Jesse Han

Strong conceptua completeness

Applications of strong conceptua completeness

A definability criterion for %0-categorical theories

Evotic functors

Strong conceptual completeness for Boolean coherent classifying toposes

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McMaster University

CT Octoberfest 2017, CMU

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Evotic functor

What is strong conceptual completeness for first-order logic?

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What is strong conceptual completeness for first-order logic?

A strong conceptual completeness statement for a logical doctrine is an assertion that a theory in this logical doctrine can be recovered from an appropriate structure formed by the models of the theory.

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What is strong conceptual completeness for first-order logic?

- A strong conceptual completeness statement for a logical doctrine is an assertion that a theory in this logical doctrine can be recovered from an appropriate structure formed by the models of the theory.
- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory T from Mod(T) equipped with structure induced by taking ultraproducts.

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What is strong conceptual completeness for first-order logic?

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- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory T from Mod(T) equipped with structure induced by taking ultraproducts.
- Before we dive in, let's look at a well-known theorem from model theory, with the same flavor, which Makkai's result generalizes: the Beth definability theorem.

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The Beth theorem

Theorem.

Let $L_0 \subseteq L_1$ be an inclusion of languages with no new sorts.

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Let $L_0 \subseteq L_1$ be an inclusion of languages with no new sorts. Let T_1 be an L_1 -theory.

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Let $L_0 \subseteq L_1$ be an inclusion of languages with no new sorts. Let T_1 be an L_1 -theory. Let $F: \mathbf{Mod}(T_1) \to \mathbf{Mod}(\emptyset_{L_0})$ be the reduct functor.

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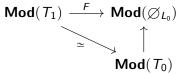
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1. There is a L_0 -theory T_0 and a factorization:

$$\begin{array}{ccc} \operatorname{\mathsf{Mod}}(T_1) & \xrightarrow{F} & \operatorname{\mathsf{Mod}}(\varnothing_{L_0}) \\ & & & \uparrow \\ & & \operatorname{\mathsf{Mod}}(T_0) \end{array}$$

2. F is full and faithful.

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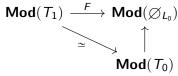
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- 2. F is full and faithful.
- 3. F is injective on objects.

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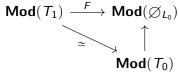
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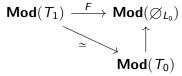
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- 2. F is full and faithful.
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- 5. F is full and faithful on $\operatorname{Hom}_{L_1}(M, M^{\mathcal{U}})$ for all $M \in \operatorname{\mathbf{Mod}}(T_1)$ and all ultrafilters \mathcal{U} .

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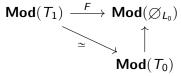
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- 5. F is full and faithful on $\operatorname{Hom}_{L_1}(M, M^{\mathcal{U}})$ for all $M \in \operatorname{\mathbf{Mod}}(T_1)$ and all ultrafilters \mathcal{U} .
- Every L₀-elementary map is an L₁-homomorphism of structures.

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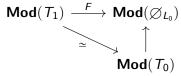
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- 5. F is full and faithful on $\operatorname{Hom}_{L_1}(M, M^{\mathcal{U}})$ for all $M \in \operatorname{\mathbf{Mod}}(T_1)$ and all ultrafilters \mathcal{U} .
- 6. Every L₀-elementary map is an L₁-homomorphism of structures.

<u>Then:</u> (*) Every L_1 -formula is T_1 -provably equivalent to an L_0 -formula.

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Corollary.

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Useful consequence of Beth's theorem

Corollary.

Let T be an L-theory, let \overline{S} be a finite product of sorts. Let $X: \mathbf{Mod}(T) \to \mathbf{Set}$ be a subfunctor of $M \mapsto \overline{S}(M)$.

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Corollary.

Let T be an L-theory, let \overline{S} be a finite product of sorts. Let $X: \mathbf{Mod}(T) \to \mathbf{Set}$ be a subfunctor of $M \mapsto \overline{S}(M)$.

<u>Then</u>: if X commutes with ultraproducts on the nose ("satisfies a Łos' theorem"), then X was definable, i.e. X is an evaluation functor for some definable set $\varphi \in \mathbf{Def}(T)$.

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Corollary.

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<u>Then</u>: if X commutes with ultraproducts on the nose ("satisfies a Łos' theorem"), then X was definable, i.e. X is an evaluation functor for some definable set $\varphi \in \mathbf{Def}(T)$.

Proof.

(Sketch): expand each model M of T by a new sort X(M). Use commutation with ultraproducts to verify this is an elementary class. Then we are in the situation of $1 \implies (*)$ from Beth's theorem.

