Strong conceptual completeness for Boolean coherent classifying toposes

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- A strong conceptual completeness statement for a logical doctrine is an assertion that a theory in this logical doctrine can be recovered from an appropriate structure formed by the models of the theory.

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Applications of strong conceptual completeness

A definability criterion for $\aleph_0$-categorical theories

Exotic functors

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- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory T from $\mathbf{Mod}(T)$ equipped with structure induced by taking ultraproducts.

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- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory T from $\mathbf{Mod}(T)$ equipped with structure induced by taking ultraproducts.
- Before we dive in, let's look at a well-known theorem from model theory, with the same flavor, which Makkai's result generalizes: the Beth definability theorem.

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Theorem.

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1. *There is a L_0-theory T_0 and a factorization:*

$$\begin{tikzcd} \mathbf{Mod}(T_1) \arrow[r, "F"] \arrow[dr, "\simeq"'] & \mathbf{Mod}(\varnothing_{L_0}) \\ & \mathbf{Mod}(T_0) \arrow[u] \end{tikzcd}$$

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6. *Every L_0-elementary map is an L_1-homomorphism of structures.*

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Then: (*) Every L_1-formula is T_1-provably equivalent to an L_0-formula.

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Then: if X commutes with ultraproducts on the nose ("satisfies a Łos' theorem"), then X was definable, i.e. X is an evaluation functor for some definable set $\varphi \in \mathbf{Def}(T)$.

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Proof.

(Sketch): expand each model M of T by a new sort $X(M)$. Use commutation with ultraproducts to verify this is an elementary class. Then we are in the situation of $1 \implies (*)$ from Beth's theorem. $\square$

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- Plain old conceptual completeness (this was one of the key results of Makkai-Reyes) says that if an interpretation $I : T_1 \to T_2$ induces an equivalence of categories $\mathbf{Mod}(T_1) \overset{I^*}{\simeq} \mathbf{Mod}(T_2)$, then I must have been a bi-interpretation.

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- Strong conceptual completeness is the following upgrade of the corollary.

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- *for every ultraproduct $\prod_{i \to \mathcal{U}} M_i$ a way to identify* $X(\prod_{i \to \mathcal{U}} M_i) \overset{\Phi_{(M_i)}}{\simeq} \prod_{i \to \mathcal{U}} X(M_i)$ *("there exists a transition isomorphism"), such that*

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Then: there exists a $\varphi(x) \in T^{\mathrm{eq}}$ *such that* $X \simeq \mathrm{ev}_{\varphi(x)}$ *as functors* $\mathbf{Mod}(T) \to \mathbf{Set}$*.*

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Then: there exists a $\varphi(x) \in T^{\mathrm{eq}}$ such that $X \simeq \mathrm{ev}_{\varphi(x)}$ as functors $\mathbf{Mod}(T) \to \mathbf{Set}$. (We call such X an ultrafunctor.)

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- That is, the specified transition isomorphisms $\Phi_{(M_i)} : X\left(\prod_{i \to \mathcal{U}} M_i\right) \to \prod_{i \to \mathcal{U}} X(M_i)$ make all diagrams of the form

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow{\Phi_{(M_i)}} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle X\left(\prod_{i\to\mathcal{U}} f_i\right)}\downarrow & & \downarrow{\scriptstyle \prod_{i\to\mathcal{U}} X(f_i)} \\ X\left(\prod_{i\to\mathcal{U}} N_i\right) & \xrightarrow[\Phi_{(N_i)}]{} & \prod_{i\to\mathcal{U}} X(N_i) \end{array}$$

commute ("transition isomorphism/pre-ultrafunctor condition").

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- For any bound node $\beta \in \Gamma^b$, we assign a triple $\langle I, \mathcal{U}, g \rangle \overset{\text{df}}{=} \langle I_\beta, \mathcal{U}_\beta, g_\beta \rangle$ where $\mathcal{U}$ is an ultrafilter on I and g is a function $g : I \to \Gamma^f$.

