(\mathcal{C}) \cong \text{Geom}(\text{Sh}(X), [\mathcal{C}^{\text{op}}, \text{Set}]). \]

Any functor \( Q : \mathcal{C} \to \text{Sh}(X) \) admits a tensor product \(- \otimes_{\mathcal{C}} Q\) extension, which preserves finite limits if, and only if, \( Q \) is a principal bundle.

This is proved in [Moe95].

In this sense, the presheaf topos \([\mathcal{C}^{\text{op}}, \text{Set}]\) classifies \(\mathcal{C}\)-principal bundles.
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Tensor Product of Presheaves

Any functor $Q : \mathcal{C} \rightarrow \mathcal{E}$ on small $\mathcal{C}$ to a cocomplete topos $\mathcal{E}$ admits a tensor product extension along the Yoneda embedding.

$$\mathcal{C} \xrightarrow{Q} \mathcal{E}.\]

$$y \downarrow \quad \mathcal{C}^{op} \times \mathcal{E} \quad Q$$

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The functor $- \otimes_C Q$ is one half of a tensor-hom adjunction

\[ \mathcal{E}(P \otimes_C Q, X) \cong [\mathcal{C}^{\text{op}}, \text{Set}](P, \mathcal{E}(Q, X)) \].

**Theorem**

The tensor-functor $- \otimes_C Q$ arising from $Q: \mathcal{C} \to \mathcal{E}$ preserves finite limits if, and only if, $Q$ is filtering.

Such a functor $Q$ is “flat.” In the case that $\mathcal{E}$ is $\text{Set}$ the functor $Q$ is flat if and only if its category of elements $\int_C Q$ is filtered.

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This is Theorem VII.5.2 of [MLM92].
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- Start with a pseudo-functor $Q : \mathcal{C} \to [\mathcal{X}^{op}, \mathcal{Cat}]$.
- Abstract conditions 2. and 3. of Moerdijk’s definition to the case of $Q$ by weakening the equalities to isomorphisms.
- Construct an extension

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- Investigate the way in which a tensor-hom adjunction, a limit-preserving extension along the Yoneda, and a classifying category are recovered.
- The recent paper [DDS] discusses a general theory of flat 2-functors.
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$$\xymatrix{ \mathcal{C} \ar[r]^-{Q} \ar[d]_-{y} & [\mathcal{X}^{\text{op}}, \mathbf{Cat}] \ar[dl] \cr [\mathcal{C}^{\text{op}}, \mathbf{Cat}] }$$

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Main Construction

• Start with pseudo-functors \( Q : \mathcal{C} \to \mathcal{Cat} \) and \( P : \mathcal{C}^{op} \to \mathcal{Cat} \).
• Set \( \Delta(P, Q) \) to be the category with objects triples
  \[(C, p, q) \quad p \in P(C)_0, \quad q \in Q(C)_0\]
  and arrows \((C, p, q) \to (D, r, s)\) the triples \((f, u, v)\) with
  \[f : C \to D \quad u : p \to f^*(r) \quad v : f_!(q) \to s.\]

• Take \( P \star Q \) to denote the category of fractions
  \[P \star Q := \Delta(P, Q)[\Sigma^{-1}]\]
  where \( \Sigma \) is the set of cartesian morphisms.
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• For any pseudo-functor \( P : \mathcal{C}^{\text{op}} \to \text{Cat} \), define another \( \mathcal{X}^{\text{op}} \to \text{Cat} \) by assigning
  \[
  X \mapsto P \star Q(-)(X)
  \]
on objects with the induced assignments on arrows and identity cells.
• This yields a 2-functor
  \[
  - \star Q : [\mathcal{C}^{\text{op}}, \text{Cat}] \longrightarrow [\mathcal{X}^{\text{op}}, \text{Cat}].
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$$X \mapsto P \ast Q(-)(X)$$

on objects with the induced assignments on arrows and identity cells.

• This yields a 2-functor

$$- \ast Q : [\mathcal{C}^{\text{op}}, \text{Cat}] \longrightarrow [\mathcal{X}^{\text{op}}, \text{Cat}].$$
Main Construction Continued

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Tensor-Hom Adjunction

In general, $- \star Q$ is a left 2-adjoint. The right adjoint is

$$[\mathcal{X}^{\text{op}}, \text{Cat}](Q, -) : [\mathcal{X}^{\text{op}}, \text{Cat}] \to [\mathcal{C}^{\text{op}}, \text{Cat}].$$

**Theorem**

For any pseudo-functor $Q$ there is an isomorphism of categories

$$[\mathcal{X}^{\text{op}}, \text{Cat}](P \star Q, F) \cong [\mathcal{C}^{\text{op}}, \text{Cat}](P, [\mathcal{X}^{\text{op}}, \text{Cat}](Q, F)).$$

natural in $P$ and $F$.