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How does strong conceptual completeness enter this picture?

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How does strong conceptual completeness enter this picture?

Plain old conceptual completeness (this was one of the key results of Makkai-Reyes) says that if an interpretation $I: T_1 \to T_2$ induces an equivalence of categories $\mathbf{Mod}(T_1) \overset{I^*}{\simeq} \mathbf{Mod}(T_2)$, then I must have been a bi-interpretation.

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- Strong conceptual completeness is the following upgrade of the corollary.

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Theorem.

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Theorem.

Let T be an L-theory. Let X be any functor $\mathbf{Mod}(T) \to \mathbf{Set}$. Suppose that you have:

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Theorem.

Let T be an L-theory. Let X be any functor $\mathbf{Mod}(T) \to \mathbf{Set}$. Suppose that you have:

for every ultraproduct $\prod_{i \to \mathcal{U}} M_i$ a way to identify $X(\prod_{i \to \mathcal{U}} M_i) \overset{\Phi_{(M_i)}}{\simeq} \prod_{i \to \mathcal{U}} X(M_i)$ ("there exists a transition isomorphism"), such that

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- (X, Φ) preserves ultraproducts of models/elementary embeddings ("is a pre-ultrafunctor"), and also

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<u>Then</u>: there exists a $\varphi(x) \in T^{eq}$ such that $X \simeq ev_{\varphi(x)}$ as functors $\mathbf{Mod}(T) \to \mathbf{Set}$.

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<u>Then</u>: there exists a $\varphi(x) \in T^{eq}$ such that $X \simeq ev_{\varphi(x)}$ as functors $Mod(T) \to Set$. (We call such X an ultrafunctor.)

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That is, the specified transition isomorphisms $\Phi_{(M_i)}: X\left(\prod_{i \to \mathcal{U}} M_i\right) \to \prod_{i \to \mathcal{U}} X(M_i)$ make all diagrams of the form

$$X\left(\prod_{i\to\mathcal{U}}M_{i}\right)\xrightarrow{\Phi_{(M_{i})}}\prod_{i\to\mathcal{U}}X(M_{i})$$

$$X\left(\prod_{i\to\mathcal{U}}f_{i}\right)\downarrow \qquad \qquad \downarrow \prod_{i\to\mathcal{U}}X(f_{i})$$

$$X\left(\prod_{i\to\mathcal{U}}N_{i}\right)\xrightarrow{\Phi_{(N_{i})}}\prod_{i\to\mathcal{U}}X(N_{i})$$

commute ("transition isomorphism/pre-ultrafunctor condition").

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What are ultramorphisms?

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What are ultramorphisms?

An **ultragraph** Γ comprises:

A directed graph whose vertices are partitioned into *free* nodes Γ^f and bound nodes Γ^b .

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What are ultramorphisms?

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- A directed graph whose vertices are partitioned into *free* nodes Γ^f and bound nodes Γ^b .
- For any bound node $\beta \in \Gamma^b$, we assign a triple $\langle I, \mathcal{U}, g \rangle \stackrel{\text{df}}{=} \langle I_{\beta}, \mathcal{U}_{\beta}, g_{\beta} \rangle$ where \mathcal{U} is an ultrafilter on I and g is a function $g: I \to \Gamma^f$.

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- An ultradiagram for Γ is a diagram of shape Γ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions g.

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- An ultradiagram for Γ is a diagram of shape Γ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions g.
- A morphism of ultradiagrams (for fixed Γ) is just a natural transformation of functors which respects the extra data: the component of the transformation at a bound node is the ultraproduct of the components for the indexing free nodes.

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Okay, but what are ultramorphisms?

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Okay, but what are ultramorphisms?

Definition.

Let $\mathsf{Hom}(\Gamma, \underline{\mathbf{S}})$ be the category of all ultradiagrams of type Γ inside $\underline{\mathbf{S}}$ with morphisms the ultradiagram morphisms defined above. Any two nodes $k, \ell \in \Gamma$ define evaluation functors $(k), (\ell) : \mathsf{Hom}(\Gamma, \underline{\mathbf{S}}) \rightrightarrows \mathbf{S}$, by

$$(k)\left(A \xrightarrow{\Phi} B\right) = A(k) \xrightarrow{\Phi_k} B(k)$$

(resp. ℓ).

An ultramorphism of type $\langle \Gamma, k, \ell \rangle$ in $\underline{\mathbf{S}}$ is a natural transformation $\delta : (k) \to (\ell)$.

