Constructing Categories of Corelations
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I. Motivation
The equational theory $\mathbb{II}$

interaction of $\mathbb{II}$ with $\mathbb{II}$

the theory $\mathbb{HA}_{k[x]}$ of $k[x]$-Hopf algebras

the theory $\mathbb{HA}_{k[x]}^{op}$ of "co"$k[x]$-Hopf algebras

interaction of $\mathbb{II}$ with $\mathbb{II}$
The equational theory $\mathbb{IH}$

This is a presentation of the category $\text{LinRel}_k(x)$. 
How do we prove this?
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$$\begin{array}{ccc}
\text{Vect} + \text{Vect}^{\text{op}} & \longrightarrow & \text{Span}(\text{Vect}) \\
\downarrow & & \downarrow \\
\text{Cospan}(\text{Vect}) & \longrightarrow & \text{LinRel}
\end{array}$$
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Linear relations interpret diagrams of linear maps

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where we may compose by function composition, pullback, and pushout.
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This leads to a presentation of LinRel.
Colimits combine systems.
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- composition (pushout),
- monoidal product (binary coproducts), and
- monoidal unit (initial object).
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Thus cospan categories provide useful language for system interconnection.
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- composition (pushout),
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Thus cospan categories provide useful language for system interconnection.

However, combining systems using colimits indiscriminately accumulates information.
Consider cospans in FinSet.
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\[
X \quad \longrightarrow \quad N \quad \longleftarrow \quad Y
\]
Consider cospans in FinSet.

If we think about these as circuits, all we care about is the induced equivalence relation on $X + Y$. 
Cospans accumulate internal structure (witnesses for 'empty equivalence classes'). Corelations forget this.
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\[ X \rightarrow N \leftarrow Y \rightarrow M \leftarrow Z \]

\[ = X \quad Z \]
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Factorisation hides internal structure.
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A factorisation system \((\mathcal{E}, \mathcal{M})\) comprises subcategories \(\mathcal{E}, \mathcal{M}\) such that
- \(\mathcal{E}\) and \(\mathcal{M}\) contain all isomorphisms
- every \(f\) admits factorisation \(f = m \circ e\).
- we have the universal property:

\[
\begin{array}{ccc}
  & e & \rightarrow \\
  u \downarrow & & \downarrow m \\
  \rightarrow e' & \rightarrow \exists! s \\
  \downarrow & & \downarrow v \\
  \rightarrow e' & \rightarrow m'
\end{array}
\]

For example, epi–mono factorisation systems (like in \(\text{FinSet}\)).
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\begin{array}{ccc}
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  u & \downarrow & \downarrow & v \\
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\end{array}
\]

For example, epi–mono factorisation systems (like in FinSet).
A **corelation** is an equivalence class of cospans, where two cospans are equivalent if

\[
\begin{array}{c}
N \\
\uparrow \quad \uparrow \quad \uparrow \\
X & & Y
\end{array}
\]

When \( M \) is stable under pushout, composition by pushout defines a category \( \text{Corel}(C) \).
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![Diagram]

We may represent each corelation by a cospan such that \( X + Y \to N \) lies in \( \mathcal{E} \).
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![Diagram of corelation]

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What is the link?
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\[ \text{Vect} + \text{Vect}^{\text{op}} \rightarrow \text{Span}(\text{Vect}) \]

\[ \text{Cospan}(\text{Vect}) \rightarrow \text{LinRel} \cong \text{Corel}(\text{Vect}) \]
What is the link?

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\begin{array}{c}
\text{Vect} + \text{Vect}^{\text{op}} \longrightarrow \text{Span} (\text{Vect}) \\
\downarrow \hspace{3cm} \downarrow \\
\text{Cospan} (\text{Vect}) \longrightarrow \text{LinRel} \cong \text{Corel} (\text{Vect})
\end{array}
\]

So we claim:

I. Corelations model system interconnection and
II. A universal property is useful for computing presentations.
What is the link?

\[
\begin{align*}
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\downarrow \ &\quad \quad \quad \downarrow \\
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\end{align*}
\]

So we claim:

I. Corelations model system interconnection and

II. A universal property is useful for computing presentations.

Does this universal construction generalise to other corelation categories?
II. A Universal Property for Corelations
A functor $\text{Span}(C) \to \text{Corel}(C)$ does not in general exist. Under what conditions might it exist?
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Define a map $\text{Span}_C(A) \rightarrow \text{Corel}(C)$ by taking pushouts.
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These two cospans represent the same corelation when the canonical map lies in $\mathcal{M}$. 

\begin{tikzpicture}
  \node (A) at (0,0) {
  \begin{tikzpicture}
    \node (a) at (0,0) {$A$};
    \node (a1) at (1,1) {$A$};
    \node (a2) at (1,-1) {$A$};
    \node (b) at (2,0) {$C$};
    \node (c) at (4,0) {$C$};
    \draw[->] (a) -- (a1);
    \draw[->] (a) -- (a2);
    \draw[->] (a1) -- (b);
    \draw[->] (a2) -- (b);
    \draw[->] (b) -- (c);
  \end{tikzpicture}
  \};
  \node (B) at (5,1) {$m$};
  \node (C) at (5,-1) {?};
  \draw[->] (A) -- (B); \draw[->] (A) -- (C);
\end{tikzpicture}
Define a map \( \text{Span}_C(\mathcal{A}) \to \text{Corel}(C) \) by taking pushouts. When is this functorial?