**Corollary**

The pseudo-functor $P \star Q$ gives a computation of the $P$-weighted pseudo-colimit of $Q$. 
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**Corollary**

The pseudo-functor $P \star Q$ gives a computation of the $P$-weighted pseudo-colimit of $Q$. 
Further Properties

- For any $C \in C_0$, there is a pseudo-natural equivalence
  \[ QC \simeq yC \star Q \]
  pseudo-natural in $C$.
- So, there is a cell
  \[ [C^{\text{op}}, \text{Cat}] \xrightarrow{\sim} \mathcal{X}^{\text{op}} \xrightarrow{Q} [C^{\text{op}}, \text{Cat}] \]
  making $- \star Q$ an extension of $Q$.

Corollary

Any pseudo-functor $P : C^{\text{op}} \to \text{Cat}$ is a pseudo-colimit of representable functors.
Further Properties

• For any $C \in \mathcal{C}_0$, there is a pseudo-natural equivalence

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pseudo-natural in $\mathcal{C}$.

• So, there is a cell

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{Q} & [\mathcal{X}^{\text{op}}, \mathcal{Cat}] \\
y & \searrow & \simeq \\
\downarrow & & \downarrow \\
[\mathcal{C}^{\text{op}}, \mathcal{Cat}] & \xrightarrow{\star} & [\mathcal{X}^{\text{op}}, \mathcal{Cat}] \\
\end{array}$$

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Any pseudo-functor $P : \mathcal{C}^{\text{op}} \to \mathcal{Cat}$ is a pseudo-colimit of representable functors.
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\[
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\]

pseudo-natural in \( C \).

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\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{Q} & [\mathcal{K}^{\text{op}}, \mathcal{C}at] \\
\downarrow y & \simeq & \downarrow - \star Q \\
[C^{\text{op}}, \mathcal{C}at] & & \\
\end{array}
\]

making \( - \star Q \) an extension of \( Q \).

Corollary

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  pseudo-natural in $C$.
- So, there is a cell

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{Q} & [\mathcal{X}^{\text{op}}, \mathcal{C}\text{at}] \\
\downarrow{y} & \simeq & \downarrow{\star Q} \\
[C^{\text{op}}, \mathcal{C}\text{at}] & \leftrightarrow & [-, \mathcal{C}\text{at}] 
\end{array}
\]

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Corollary

Any pseudo-functor $P : C^{\text{op}} \to \mathcal{C}\text{at}$ is a pseudo-colimit of representable functors.
Further Properties

- For any $C \in \mathcal{C}_0$, there is a pseudo-natural equivalence
  
  $$QC \simeq yC \star Q$$

  pseudo-natural in $C$.
- So, there is a cell

  $\mathcal{C} \xrightarrow{Q} [\mathcal{X}^{\text{op}}, \mathcal{C}at]$

  $y \downarrow \simeq \nearrow$

  $[\mathcal{C}^{\text{op}}, \mathcal{C}at]$

  making $- \star Q$ an extension of $Q$.

Corollary

*Any pseudo-functor $P: \mathcal{C}^{\text{op}} \to \mathcal{C}at$ is a pseudo-colimit of representable functors.*
Pseudo-Coequalizers

The tensor product $P \otimes_{\mathcal{C}} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

$$
\begin{array}{ccc}
P \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} Q & \xrightarrow{1 \times \alpha} & P \times_{\mathcal{C}_0} Q \\
& \alpha' \times 1 & \end{array}
\longrightarrow
\begin{array}{c}
P \times_{\mathcal{C}} Q
\end{array}
.$$ 

Theorem

For pseudo-functors $P$ and $Q$, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

$$
\begin{array}{ccc}
P \times_{\mathcal{C}} \mathcal{C}^2 \times_{\mathcal{C}} \mathcal{D} & \xrightarrow{\mu \times 1} & P \times_{\mathcal{C}} \mathcal{D} \\
& 1 \times \nu & \end{array}
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$$

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For pseudo-functors $P$ and $Q$, the category of fractions $P \star \mathcal{C} Q$ fits into a pseudo-coequalizer diagram

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The tensor product $P \otimes \mathcal{C} Q$ of ordinary presheaves fits into a coequalizer diagram of the form