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It's sufficient to consider the ultramorphisms which come from universal properties of colimits of products in **Set**.

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Now, what's changed between this statement and that of the useful corollary to Beth's theorem?

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Strong conceptual completeness, II

Now, what's changed between this statement and that of the useful corollary to Beth's theorem?

We dropped the *subfunctor* assumption! We don't have such a nice way of knowing exactly how X(M) is obtained from M. We only have the invariance under ultra-stuff. We've left the placental warmth of the ambient models and we're considering some kind of abstract permutation representation of $\mathbf{Mod}(T)$.

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- Yet, if X respects enough of the structure induced by the ultra-stuff, then X must have been constructible from our models in some first-order way ("is definable").

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- Yet, if X respects enough of the structure induced by the ultra-stuff, then X must have been constructible from our models in some first-order way ("is definable").
- (With this new language, the corollary becomes: "strict sub-pre-ultrafunctors of definable functors are definable.")

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Actually, Makkai proved something more, by doing the following:

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Strong conceptual completeness, III

Actually, Makkai proved something more, by doing the following:

Introduce the notions of ultracategory and ultrafunctors by requiring all this extra ultra-stuff to be preserved.

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Actually, Makkai proved something more, by doing the following:

- Introduce the notions of ultracategory and ultrafunctors by requiring all this extra ultra-stuff to be preserved.
- Develop a general duality theory between pretoposes ("Def(T)") and ultracategories ("Mod(T)") via a contravariant 2-adjunction ("generalized Stone duality").

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- Develop a general duality theory between pretoposes (" $\mathbf{Def}(T)$ ") and ultracategories (" $\mathbf{Mod}(T)$ ") via a contravariant 2-adjunction ("generalized Stone duality").
- In particular, from this adjunction we get $Pretop(T_1, T_2) \simeq Ult(Mod(T_2), Mod(T_1)).$

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Actually, Makkai proved something more, by doing the following:

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- Develop a general duality theory between pretoposes (" $\mathbf{Def}(T)$ ") and ultracategories (" $\mathbf{Mod}(T)$ ") via a contravariant 2-adjunction ("generalized Stone duality").
- In particular, from this adjunction we get $\mathbf{Pretop}(T_1, T_2) \simeq \mathbf{Ult}(\mathbf{Mod}(T_2), \mathbf{Mod}(T_1)).$

Therefore, SCC tells us how to recognize a reduct functor in the wild between two categories of models—i.e., if there is some uniformity underlying a functor $\mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ due to a purely syntactic assignment $T_1 \to T_2$. Just check if the ultra-structure is preserved!

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A definability criterion for ℵ₀-categorica theories

Exotic functor

Caveat. Of course, one has an infinite list of conditions to verify here.

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Evotic functor

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Exotic functor

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- So the only way to actually do this is to recognize some kind of uniformity in the putative reduct functor which lets you take care of all the ultramorphisms at once.
- But it gives you another way to think about uniformities you need.
- It also gives you a way to check that something can never arise from any interpretation!

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Ultramorphisms, I

Part of the critera for (X, Φ) (a functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ plus a choice of transition isomorphism Φ) to be definable was "preserving ultramorphisms."

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- What are ultramorphisms? Loosely speaking, ultraproducts are a kind of universal construction in Set, and so there are certain canonical comparison maps between them induced by their universal properties. (By the Los theorem, these things are "absolute" in the sense that no matter what first-order structure you put on a set, these maps will always be elementary embeddings.)

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- Out of mercy, I will spare you the formal definition (because then I'd have to define ultragraphs, ultradiagrams, and ultratransformations...)
- Keep in mind these two examples:



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Ultramorphisms, II

Examples.

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Continuous

Ultramorphisms, II

Examples.

► The diagonal embedding into an ultrapower.

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Ultramorphisms, II

Examples.

- ► The diagonal embedding into an ultrapower.
- Generalized diagonal embeddings. More generally, let f: I → J be a function, let U be an ultrafilter on I and let V be the pushforward ultrafilter on J.

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Ultramorphisms, II

Examples.

- ► The diagonal embedding into an ultrapower.
- Generalized diagonal embeddings. More generally, let $f: I \to J$ be a function, let $\mathcal U$ be an ultrafilter on I and let $\mathcal V$ be the pushforward ultrafilter on J. Then for any I-indexed sequence of structures $(M_i)_{i\in I}$, there is a canonical map $\delta_f: \prod_{j\to \mathcal V} M_{f(i)} \to \prod_{i\to \mathcal U} M_i$ given by taking the diagonal embedding along each fiber of f.