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- An ultradiagram for Γ is a diagram of shape Γ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions g.

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- An ultradiagram for Γ is a diagram of shape Γ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions g.
- A *morphism* of ultradiagrams (for fixed Γ) is just a natural transformation of functors which respects the extra data: the component of the transformation at a bound node is the ultraproduct of the components for the indexing free nodes.

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Let $\mathrm{Hom}(\Gamma, \underline{\mathbf{S}})$ *be the category of all ultradiagrams of type* Γ *inside* $\underline{\mathbf{S}}$ *with morphisms the ultradiagram morphisms defined above. Any two nodes* $k, \ell \in \Gamma$ *define evaluation functors* $(k), (\ell) : \mathrm{Hom}(\Gamma, \underline{\mathbf{S}}) \rightrightarrows \mathbf{S}$, *by*

$$(k)\left(A \xrightarrow{\Phi} B\right) = A(k) \xrightarrow{\Phi_k} B(k)$$

(resp. ℓ*).*

An **ultramorphism** *of type* $\langle \Gamma, k, \ell \rangle$ *in* $\underline{\mathbf{S}}$ *is a natural transformation* $\delta : (k) \to (\ell)$.

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It's sufficient to consider the ultramorphisms which come from universal properties of colimits of products in **Set**.

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- We dropped the *subfunctor* assumption! We don't have such a nice way of knowing exactly how $X(M)$ is obtained from M. We only have the invariance under ultra-stuff. We've left the placental warmth of the ambient models and we're considering some kind of abstract permutation representation of $\mathbf{Mod}(T)$.

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- Yet, if X respects enough of the structure induced by the ultra-stuff, then X must have been constructible from our models in some first-order way ("is definable").

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- Yet, if X respects enough of the structure induced by the ultra-stuff, then X must have been constructible from our models in some first-order way ("is definable").
- (With this new language, the corollary becomes: "strict sub-pre-ultrafunctors of definable functors are definable.")

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- In particular, from this adjunction we get $\mathbf{Pretop}(T_1, T_2) \simeq \mathbf{Ult}(\mathbf{Mod}(T_2), \mathbf{Mod}(T_1))$.

Therefore, SCC tells us how to recognize a reduct functor in the wild between two categories of models—i.e., if there is some uniformity underlying a functor $\mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ due to a purely syntactic assignment $T_1 \to T_2$. Just check if the ultra-structure is preserved!

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- So the only way to actually do this is to recognize some kind of uniformity in the putative reduct functor which lets you take care of all the ultramorphisms at once.
- But it gives you another way to think about uniformities you need.
- It also gives you a way to check that something can never arise from any interpretation!

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- What are ultramorphisms? Loosely speaking, ultraproducts are a kind of universal construction in **Set**, and so there are certain canonical comparison maps between them induced by their universal properties. (By the Los theorem, these things are "absolute" in the sense that no matter what first-order structure you put on a set, these maps will always be elementary embeddings.)

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- Keep in mind these two examples:

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- *The* *diagonal embedding* *into an ultrapower.*
- *Generalized diagonal embeddings. More generally, let* $f : I \to J$ *be a function, let* $\mathcal{U}$ *be an ultrafilter on* I *and let* $\mathcal{V}$ *be the pushforward ultrafilter on* J*. Then for any* I*-indexed sequence of structures* $(M_i)_{i \in I}$*, there is a canonical map* $\delta_f : \prod_{j \to \mathcal{V}} M_{f(i)} \to \prod_{i \to \mathcal{U}} M_i$ *given by taking the diagonal embedding along each fiber of* f*.*

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Equip automorphism groups with the topology of pointwise convergence.

Δ-functors induce continuous maps on automorphism groups

- Why should we expect ultramorphisms to help us identify evaluation functors in the wild?
- Here's an result which might indicate that knowing that they're preserved tells us something nontrivial.

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Equip automorphism groups with the topology of pointwise convergence.

Theorem.