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For pseudo-functors $P$ and $Q$, the category of fractions $P \star Q$ fits into a pseudo-coequalizer diagram

$$
P \otimes \mathcal{C} \mathcal{C}^2 \times \mathcal{C} \mathcal{D} \xrightarrow{\mu \times 1} P \otimes \mathcal{C} \mathcal{D} \xrightarrow{1 \times \nu} P \star Q.
$$
Definition

A pseudo-functor $Q : \mathcal{C} \to [\mathcal{X}^{\text{op}}, \text{Cat}]$ is a $\mathcal{C}$-principal bundle over $\mathcal{X}$ provided that for each $X \in \mathcal{X}_0$, each $Q(C)(X)$ is in $\text{Grpd}$ and

1. there is $C \in \mathcal{C}_0$ such that $Q(C)(X)$ is nonempty;

2. for $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$, there is a span $C \xleftarrow{f} E \xrightarrow{g} D$ in $\mathcal{C}$ and $y \in Q(E)(X)_0$ such that $f!y \simeq q$ and $g!y \simeq r$;

3. and given two arrows $f, g : C \Rightarrow D$ of $\mathcal{C}$ and objects $q \in Q(C)(X)_0$ and $r \in Q(D)(X)_0$ with isomorphisms

$$u : f!q \simeq r \quad v : g!q \simeq r$$

of $Q(D)(X)$, there is an arrow $h : E \to C$ equalizing $f$ and $g$ with an object $y \in Q(E)(X)$ and isomorphism $w : h!y \simeq q$ making the arrows

$$(fh)!y \xrightarrow{\sim} f!h!(y) \xrightarrow{f!w} f!q \xrightarrow{u} r \quad (gh)!y \xrightarrow{\sim} g!h!(y) \xrightarrow{g!w} g!q \xrightarrow{v} r$$

equal in $Q(D)(X)$. 
Definition

A pseudo-functor $Q : \mathcal{C} \to [\mathcal{X}^{op}, \mathbf{Cat}]$ is a $\mathcal{C}$-principal bundle over $\mathcal{X}$ provided that for each $X \in \mathcal{X}_0$, each $Q(C)(X)$ is in $\mathbf{Grpd}$ and

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$$(f h) ! y \xrightarrow{u} f ! h ! (y) \xrightarrow{f ! w} f ! q \xrightarrow{u} r \quad (g h) ! y \xrightarrow{v} g ! h ! (y) \xrightarrow{g ! w} g ! q \xrightarrow{v} r$$

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   \end{align*}$$
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Definition

A pseudo-functor $Q: \mathcal{C} \to [\mathcal{X}^{\text{op}}, \text{Cat}]$ is a $\mathcal{C}$-principal bundle over $\mathcal{X}$ provided that for each $X \in \mathcal{X}_0$, each $Q(C)(X)$ is in $\text{Grpd}$ and

1. there is $C \in \mathcal{C}_0$ such that $Q(C)(X)$ is nonempty;
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A pseudo-functor \( Q : \mathcal{C} \to [\mathcal{X}^{op}, \text{Cat}] \) is a \( \mathcal{C} \)-principal bundle over \( \mathcal{X} \) provided that for each \( X \in \mathcal{X}_0 \), each \( Q(C)(X) \) is in \( \text{Grpd} \) and

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equal in $Q(D)(X)$. 

Remarks

• The definition is essentially that each $Q(C)(X)$ is a groupoid and for each $X \in \mathcal{X}_0$, the Grothendieck completion

$$\int_{\mathcal{E}} Q(-)(X)$$

is filtered.

• In the case $\mathcal{X} = 1$, the construction $P * Q$ admits a right calculus of fractions if $\int_{\mathcal{E}} Q$ is filtered.

• The fibers of $Q$ are preordered. So, a principal bundle is basically a system of discrete opfibrations each of which is a cofiltered colimit.