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Δ -functors induce continuous maps on automorphism groups

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Δ -functors induce continuous maps on automorphism groups

Why should we expect ultramorphisms to help us identify evaluation functors in the wild?

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- Here's an result which might indicate that knowing that they're preserved tells us something nontrivial.

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Exotic functo

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- Why should we expect ultramorphisms to help us identify evaluation functors in the wild?
- Here's an result which might indicate that knowing that they're preserved tells us something nontrivial.

Definition.

Say that $X: \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is a Δ -functor if it preserves ultraproducts and diagonal maps into ultrapowers.

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Say that $X : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is a Δ -functor if it preserves ultraproducts and diagonal maps into ultrapowers. Equip automorphism groups with the topology of pointwise convergence.

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Exotic functor

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- Here's an result which might indicate that knowing that they're preserved tells us something nontrivial.

Definition.

Say that $X : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is a Δ -functor if it preserves ultraproducts and diagonal maps into ultrapowers. Equip automorphism groups with the topology of pointwise convergence.

Theorem.

If X is a Δ -functor from $\mathbf{Mod}(T)$ to $\mathbf{Mod}(T')$, then X restricts to a continuous map $\mathrm{Aut}(M) \to \mathrm{Aut}(X(M))$ for every $M \in \mathbf{Mod}(T)$.

Proof.

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Proof.

The topology of pointwise convergence is sequential, so to check continuity it suffices to check convergent sequences of automorphisms are preserved.

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Evotic functor

- The topology of pointwise convergence is sequential, so to check continuity it suffices to check convergent sequences of automorphisms are preserved.
- If $f_i \to f$ in $\operatorname{Aut}(M)$, then since the cofinite filter is contained in any ultrafilter, $\prod_{i \to \mathcal{U}} f_i$ agrees with $\prod_{i \to \mathcal{U}} f$ over the diagonal copy of M in $M^{\mathcal{U}}$.

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- Applying X and using that X is a Δ -functor, conclude that $\prod_{i \to \mathcal{U}} X(f_i)$ agrees with $\prod_{i \to \mathcal{U}} X(f)$ over the diagonal copy of X(M) inside $X(M)^{\mathcal{U}}$.

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- For any point $a \in X(M)$, the above says the sequence $(X(f_i)(a))_{i \in I} = \mathcal{U}(X(f)(a))_{i \in I}$.

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- Applying X and using that X is a Δ -functor, conclude that $\prod_{i \to \mathcal{U}} X(f_i)$ agrees with $\prod_{i \to \mathcal{U}} X(f)$ over the diagonal copy of X(M) inside $X(M)^{\mathcal{U}}$.
- For any point $a \in X(M)$, the above says the sequence $(X(f_i)(a))_{i \in I} =_{\mathcal{U}} (X(f)(a))_{i \in I}$.
- Since \mathcal{U} was arbitrary and the cofinite filter on I is the intersection of all non-principal ultrafilters on I, we conclude that the above equation holds cofinitely. Hence, $X(f_i) \to X(f)$.

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\aleph_0 -categorical theories

A first-order theory T is ℵ₀-categorical if it has one countable model up to isomorphism.

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- A first-order theory T is \aleph_0 -categorical if it has one countable model up to isomorphism.
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- A first-order theory T is \aleph_0 -categorical if it has one countable model up to isomorphism.
- A theorem of Coquand, Ahlbrandt and Ziegler says that, given two \aleph_0 -categorical theories T and T' with countable models M and M', a topological isomorphism $\operatorname{Aut}(M) \simeq \operatorname{Aut}(M')$ induces a bi-interpretation $M \simeq M'$.

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- Since we know Δ-functors induce continuous maps on automorphism groups, they're a good candidate for definable functors.

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- Since we know Δ-functors induce continuous maps on automorphism groups, they're a good candidate for definable functors.
- Boolean coherent toposes split into a finite coproduct of $\mathscr{E}(T_i)$, where each T_i is \aleph_0 -categorical.

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Theorem.

Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$. If T is \aleph_0 -categorical, the following are equivalent:

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Theorem.

Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$. If T is \aleph_0 -categorical, the following are equivalent:

1. For some transition isomorphism, (X, Φ) is a Δ -functor (preserves ultraproducts and diagonal maps).

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Theorem.

Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$. If T is \aleph_0 -categorical, the following are equivalent:

- 1. For some transition isomorphism, (X, Φ) is a Δ -functor (preserves ultraproducts and diagonal maps).
- 2. For some transition isomorphism, (X, Φ) is definable.