These two cospans represent the same corelation when the canonical map lies in \( \mathcal{M} \).
Call this the **pullback–pushout property** (with respect to \( \mathcal{M} \)).
Define a map \( \text{Span}_C(\mathcal{A}) \to \text{Corel}(\mathcal{C}) \) by taking pushouts. When is this functorial?

These two cospans represent the same corelation when the canonical map lies in \( \mathcal{M} \).
Call this the **pullback–pushout property** (with respect to \( \mathcal{M} \)).

When \( \mathcal{A} \) obeys the pullback–pushout property, then there exists a functor \( \text{Span}_C(\mathcal{A}) \to \text{Corel}(\mathcal{C}) \).
Theorem

Suppose a category $C$ has pushouts and pullbacks, a factorisation system $(E, M)$ with $M \subseteq \text{monos}$, stable under pushout such that $M$ obeys the pullback–pushout property. Then we have a pushout square in $\text{Cat}$:
Theorem

Suppose a category $\mathcal{C}$ has
- pushouts and pullbacks
- a factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M} \subseteq$ monos, stable under pushout
- such that $\mathcal{M}$ obeys the pullback–pushout property.
Theorem

Suppose a category $\mathcal{C}$ has

- pushouts and pullbacks
- a factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M} \subseteq$ monos, stable under pushout
- such that $\mathcal{M}$ obeys the pullback–pushout property.

Then we have a pushout square in $\mathbf{Cat}$:

\[
\begin{array}{ccc}
\mathcal{M} +_{|\mathcal{M}|} \mathcal{M}^{\text{op}} & \longrightarrow & \text{Span}_{\mathcal{C}}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\text{Cospan}(\mathcal{C}) & \longrightarrow & \text{Corel}(\mathcal{C})
\end{array}
\]
Theorem: generalising $\mathcal{M}$

Suppose $\mathcal{C}$ has

- pushouts and pullbacks
- a factorisation system with $\mathcal{M} \subseteq \text{monos}$, stable under pushout
- a subcategory $\mathcal{A} \supseteq \mathcal{M}$, stable under pullback, obeying the pullback–pushout property.

Then we have a pushout square in $\text{Cat}$:

$$
\begin{array}{ccc}
\mathcal{A} + \vert \mathcal{A} \vert \mathcal{A}^{\text{op}} & \to & \text{Span}_\mathcal{C}(\mathcal{A}) \\
\downarrow & & \downarrow \\
\text{Cospan}(\mathcal{C}) & \to & \text{Corel}(\mathcal{C})
\end{array}
$$
Corollary: abelian case

Let $C$ be an abelian category. This has a (co)stable epi–mono factorisation system.

The theorem can also be extended to monoidal categories, by requiring that the monoidal product preserve pushouts in $C$ and pullbacks in $A$, and that $M$ and $A$ are closed under the monoidal product.
Corollary: abelian case

Let $\mathcal{C}$ be an abelian category. This has a (co)stable epi–mono factorisation system.

We have a pushout square in $\textbf{Cat}$:

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\begin{array}{ccc}
\mathcal{C} + |\mathcal{C}| & \rightarrow & \text{Span}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Cospan}(\mathcal{C}) & \rightarrow & \text{Corel}(\mathcal{C}) \cong \text{Rel}(\mathcal{C})
\end{array}
\]
Let $C$ be an abelian category. This has a (co)stable epi–mono factorisation system.
We have a pushout square in $\text{Cat}$:

$$
\begin{array}{ccc}
C + |C| \times C^{\text{op}} & \rightarrow & \text{Span}(C) \\
\downarrow & & \downarrow \\
\text{Cospan}(C) & \rightarrow & \text{Corel}(C) \cong \text{Rel}(C)
\end{array}
$$

The theorem can also be extended to monoidal categories, by requiring that the monoidal product preserve pushouts in $C$ and pullbacks in $\mathcal{A}$, and that $\mathcal{M}$ and $\mathcal{A}$ are closed under the monoidal product.
Examples

Corelations (Equivalence relations):

\[ \text{Inj} + \text{Inj}^{\text{op}} \longrightarrow \text{Span} (\text{Inj}) \]
\[ \downarrow \]
\[ \text{Cospan} (\text{FinSet}) \longrightarrow \text{Corel} (\text{FinSet}) \]