If X is a Δ-functor from $\mathbf{Mod}(T)$ to $\mathbf{Mod}(T')$, then X restricts to a continuous map $\mathrm{Aut}(M) \to \mathrm{Aut}(X(M))$ for every $M \in \mathbf{Mod}(T)$.

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- Applying X and using that X is a Δ-functor, conclude that $\prod_{i \to \mathcal{U}} X(f_i)$ agrees with $\prod_{i \to \mathcal{U}} X(f)$ over the diagonal copy of $X(M)$ inside $X(M)^{\mathcal{U}}$.

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- For any point $a \in X(M)$, the above says the sequence $(X(f_i)(a))_{i \in I} =_{\mathcal{U}} (X(f)(a))_{i \in I}$.

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- Since $\mathcal{U}$ was arbitrary and the cofinite filter on I is the intersection of all non-principal ultrafilters on I, we conclude that the above equation holds cofinitely. Hence, $X(f_i) \to X(f)$.

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- A theorem of Coquand, Ahlbrandt and Ziegler says that, given two $\aleph_0$-categorical theories T and T' with countable models M and M', a topological isomorphism $\mathrm{Aut}(M) \simeq \mathrm{Aut}(M')$ induces a bi-interpretation $M \simeq M'$.

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- Since we know Δ-functors induce continuous maps on automorphism groups, they're a good candidate for definable functors.
- Boolean coherent toposes split into a finite coproduct of $\mathscr{E}(T_i)$, where each T_i *is* $\aleph_0$-categorical.

A definability criterion for $\aleph_0$-categorical theories

Theorem.

Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$. If T is $\aleph_0$-categorical, the following are equivalent:

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1. *For some transition isomorphism, (X, Φ) is a Δ-functor (preserves ultraproducts and diagonal maps).*
2. *For some transition isomorphism, (X, Φ) is definable.*

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Proof.

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- Let M be the countable model. Use the lemma about Δ-functors (X, Φ) inducing continuous maps on the automorphism groups (equivalently, (X, Φ) has the finite support property) to cover each $\mathrm{Aut}(M)$-orbit of $X(M)$ by a projection from an $\mathrm{Aut}(M)$-orbit of M. By ω-categoricity, the kernel relation of this projection is definable, so we know that $X(M)$ looks like an (*a priori*, possibly infinite) disjoint union of types.

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- By $\mathrm{Aut}(M)^{\mathcal{U}}$ orbit-counting, there are actually only finitely many types.
- Invoke the Keisler-Shelah theorem to transfer to all $N \models T$.

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Corollary.

Let T and T' be $\aleph_0$-categorical. Let X be an equivalence of categories

$$\mathbf{Mod}(T_1) \overset{X}{\simeq} \mathbf{Mod}(T_2).$$

Then X was induced by a bi-interpretation $T_1 \simeq T_2$ if and only if X was a Δ-functor.

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In particular, Bodirsky, Evans, Kompatscher and Pinkser gave an example of two $\aleph_0$-categorical theories T, T' with abstractly isomorphic but not topologically isomorphic automorphism groups of the countable model. This abstract isomorphism induces an equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ and since it can't come from an interpretation, from the corollary we conclude that it fails to preserve an ultraproduct or a diagonal map was not preserved.

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The previous definability criterion fails for general T. That is:

- *There exists a theory T and a Δ-functor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ which is not definable.*
- *There exists a theory T and a pre-ultrafunctor (X, Φ) which is not a Δ-functor (hence, is also not definable.)*

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- Taking ultraproducts creates external realizations ("infinite/infinitesimal points") of either one.
- You can either try to construct a transition isomorphism which turns it into a pre-ultrafunctor (creating a non-Δ pre-ultrafunctor) or obtain one non-constructively (creating a non-definable Δ-functor).

$\square$

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- How do ultramorphisms relate to the Awodey-Forssell duality?
- Conjecture: the pre-ultrafunctor part of the data ensures compactness after you get inside the classifying topos, i.e. if you start with $A \in \mathscr{E}$ and ev_A is an ultrafunctor, then A was compact.

Thank you!