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Proof.

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Proof.

(Sketch.)

• One direction is immediate by SCC: definable functors are ultrafunctors are at least Δ -functors.

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Proof.

- One direction is immediate by SCC: definable functors are ultrafunctors are at least Δ -functors.
- Let M be the countable model. Use the lemma about Δ -functors (X,Φ) inducing continuous maps on the automorphism groups (equivalently, (X,Φ) has the finite support property) to cover each $\mathrm{Aut}(M)$ -orbit of X(M) by a projection from an $\mathrm{Aut}(M)$ -orbit of M. By ω -categoricity, the kernel relation of this projection is definable, so we know that X(M) looks like an (a priori, possibly infinite) disjoint union of types.

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- By $Aut(M)^{\mathcal{U}}$ orbit-counting, there are actually only finitely many types.

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- By $Aut(M)^{\mathcal{U}}$ orbit-counting, there are actually only finitely many types.
- Invoke the Keisler-Shelah theorem to transfer to all $N \models T$.



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Corollary.

Let T and T' be \aleph_0 -categorical. Let X be an equivalence of categories

$$\mathsf{Mod}(\mathcal{T}_1) \overset{X}{\simeq} \mathsf{Mod}(\mathcal{T}_2).$$

Then X was induced by a bi-interpretation $T_1 \simeq T_2$ if and only if X was a Δ -functor.

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Then X was induced by a bi-interpretation $T_1 \simeq T_2$ if and only if X was a Δ -functor.

In particular, Bodirsky, Evans, Kompatscher and Pinkser gave an example of two \aleph_0 -categorical theories T,T' with abstractly isomorphic but not topologically isomorphic automorphism groups of the countable model. This abstract isomorphism induces an equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ and since it can't come from an interpretation, from the corollary we conclude that it fails to preserve an ultraproduct or a diagonal map was not preserved.

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Exotic pre-ultrafunctors

In light of the previous result, a natural question to ask is:

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In light of the previous result, a natural question to ask is:

Question.

Is being a Δ -functor enough for SCC? That is, do non-definable Δ -functors exist?

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The previous definability criterion fails for general T. That is:

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Is being a Δ -functor enough for SCC? That is, do non-definable Δ -functors exist?

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The previous definability criterion fails for general T. That is:

There exists a theory T and a Δ -functor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ which is not definable.

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Is being a Δ -functor enough for SCC? That is, do non-definable Δ -functors exist?

Theorem.

The previous definability criterion fails for general T. That is:

- There exists a theory T and a Δ -functor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ which is not definable.
- There exists a theory T and a pre-ultrafunctor (X, Φ) which is not a Δ -functor (hence, is also not definable.)

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Proof.

 $(\mathsf{Sketch.})$

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Proof.

(Sketch.)

Complete types won't work, so take a complete type and cut it in half into two partial types, one of which refines the other. Define X(M) to be the realizations in M of the coarser one.

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Proof.

- Complete types won't work, so take a complete type and cut it in half into two partial types, one of which refines the other. Define X(M) to be the realizations in M of the coarser one.
- Taking ultraproducts creates external realizations ("infinite/infinitesimal points") of either one.

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Proof.

- Complete types won't work, so take a complete type and cut it in half into two partial types, one of which refines the other. Define X(M) to be the realizations in M of the coarser one.
- Taking ultraproducts creates external realizations ("infinite/infinitesimal points") of either one.
- You can either try to construct a transition isomorphism which turns it into a pre-ultrafunctor (creating a non- Δ pre-ultrafunctor) or obtain one non-constructively (creating a non-definable Δ -functor).

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Future work

Is the above X(M) isomorphic to ev_A for some $A \in \mathcal{E}(T)$?

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Future work

- Is the above X(M) isomorphic to ev_A for some $A \in \mathcal{E}(T)$?
- Which parts of Makkai's ultra-data ensure X : Mod(T) → Set is ev_A for A ∈ ℰ and which parts make sure that A is compact?

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- How do ultramorphisms relate to the Awodey-Forssell duality?

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- Which parts of Makkai's ultra-data ensure X : Mod(T) → Set is ev_A for A ∈ ℰ and which parts make sure that A is compact?
- How do ultramorphisms relate to the Awodey-Forssell duality?
- Conjecture: the pre-ultrafunctor part of the data ensures compactness after you get inside the classifying topos, i.e. if you start with $A \in \mathscr{E}$ and ev_A is an ultrafunctor, then A was compact.

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Thank you!