Partial equivalence relations:

\[ \text{PInj} + \text{PInj}^{\text{op}} \longrightarrow \text{Span} (\text{PInj}) \]
\[ \downarrow \]
\[ \text{Cospan} (\text{ParFunc}) \longrightarrow \text{PER} \cong \text{Corel} (\text{ParFunc}) \]
Examples

Linear relations:

\[
\begin{align*}
\text{Vect} + \bullet \text{Vect}^{\text{op}} & \longrightarrow \text{Span}(\text{Vect}) \\
\downarrow & \\
\text{Cospan}(\text{Vect}) & \longrightarrow \text{Corel}(\text{Vect}) \cong \text{LinRel}
\end{align*}
\]

Discrete time, linear, time-invariant, dynamical systems over \( k \):

\[
\begin{align*}
\text{SpltM} + \bullet \text{SpltM}^{\text{op}} & \longrightarrow \text{Span}(\text{SpltM}) \\
\downarrow & \\
\text{Cospan}(\text{Mat}_k[s,s^{-1}]) & \longrightarrow \text{Corel}(\text{Mat}_k[s,s^{-1}])
\end{align*}
\]
Examples

Let $T$ be a comonad on $\text{Set}$ such that $T$ and $T^2$ both preserve pullbacks of regular monos. Then the category $\text{Set}^T$ of coalgebras over $T$ obeys the theorem with respect to (epis, regular monos).
Examples

Let $T$ be a comonad on $\text{Set}$ such that $T$ and $T^2$ both preserve pullbacks of regular monos. Then the category $\text{Set}^T$ of coalgebras over $T$ obeys the theorem with respect to (epis, regular monos).

This property is obeyed by the cofree comonad on the double finite power set functor, which has been used to model logic programs.
Theorem: dual case

Suppose a category $\mathcal{C}$ has

- pushouts and pullbacks
- a factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} \subseteq \text{epis}$, stable under pullback
- such that $\mathcal{E}$ obeys the pullback–pushout property.

Then we have a pushout square in $\text{Cat}$:

\[
\begin{array}{ccc}
\mathcal{E} + |\mathcal{E}| \mathcal{E}^{\text{op}} & \longrightarrow & \text{Span}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Cospan}(\mathcal{E}) & \longrightarrow & \text{Rel}(\mathcal{C})
\end{array}
\]
Non-example: Relations

Surj does not obey pushout–pullback property.

\[
\text{Surj} + \text{Surj}^{\text{op}} \longrightarrow \text{Span}(\text{FinSet})
\]

\[
\downarrow \quad \downarrow
\]

\[
\text{Cospan}(	ext{Surj}) \longrightarrow \text{Rel}(\text{FinSet})
\]
Non-example: Relations

Surj does not obey pushout–pullback property.

\[
\begin{align*}
\text{Surj + Surj}^{\text{op}} & \longrightarrow \text{Span}(\text{FinSet}) \\
\downarrow & \\
\text{Cospan(Surj)} & \longrightarrow \text{Rel}(\text{FinSet})
\end{align*}
\]
Non-example: Relations

Surj does not obey pushout–pullback property.

\[ \text{Surj} + \text{Surj}^{\text{op}} \longrightarrow \text{Span}(\text{FinSet}) \]

\[ \downarrow \quad \downarrow \]

\[ \text{Cospans}(\text{Surj}) \longleftarrow \text{Rel}(\text{FinSet}) \]

\[ \begin{array}{ccc}
3 & \longrightarrow & 2 \\
\downarrow & & \downarrow \\
2 & \longleftarrow & 2 \\
\downarrow & & \downarrow \\
1 & \longleftarrow & 1
\end{array} \]
Non-example: Relations

Surj does not obey pushout–pullback property.

\[
\begin{align*}
\text{Surj} + \text{Surj}^{\text{op}} & \longrightarrow \text{Span}(\text{FinSet}) \\
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Non-example: Relations

Surj does not obey pushout–pullback property.

\[
\begin{array}{c}
\text{Surj + Surj}^{\text{op}} \\
\downarrow \\
\text{Cospan(Surj)}
\end{array} \rightarrow \begin{array}{c}
\text{Span(FinSet)} \\
\downarrow \\
\text{Rel(FinSet)}
\end{array}
\]

Not an epi!
Non-example: Relations

Surj does not obey pushout–pullback property.

\[
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\end{array}
\]

Not an epi! (We cannot construct \(\text{Rel} = \text{Rel}(\text{FinSet})\) as a pushout.)
To recap:

I. Corelations model system interconnection

II. Categories of corelations can be constructed as a pushout of span and cospan categories.

III. This helps derive presentations.
I thank Fabio Zanasi for collaborating on this work. Thank you for listening.

For more: http://www.brendanfong.com/