• When a $\mathcal{C}$-principal bundle $Q: \mathcal{C} \to \mathsf{Cat}$ takes sets as values, it is essentially just a flat $\mathsf{Set}$-valued functor.
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Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with $2$.
- For $F$ valued in $[\mathcal{K}^{\text{op}}, \text{Cat}]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

\[
\begin{array}{ccc}
F(C \times D) & \xrightarrow{\pi_C} & \Theta & \xleftarrow{\pi_D} \\
\downarrow & & \downarrow & & \downarrow \\
FC & \xleftarrow{\pi_{FC}} & FC \times FD & \xrightarrow{\pi_{FD}} & FD
\end{array}
\]

- Say that a pseudo-functor (valued in $[\mathcal{K}^{\text{op}}, \text{Cat}]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.
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\[
\begin{array}{c}
\text{FC} \\ F_{\pi_{FC}}
\end{array}
\begin{array}{c}
\downarrow \\
\Theta
\end{array}
\begin{array}{c}
\text{FC} \times \text{FD} \\ F\pi_{FD}
\end{array}
\begin{array}{c}
\downarrow \\
\text{FD}
\end{array}
\begin{array}{c}
\text{FC} \times D \\ F(C \times D)
\end{array}
\begin{array}{c}
F\pi_C \\
\downarrow
\end{array}
\begin{array}{c}
\text{FC} \\ \pi_{FC}
\end{array}
\begin{array}{c}
\leftarrow
\end{array}
\begin{array}{c}
\text{FC} \times \text{FD} \\ \pi_{FD}
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
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Set-Up for Statement of Main Result

- Weighted pseudo-limits can be constructed from finite products, pseudo-equalizers, and cotensors with $2$.
- For $F$ valued in $[\mathcal{X}^{op}, \mathcal{C}at]$, there is an induced canonical functor from the image of a limit to the limit of the images. For example, binary products

\[
\begin{array}{ccc}
F(C \times D) & \xrightarrow{\Theta} & F\pi_D \\
F\pi_C & \downarrow & \downarrow \\
FC & \xleftarrow{\pi_{FC}} & FC \times FD & \xrightarrow{\pi_{FD}} & FD
\end{array}
\]

- Say that a pseudo-functor (valued in $[\mathcal{X}^{op}, \mathcal{C}at]$) preserves a type of finite pseudo-limit if (the components of) the corresponding canonical functors are weak equivalences.
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Theorem

A pseudo-functor $Q : \mathcal{C} \to [\mathcal{X}^{\text{op}}, \text{Cat}]$ is a $\mathcal{C}$-principal bundle over $\mathcal{X}$ if, and only if, the extension $- \star Q$ preserves all finite weighted pseudo-limits.
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Remarks on the Proof

- Can reduce to the case where $\mathcal{X}$ is $1$.
- The definition implies that $1 \star Q \simeq 1$. From this it can be seen that all the canonical maps are fully faithful.
- The proof of essential surjectivity a pattern: fibred in $\mathcal{G}rp$ corresponds to cotensors with $2$; nontriviality corresponds to $1$; transitivity to binary products; and freeness to equalizers.
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Generalizing Principal Bundles

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Pseudo-Functors Classify Generalized Principal Bundles

- Let $\text{Prin}(\mathcal{C})$ denote the 2-category of $\mathcal{C}$-principal bundles.
- Let $\text{Hom}(\text{Cat}, [\mathcal{C}^{\text{op}}, \text{Cat}])$ denote the 2-category of 2-adjunctions
  \[ [\mathcal{C}^{\text{op}}, \text{Cat}] \rightleftarrows \text{Cat} \]

  whose left adjoints preserve finite limits.

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There is a 2-categorical equivalence

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Generalizing Principal Bundles

What is the tensor product, then?

• For \( \text{Cat} \)-valued pseudo-functors \( P \) and \( Q \) as above define

\[
P \otimes_C Q := \mathcal{C}^{\text{op}}(-, -) \star P \times Q.
\]

• So, \( P \otimes_C Q \) has as objects triples \( (f, p, q) \) for \( f : C \to D \) with \( p \in PD \) and \( q \in QC \) and as arrows those \( (h, k, u, v) : (f, p, q) \to (g, r, s) \) with \( f = kgh \) and \( u : k^*p \to r \) and \( v : h^!q \to s \).

• There is an equivalence of categories

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\text{Cat}(P \otimes_C Q, \mathcal{A}) \simeq [\mathcal{C}^{\text{op}}, \text{Cat}](P, \text{Cat}(Q, \mathcal{A}))
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exhibiting \( P \otimes_C Q \) as the bicolimit of \( Q \) weighted by \( P \).

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- A definition of a principal bundle for an indexed category-valued pseudo-functor on a 1-category modeled on Moerdijk’s definition can be made.
- A tensor-hom adjunction can be recovered.
- A bimodule is a principal bundle if, and only if, its corresponding extension along the Yoneda embedding preserves finite weighted pseudo-limits.
- Pseudo-functors “classify” principal bundles.
- Thank you for your attention!
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